Critical Balls

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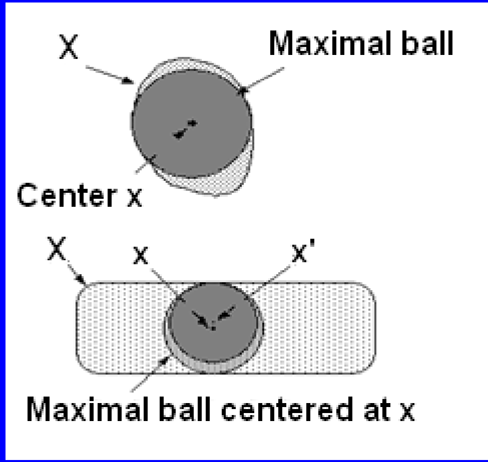
Maximal balls and skeleton

A ball $B_n(x)$, centered at x and of size n, is maximal in set X if there exists no value k and no center y such that:

$$B_n(x) \subset B_k(y) \subset X \quad n \leq k$$



The skeleton of a set X is defined as the set of the centers of its maximal balls:

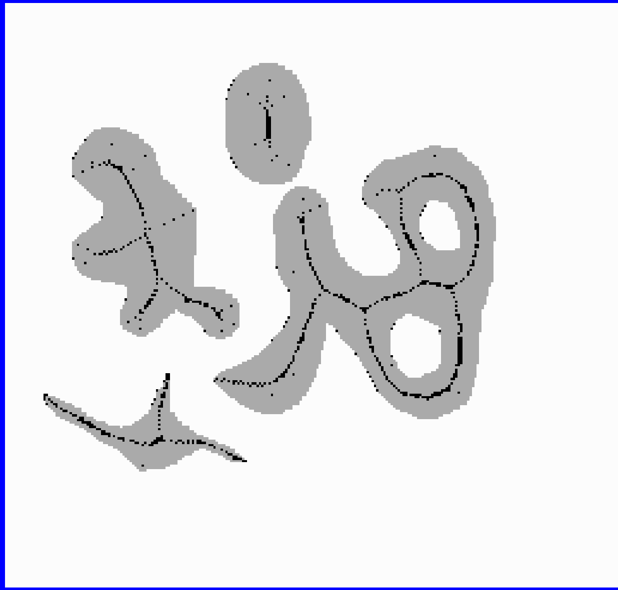
$$S(X) = \{x \in X : \exists B_n(x) \text{maximal}\}$$

The maximal ball skeleton corresponds to the residues of the openings of the successive erosions of X:

$$S(X) = \bigcup_{i \in N} [\varepsilon_i(X) \setminus (\gamma \circ \varepsilon_i(X))]$$

The set X can be entirely rebuilt from its skeleton:

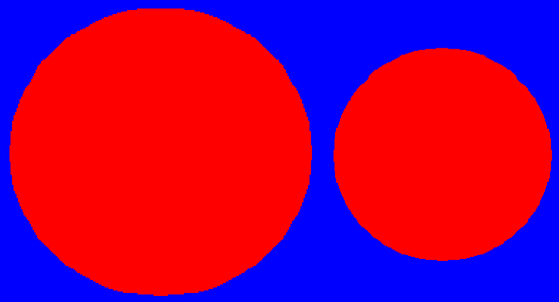
$$X = \bigcup_{i \in N} \delta^i(S_i(X))$$



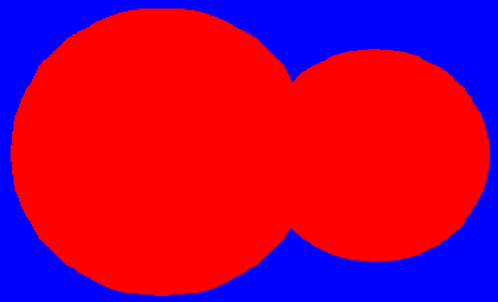
Skeleton as shape descriptor ?

The maximal balls skeleton is not a good shape descriptor.

One generally prefer to describe a complicated shape as an assembly of simpler shapes:

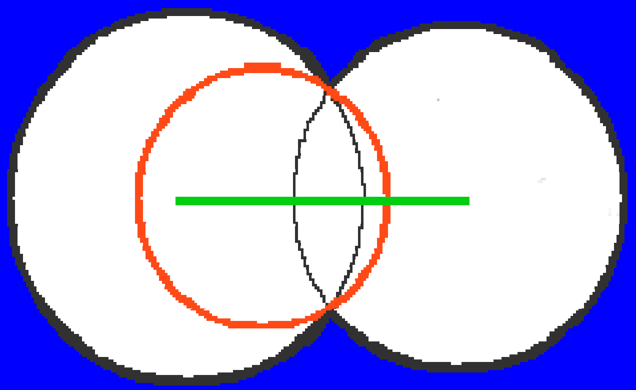


2 disks



This shape is simply the union of two disks

However, the skeleton of this set exhibits an <u>infinity</u> of new disks!



The skeleton is not stable for the union of sets:

$$S(X \cup Y) \neq S(X) \cup S(Y)$$

Critical balls, definition and properties

The skeleton (position and radius of the maximal balls) of a set X is redundant to rebuilt it. Only a subset of it is compulsory, the critical balls set.

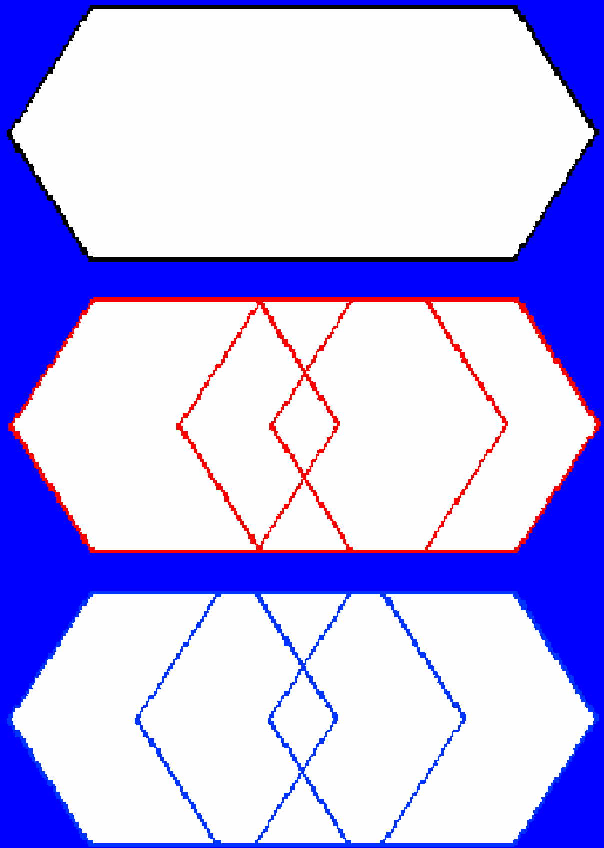
A maximal B of a set X is critical when there is no combination of other maximal balls which covers B:

$$\nexists \{B_k\}, B_k \neq B : B \subset \bigcup_k B_k$$

If B, critical ball of X, there is at least one point x of X which is covered <u>only</u> by this ball (this point is likely to belong to the boundary of X).

For any bounded and closed set X, the critical skeleton S_c (centers of the critical balls) exists and is unique.

Digital critical balls, definition and problems



Unicity is not preserved because of « balls » (polygons) of same size.

Definition of a digital critical ball:
A digital maximal ball B_i of size i is critical if there is no combination of maximal balls B_j , *with i ≠ j,* which covers B_i.

$$B_i \text{ critical} : \nexists J = \{j_1, \ldots, j_n : j_k \neq i\} \text{ such that } B_i \subset \bigcup_{j \in J} B_j$$

Building the critical skeleton

The critical balls can be obtained in two steps:

- selection of the maximal balls not covered by larger balls.
- selection of the maximal balls not covered by smaller balls.

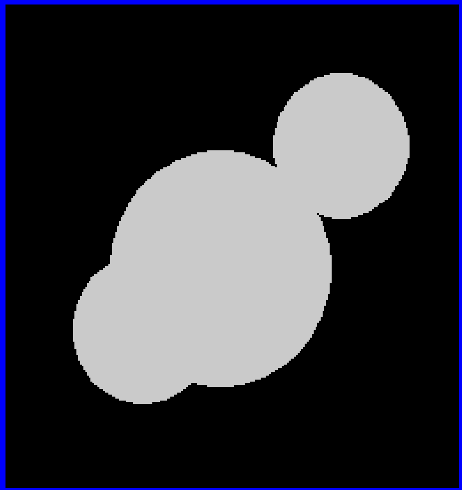
The first step can be achieved by using a residual operator, the ultimate opening and its associated function, the granulometric function.

Ultimate opening

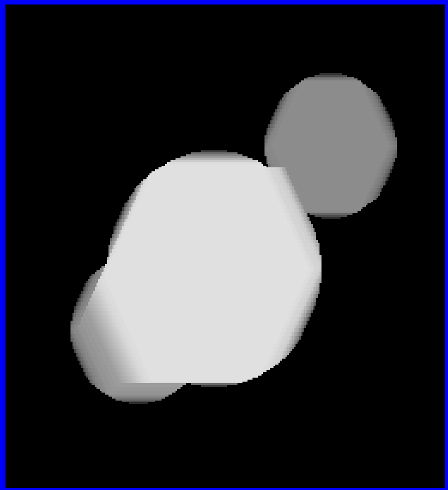
$$\theta = \underset{i \in I}{\mathrm{Sup}}\left(\gamma_{i-1} - \gamma_i\right)$$

Granulometric function

$$c = \arg\max\left(\gamma_{i-1} - \gamma_i\right)$$



Set X



Granulometric function

Granulometric function and quench function

Maximal balls which are not totally covered by larger balls are appearing in the granulometric function.

The quench function q of the skeleton can be used to build the granulometric function:

$$X_i = \{x : q(x) = i\}$$

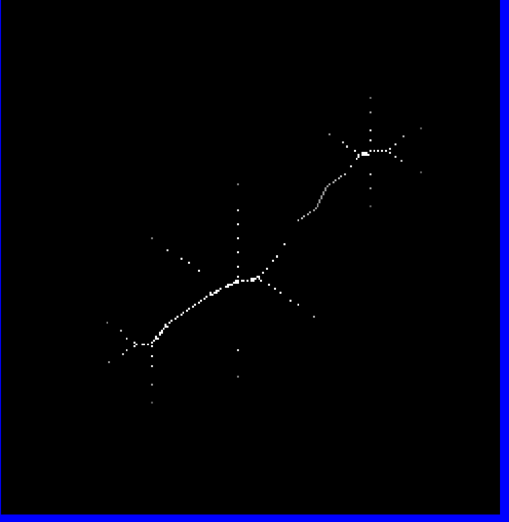
- X_i , centers of balls of size i
- k_{X_i}, valued indicator function of X_i:

$$k_{X_i}(x) = i \ \text{ if } x \in X_i$$

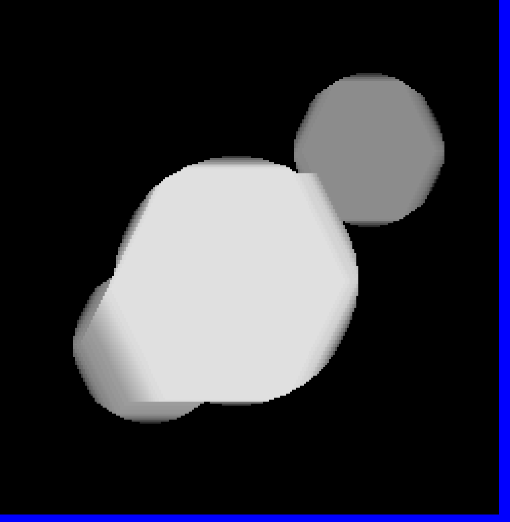
$$k_{X_i}(x) = 0 \ \text{ if not}$$

- Granulometric function c

$$c = \sup_i\left(\delta_i\left(k_{X_i}\right)\right)$$



q, quench function



c, granulometric function

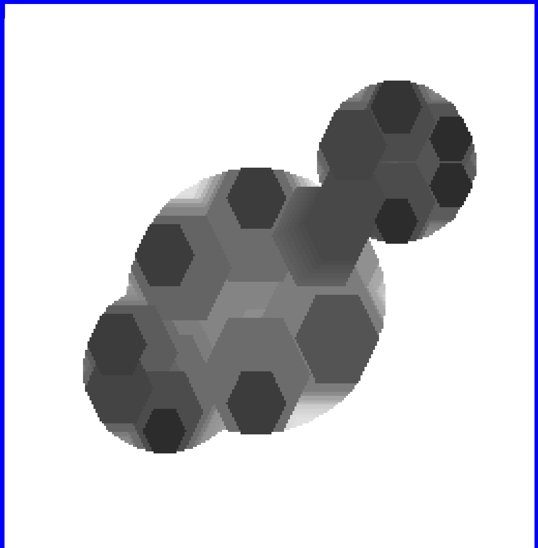
Definition of a dual function

Extracting the maximal balls not covered by smaller balls (2nd step) is made by defining a « dual function » c' from the « dual indicator function » k'_{X_i}:

$$k'_{X_i}(x) = i \quad \text{if } x \in X_i \; ; \; k'_{X_i}(x) = +\infty \quad \text{if not}$$

Maximal balls which are not entirely covered by smaller balls are still visible in c':

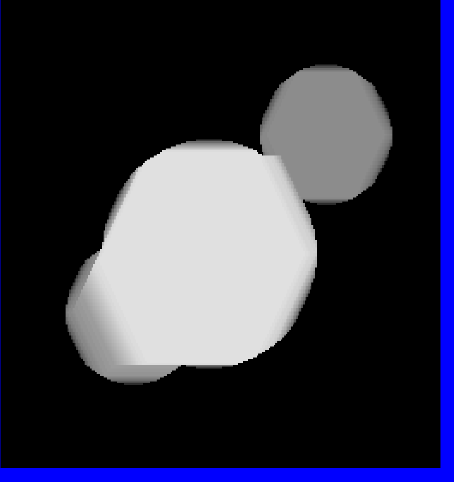
$$c' = \inf_i \left(\varepsilon_i \left(k'_{X_i} \right) \right)$$



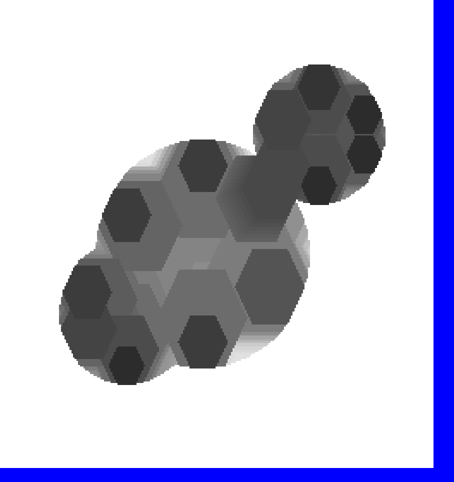
Points of X covered by balls which themselves are not covered by larger or smaller balls are therefore only covered by critical balls.

Extracting the critical skeleton

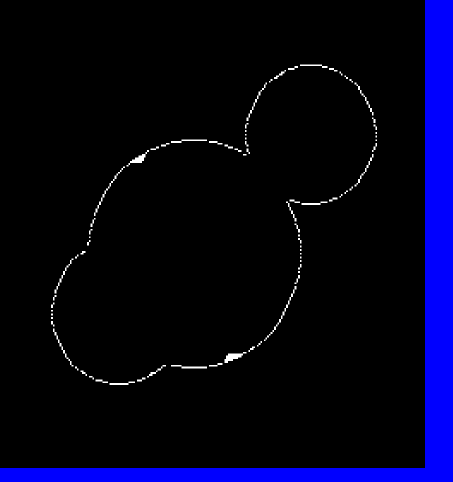
The points x of X covered by critical balls fulfil the equality c(x) = c'(x). We can define e = c = c' when c = c'.



c



c'



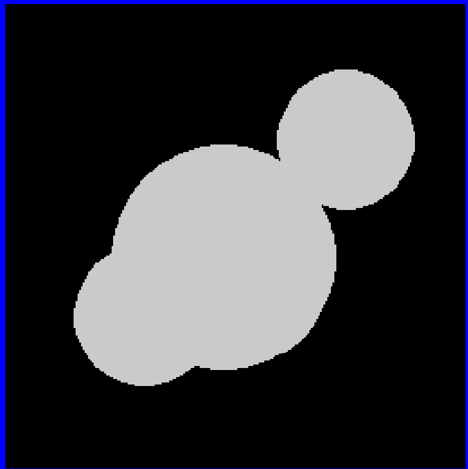
Points of $e \neq 0$

The centers are obtained from the sets Z_i and Y_i:

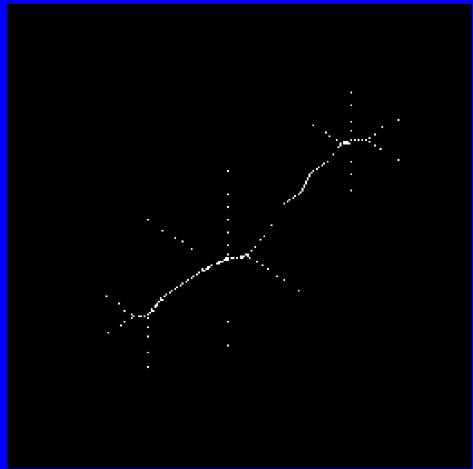
$$Z_i = \{x : q(x) = i\} \quad \text{and} \quad Y_i = \{y : e(y) = i\}$$

The intersection $Z_i \cap \delta_i(Y_i)$ gives the centers of the critical balls of size i.

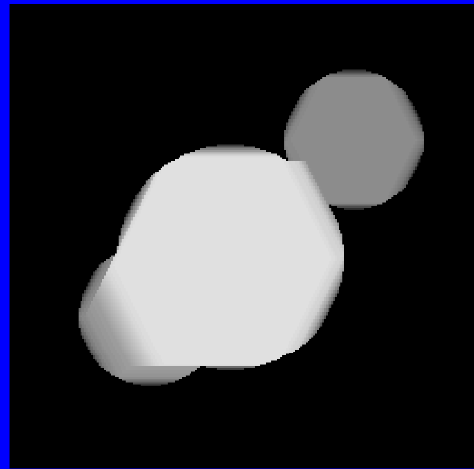
Digital critical skeleton



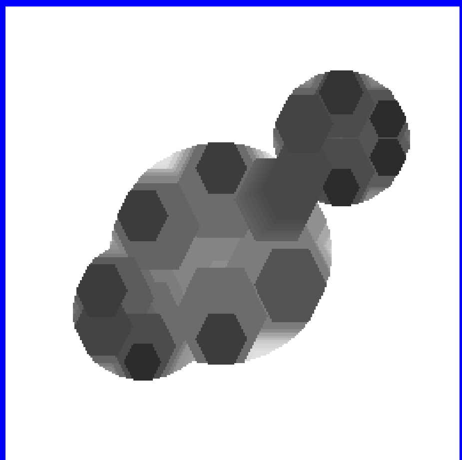
Initial set



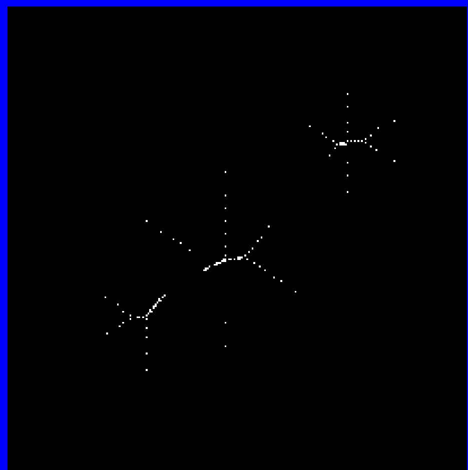
Quench function



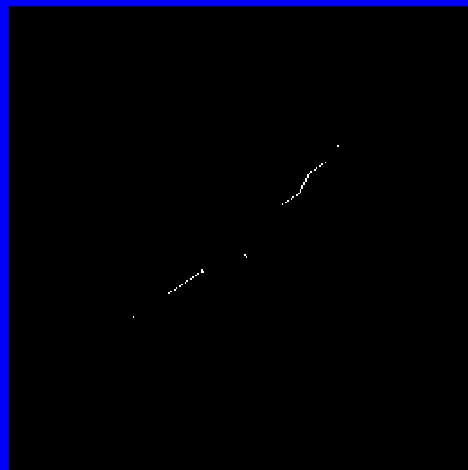
Granulometric function



Dual function

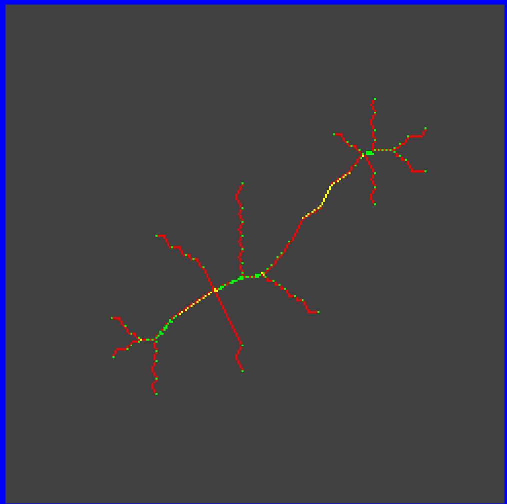


Center of critical balls



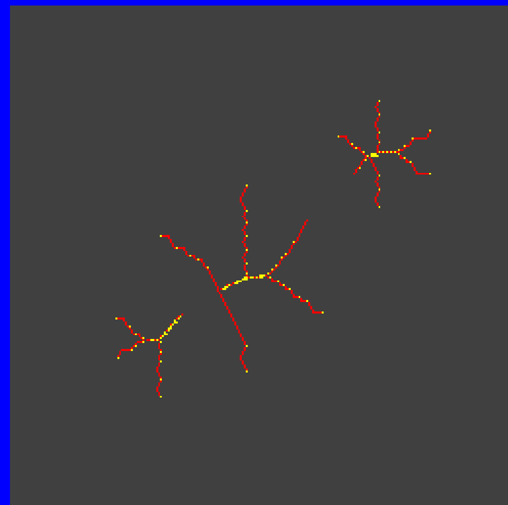
Non critical centers

Connected critical skeletons

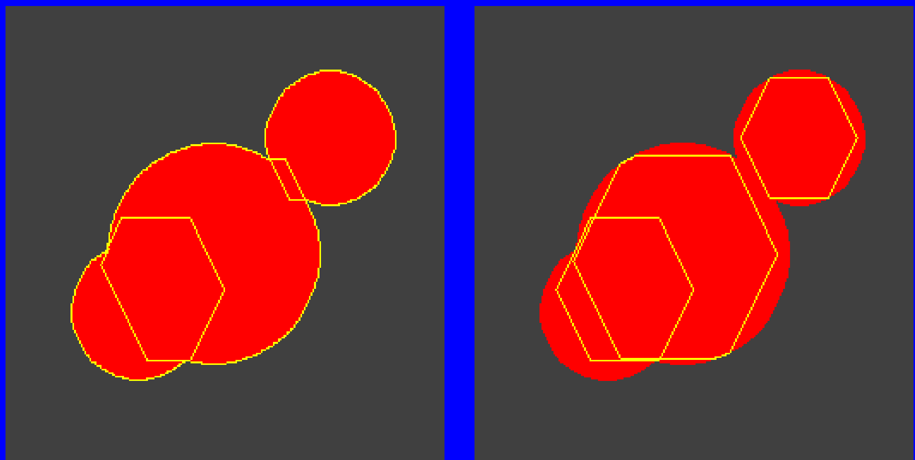


The maximal balls skeleton can be connected (by geodesic thinning)

- In green, critical skeleton S_c
- in yellow, centers of non critical balls
- In red, junctions



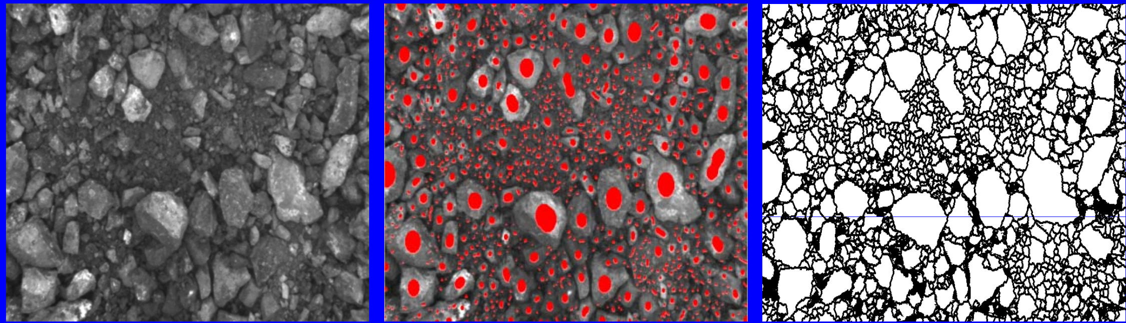
Connected critical skeleton



Critical sets (left) and biggest ball (hexagon in this example) embedded in each one (right).

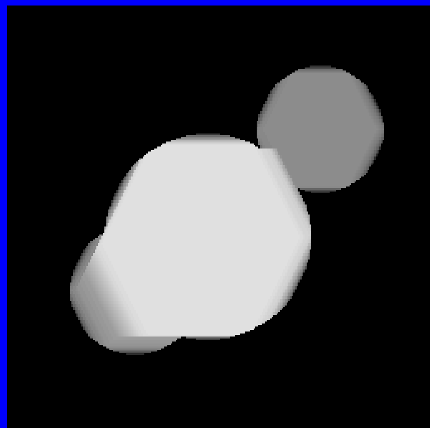
Usage and next steps

- Critical balls can be used to enhance the segmentation of « potatoe shaped » sets (better markers definition).

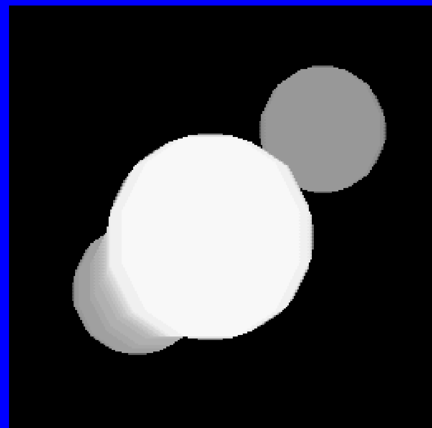


Segmentation of rocks with markers selected among the critical balls.

- Dodecagonal critical balls
- Sets shape filtering based on the level of criticity of the maximal balls
- Super-critical sets (X super-critical if all its openings are critical)



Critical hexagons



Critical dodecagons