CRITICAL BALLS

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ABSTRACT

This paper introduces the concept of critical ball. Critical balls are maximal balls which are necessary and sufficient to describe and rebuild sets, contrary to maximal balls where some redundancy exists. A general definition of a critical ball is given in continuous spaces and some of its main properties are depicted. Thanks to a slight modification of their definition, critical balls can also be used in digital spaces. Then, we explain how to extract them rapidly through the use of two residual transformations. Finally, some examples of use of critical balls for shape description and image segmentation are briefly presented.

KEYWORDS: Skeletons; maximal balls; critical balls; residual transforms

I. MAXIMAL AND CRITICAL BALLS

Maximal balls and skeleton

A ball B(x, λ) centered at point x of radius λ included in a set X is maximal if this ball is not covered by any other ball included in X. Obviously, if X is part of an Euclidean space, the balls B(y, μ) which cover B(x, λ) have a larger radius (μ>λ) but this is not compulsory in non Euclidean spaces.

The set of all the maximal balls center points constitute the well-known skeleton by maximal balls S(X) of X:

$$S(X) = \{x \in X: \exists B(x,\lambda)\ maximal\} \qquad (1)$$

Critical balls, general definition

Through its skeleton, a particular function q, called quench function, can be associated to any set X. The support of q is the skeleton S(X). For each point x belonging to S(X), q(x) is equal to λ, radius of the maximal ball centered at x. The quench function q(x) can be considered as a descriptor of X as the initial set can be entirely rebuilt only from q(x). Indeed, we have:

$$X = \cup_{x\in S(X)} \delta_{q(x)}\{x\} \qquad (2)$$

where δ_r is the dilation by a ball of radius r.

However, all the maximal balls are not necessary to rebuild a set. Figure 1 illustrates this. Only two maximal balls are necessary to rebuild and describe the set X (in fact, X is obviously the union of two balls) whereas an infinity of maximal balls are considered in S(X). The maximal balls of X which are necessary and sufficient to describe and rebuild X are called critical balls and their centers define the skeleton by critical balls $S_C(X)$ of X [1].

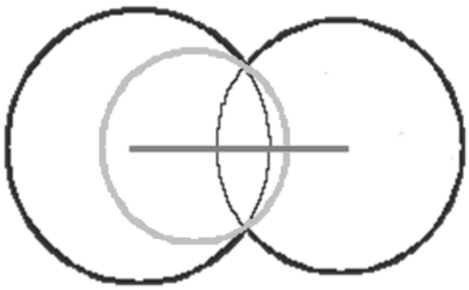

Figure 1 : X is the union of two balls, which are the only critical balls, but an infinity of maximal balls exists (light grey), their centers corresponding to the skeleton of X (dark grey).

A critical ball $B_C(x, \lambda)$ of X is a maximal ball which is not covered by any combination of other maximal balls of X.

$$\nexists\{B_k\}, B_k \neq B_C: B_C \subset \cup_k B_k \qquad (3)$$

We can show that the critical skeleton $S_C(X)$ of X exists (to prove this, it is simpler to use topological closed balls, contrary to the maximal balls skeleton which is generally defined with topological open balls). It is also easy to prove that a critical skeleton is unique. It is impossible to find two different sets of critical balls which cover the same set X. Another interesting property is the following: if a point of X is covered by a unique maximal ball, this ball is critical. Conversely, if a ball is critical, there exists at least one point of X which is included only in this critical ball (this point is likely a boundary point, hence the use of closed sets).

This definition of a critical ball suffers from two major defects. Firstly, it is not appropriate for digital spaces as the uniqueness property is not verified anymore. Secondly, this definition is not very useful to design an efficient and fast procedure to extract them.

II. DIGITAL CRITICAL BALLS

Covering in digital spaces

In digital spaces, critical balls are polygons: squares, hexagons or dodecagons and octagons depending of the digital grid in use (more sophisticated polygons may also be used).

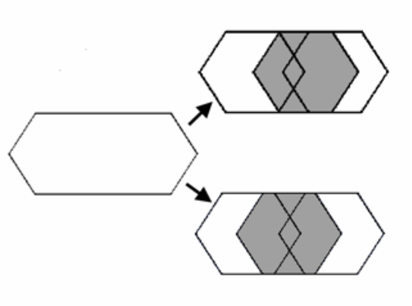

Figure 2: Two different sets of critical hexagons (grey hexagons are different) cover the initial set (at left). The uniqueness property is not fulfilled with critical polygons of same size.

However, in this case, the uniqueness of the covering is no longer verified as illustrated in Figure 2, where different sets of critical hexagons can be used to cover the initial set. This is due to the fact that, contrary to the Euclidean disk, any maximal polygon can be covered by other polygons of the same size.

Digital critical balls

The previous problem can be overcome by a new definition of a critical ball in a digital space.

A digital ball B_i, of size i, is critical if there exists no combination of maximal balls B_j, with $i \neq j$, which is covering B_i:

$$\nexists J = \{j_1, \dots, j_n : j_k \neq i\} \text{ such that } B_i \subset \cup_{j \in J} B_j \quad (4)$$

This slight relaxation of the definition (which changes nothing in Euclidean spaces) avoids choosing between critical balls of equal sizes. As this choice is not really possible, we keep all of them.

III. CRITICAL BALLS EXTRACTION

Residual operators

How critical balls can be extracted? The simplest way, but also the more tedious one, would consist in extracting each maximal ball and testing that it cannot be covered by the others. Fortunately, a faster approach can be designed based on residual operators [2].

A residual operator is built from two sequences $\{\psi_i\}$ and $\{\zeta_i\}$ (with $\psi_i \geq \zeta_i$) of operators called primitives. The index i may be any integer parameter taken from a set I: a size, an ordering value, etc.

A residual transformation is therefore a doublet of operators defined as follows:

$$\begin{cases} \theta = \cup_{i \in I}(\psi_i - \zeta_i) \\ q = argmax(\psi_i - \zeta_i) \end{cases} \quad (5)$$

$(\psi_i - \zeta_i)$ is called residue of index i. θ contains the highest residue appeared at each point, whereas q, called associated function, indicates for each point, the index value of this highest residue.

Note that the skeleton S of a set X and its corresponding quench function are residual operators, according to Lantuejoul's formula where $\psi_i = \varepsilon_i$ is an erosion of size i of X and $\zeta_i = \gamma \circ \varepsilon_i$ the elementary opening of this erosion [4]:

$$S(X) = \cup_{i \in \mathbb{N}}\left[\epsilon_i(X) - \gamma\big(\epsilon_i(X)\big)\right] \quad (6)$$

The quench function corresponds to the value i for which the corresponding skeleton point appeared.

Ultimate opening, dual operator

Two other residual operators can be used to sort critical balls of a set X. The first one is called ultimate opening and its associated function, granulometric function. In fact, it is this latter one which will be used.

The ultimate opening of a set X is defined by:

$$\theta = \cup_{i \in \mathbb{N}}(\gamma_{i-1} - \gamma_i) \quad (7)$$

Each residue corresponds to the difference between two successive openings γ_{i-1} and γ_i of X. We can see easily that this ultimate opening, when defined on a set X, is equal to the set itself. The granulometric function is more interesting. It is defined by:

$$c = argmax(\gamma_{i-1} - \gamma_i) \quad (8)$$

For each point $x \in X$, c(x) indicates the size of the greatest maximal ball which contains x and which is not entirely covered by balls of larger sizes.

The granulometric function can be obtained with the quench function of the skeleton. Let us define:

$$S_i = \{x \in S(X) : q(x) = i\} \quad (9)$$

The valued indicator function k_{S_i} of S_i is given by:

$$\begin{cases} k_{S_i}(x) = i \text{ if } x \in S_i \\ k_{S_i}(x) = 0 \text{ if not} \end{cases} \quad (10)$$

We have:

$$c = sup_{i\in\mathbb{N}}\left(\delta_i\left(k_{S_i}\right)\right) \quad (11)$$

where δ_i is the size i dilation of a function.

A dual operator c', called anti-granulometric function, can also be defined. It is simpler to define it from the various sets S_i.

Let k'_{S_i} be the dual indicator function:

$$\begin{cases} k'_{S_i}(x) = i \; if \; x \in S_i \\ k'_{S_i}(x) = +\infty \; if \; not \end{cases} \quad (12)$$

Then:

$$c' = inf_{i\in\mathbb{N}}\left(\varepsilon_i\left(k'_{S_i}\right)\right) \quad (13)$$

where ε_i is the size i erosion of a function.

This operator indicates the size of the maximal balls which are not totally covered by balls of smaller sizes (Figure 3).

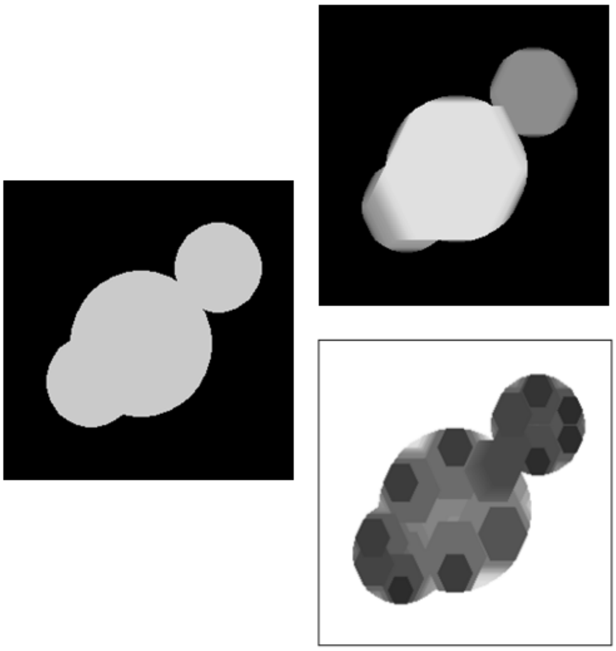

Figure 3: Initial set X (left), granulometric function c (upper right) and anti-granulometric function c' (lower right), both of them are built with maximal hexagons.

Critical balls sorting

Then, according to the previous digital definition of critical balls, points x of X which are covered by critical balls verify the following identity:

$$c(x) = c'(x) \quad (14)$$

We define:

$$e = c = c' when \; c = c' \quad (15)$$

The centers of the critical balls of X can be extracted by the following procedure. Let us consider again S_i, as it is defined in Eq.(9) and let us define:

$$E_i = \{y \in X: e(y) = i\} \quad (16)$$

Then the critical skeleton $S_C(X)$ is given by:

$$S_C(X) = \cup_{i\in\mathbb{N}}[S_i \cap \delta_i(E_i)] \quad (17)$$

Figure 4 illustrates the different steps of the extraction of the critical skeleton S_C with the help of the quench function q and of the function e.

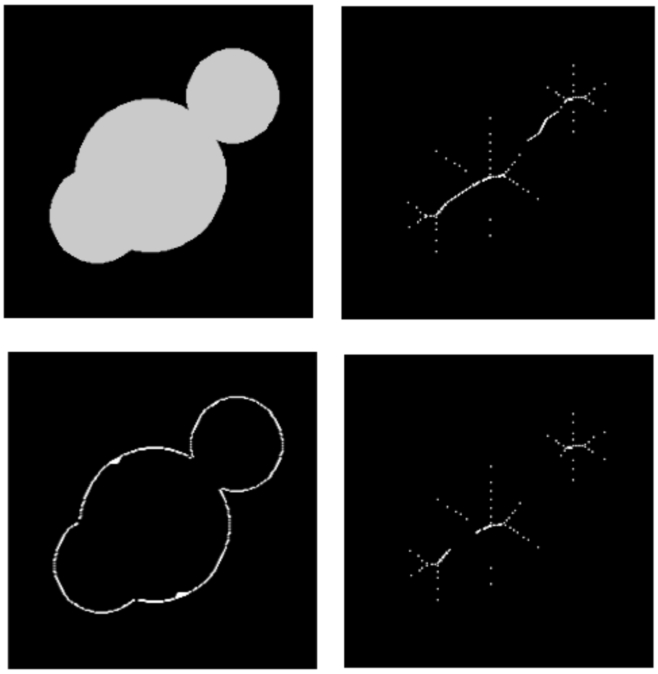

Figure 4: Initial set X(upper left), quench function (upper right), the function e containing points of X covered by a critical hexagon (lower left) and critical skeleton $S_C(X)$ (lower right).

IV. APPLICATIONS

Shape description

The skeleton S(X) of a set X is not a very good shape descriptor as too many maximal balls are involved. The critical skeleton $S_C(X)$ is a better shape descriptor. Two kinds of sets can be considered: sets X where $S_C(X) = S(X)$ – these sets are called critical sets – and sets where this identity is not verified. Note that a convex set is always critical. The reverse is not true. This description of any set X by its critical balls allows a faster implementation of basic morphological transformations which use this set as structuring element.

Image segmentation

When a set X is not critical, it is interesting to determine the minimal assembly of connected critical sets X_i allowing its description and eventually its segmentation:

$$X = \cup_{i\in I} X_i \,, I \; minimal, X_i \; critical \quad (18)$$

We have in this case:

$$S_C(X) = \cup_{i \in I} S_C(X_i) \quad (19)$$

To achieve this, a difficulty comes from the fact that the critical skeleton S_C (together with the skeleton S) is not connected. Fortunately, this connection can be obtained by geodesic thinnings [3]. Then, each connected component of S_C (after connection) can be used to rebuild the corresponding critical set. A new segmentation of the initial set X is obtained by this means (Figure 5).

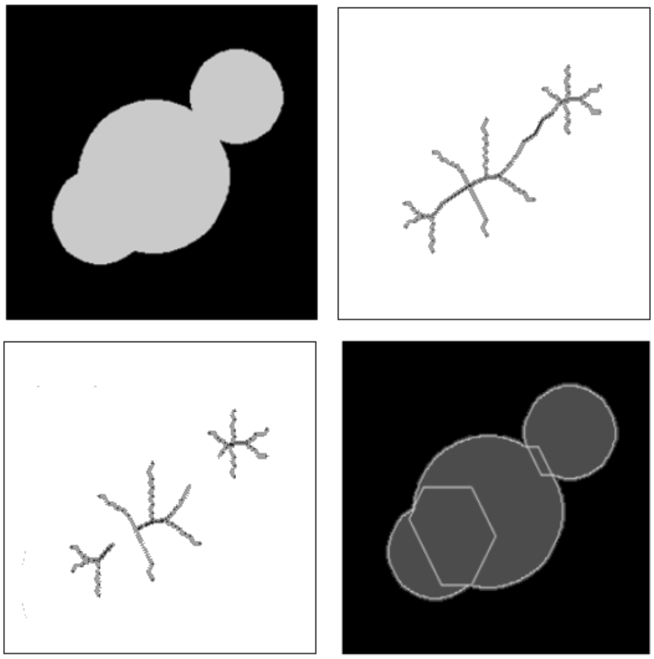

Figure 5: A connected skeleton (upper right) of the initial set (upper left) containing all the centers of maximal balls is obtained by geodesic thinning. A connected critical skeleton can be extracted from the previous one (lower left) and each of its connected components allows to rebuild the critical sets which compose the initial set (lower right).

This approach allows better segmentations through a better definition of markers for the objects to be separated. It is in particular the case when the objects present irregular shapes, as illustrated in Figure 6. Critical balls provide better markers than more classical approaches based on distance functions and ultimate erosions [5].

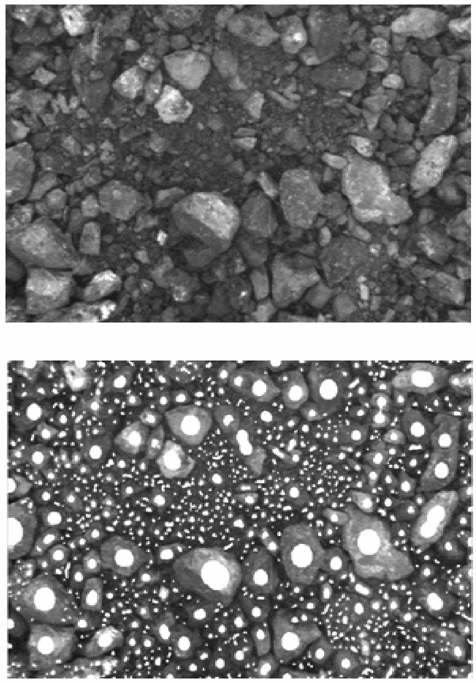

Figure 6: Heap of rocks (upper image) and result of the marking process after use of a granulometric function and extraction of the critical balls. These balls are eroded, the size of erosion is proportional to the size of each critical ball.

V. CONCLUSION

The concept of critical ball and of critical skeleton leads to a better description and classification of shapes. The notion of critical set can also be refined. It is interesting, for instance, to analyze the successive openings of critical sets in order to verify whether these openings are critical or not. Through the use of residual operators, extracting the critical balls of a set is rather simple, even when sophisticated polygons are used. Note also that this notion can be easily extended to 3D shapes and also to grey scale images (as shown in the last example), as residual transformations can equally be defined on these kinds of images.

VI. REFERENCES

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