N-630

GEODESIC DISTANCE AND IMAGE ANALYSIS

Ch. Lantuéjoul S. Beucher

Fontainebleau Novembre 1979

GEODESIC DISTANCE AND IMAGE ANALYSIS

Christian LANTUEJOUL Serge BEUCHER

Centre de Morphologie Mathématique
35, rue Saint-Honoré - 77305 Fontainebleau (FRANCE)

0) ABSTRACT

Let X be a population of objects. Given two arbitrary points x and y belonging to X, let us define the number $d_X(x,y)$ as follows : $d_X(x,y)$ is the greatest lower bound of the lengths of the arcs in X ending at points x and y, if such arcs exist, and $+\infty$ if not. The function d_X is a distance function, called "geodesic distance". Note that if x and y belong to two distinct objects of X, $d_X(x,y) = +\infty$. In other words, d_X seems to be an appropriate distance function to deal with connectivity problems. Furthermore, since d_X is defined without any reference to the background in which the objects are embedded, it is possible with this distance function to provide an objective meaning for intrinsic properties of an object, such as the centre of its circumscribed ball, or its ends.

1) INTRODUCTION

Many specimens in image analysis have two different phases. The first phase contains the objects under study (biological cells, stringers in steel, etc.). The second phase is the background.

Sometimes, the background is not significant; it may be none other than the support of the objects under study (e.g. a slide with a smear). In this case, the background must have no influence on the quantization of the objects under study.

The aim of this paper is to present a method for studying objects in an intrinsic way, that is, independently of the background in which they are embedded. This method requires the introduction of a special metric on the objects under study, which is the geodesic distance function. Before defining this metric, we first consider the intuitive problem of measuring the length of a fibre. Afterwards, it is possible to give an objective meaning to notions like the centre of the circumscribed ball, or the ends of a particle.

2) HOW LONG IS A FIBRE ?

In image analysis, many specimens containing long narrow particles can occur (e.g. glass or asbestos fibres, eutectic alloy lamellae, neuron arms, etc.). People concerned with such specimens usually want to know the distribution function of the lengths of these particles. But at first, we must ponder the question: what is the length of a particle ?

Let us consider a particle X with an arbitrary shape, and let x and y be two points of X. There exist several arcs in X linking x and y (cf. Fig. 1-1); the shortest among them is called "geodesic arc" (cf. Fig. 1-2), and its length is denoted $d_X(x,y)$.

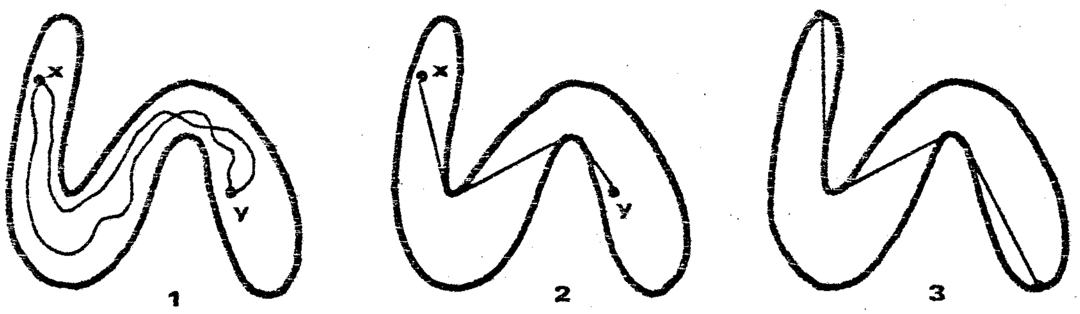

Figure 1

Now we choose to define the length of the particles as the length of the longest geodesic arc of this particle (cf. Fig. 1-3) :

$$L(X) = \sup_{x,y \in X} d_X(x,y)$$

This definition may seem somewhat arbitrary. But it leads to results which are quite consistent with intuition, and moreover, it has three potent advantages :

i) this definition is general, for it is applicable to particles of any shape.

ii) this definition is robust, in the sense that a slight deformation of a particle scarcely modifies its length.

iii) this definition is operational, insofar as the technology to apply it either exists or can be devised.

3) THE GEODESIC DISTANCE FUNCTION

Assume now that X is not a particle, but a population of particles. If the two points x and y belong to two distinct particles, there is no arc in X linking x and y, and we write $d_X(x,y) = +\infty$. It can be then easily shown that the function d_X satisfies all the properties of a distance function:

i) $d_X(x,y) \geq 0$ and $d_X(x,y) = 0$ if and only if $x = y$

ii) $d_X(x,y) = d_X(y,x)$

iii) $d_X(x,z) \leq d_X(x,y) + d_X(y,z)$

In what follows, the function d_X is called "geodesic distance function". In Fig. 2-1 and 2-2, a comparison is drawn between the circumferences of balls of centres x and radii λ, derived from the two definitions of distance : $\partial B_X(x,\lambda)$ is derived from the geodesic metric d_X, and $\partial B(x,\lambda)$ from the natural metric of the space $\mathbb{R}^2$ in which X is embedded. In order to have a better understanding of Fig. 2-1, imagine that the particles are a string of ponds, and that a stone is thrown into one of them. A front of ripples appears, and we observe them at successive moments.

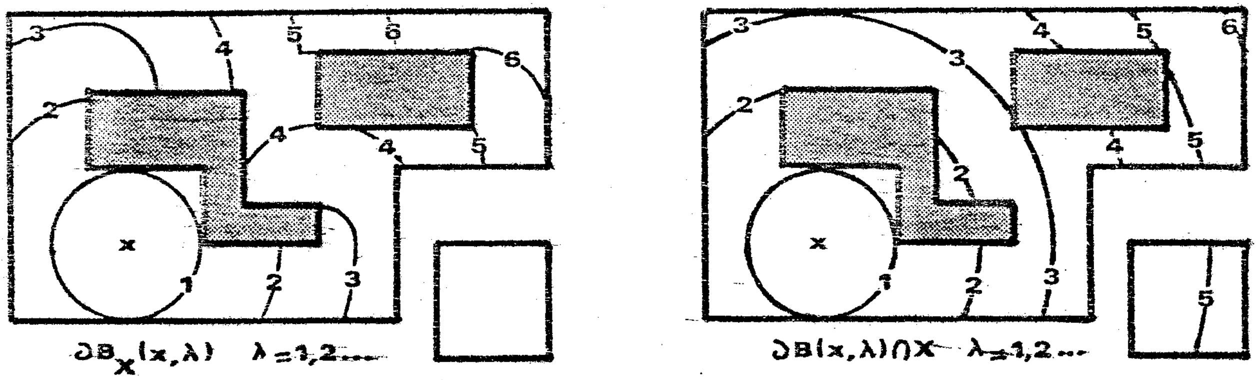

Figure 2

With the metric d_X, we can define the geodesic distance between a point x of X and a subset Y of X : $d_X(x,Y)$ is the smallest geodesic distance between x and any point y of Y :

$$d_X(x,Y) = \inf_{y \in Y} d_X(x,y)$$

The main interest of the geodesic distance function lies in the fact that it is perfectly adapted to dealing with connectivity problems. A concrete illustration of this remark is given by the following example. Consider the biological images X and Y. X is a population of cells with parts of broken cells, artifacts, etc.; Y is the population of the nuclei. X and Y are obtained using a double staining technique. The only cells that must be studied are the complete cells containing a nucleus. The other objects are just artifacts and must be disregarded. How can we detect cells of X which have a nucleus ?

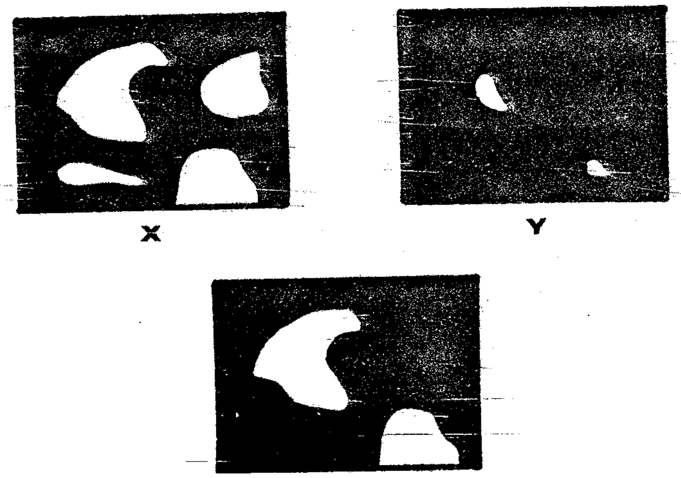

Figure 3

Let x be a point in a cell that contains a nucleus. There exists an arc in the cell linking x and a point y of the nucleus. In other words, the geodesic distance between x and the nuclei is finite. Mathematically speaking, the population of cells with a nucleus is :

$$\{x \in X \mid d_X(x,Y) < +\infty\}$$

4) CENTRE OF THE CIRCUMSCRIBED BALL OF A PARTICLE WITHOUT HOLES

It is often necessary to mark a particle with a point. In order to increase the speed of subsequent processing, the point must not be arbitrarily located within the particle. In what follows, we are going to associate with each simply connected particle a point which is none other than the centre of the circumscribed circle when the particle is a triangle with non-obtuse angles.

Let us again imagine that X is a pond. When a stone is thrown into the pond at point x, we can measure the first time $\lambda(x)$ at which all the shores have been reached by the ripples :

$$\lambda(x) = \sup_{y \in X} d_X(x,y)$$

We thus introduce a function λ which is represented by time-level lines on Fig. 4.

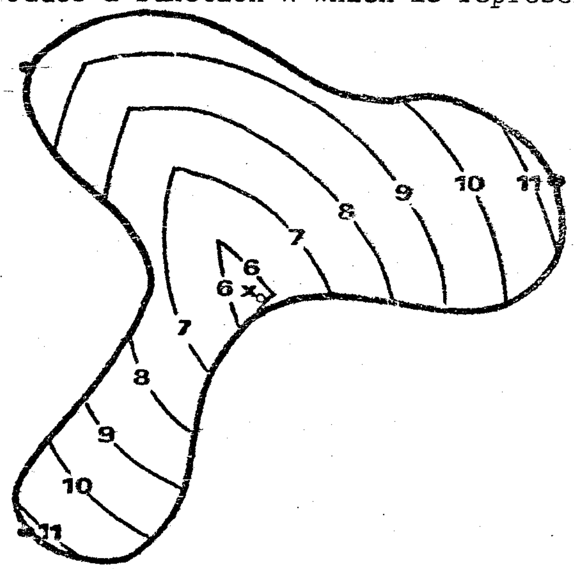

Figure 4

It can be shown (cf. Appendix) that this function is continuous on X and has a single minimum. The point x_o such that $\lambda(x_o)$ is minimum, is called "centre of the circumscribed ball of the particle" and the numerical value $\lambda(x_o)$ is the radius of this ball.

If the particle is not simply connected, the function X does not have a minimum which is a single point (e.g. in an annulus, λ is minimum on a circle surrounding the hole). This remark can be helpful in detecting simply-connected particles.

5) ENDS OF A PARTICLE

Unless X is reduced to a single point, the function λ always has several maxima (cf. Fig. 4). The points x such that $\lambda(x)$ is a maximum are located on the boundary of X. They are called the ends of the particle X. In the case where X is a triangle with acute angles, the ends are their vertices.

ACKNOWLEDGEMENT :

The authors wish to thank John PIRET for his kind help during the preparation of this paper.

APPENDIX : Proof that the centre of the circumscribed disk of a simply connected particle exists and is unique.

In order to establish the proof, we make the following assumptions :

i) X is a simply connected compact set for the metric d_X.

ii) for every $x \in X$, $\lambda(x)$ is finite.

These two assumptions are realistic in practice, and avoid pathological sets which could occur mathematically. For instance, let X be the hyperbolic spiral of polar equation $\rho = 1/\theta$, with $\theta \in [1, +\infty]$; the point whose polar coordinates are $(+\infty, 0)$ is at an infinite distance from any other point of X. X is a compact set in the euclidian sense, but not in the geodesic sense. Moreover, X is not geodesically connected !

So, let us assume $\lambda < +\infty$. λ is continuous, for $|\lambda(x) - \lambda(y)| \leq d_X(x,y)$. Since X is compact, λ attains its minimum value at at least one point x_o of X. It now remains to see that x_o is unique.

Let x, y and z be three points of X. Suppose that the domain within X that is bounded by the three geodesic arcs G(x,y), G(y,z) and G(z,x) is simply connected (which always occurs if X is itself simply connected). Then for any point $t \in G(y,z)$ different from y and z, the following convexity inequality holds :

$$d_X(t,x) < \sup \left[d_X(y,x) \ , \ d_X(z,x) \right]$$

Let us now suppose that the function λ has two minimal points x_o and x_1. Let x be a point of the geodesic $G(x_o, x_1)$ different from x_o and x_1. There exists a point y X such that $\lambda(x) = d_X(x,y)$. Using the convexity inequality, we obtain :

$$\lambda(x) = d_X(x,y) < \sup \left[d_X(x_o \ , y) \ , \ d_X(x_1 \ , y) \right] \leq \lambda(x_o)$$

which is a contradiction.