# MATHEMATICAL MORPHOLOGY AND FUNCTIONS



S. BEUCHER

FONTAINEBLEAU NOVEMBRE 1984

# MATHEMATICAL MORPHOLOGY AND FUNCTIONS

## INTRODUCTION

Mathematical morphology seems to be closely related to a class of mathematical concepts : the sets. All the classical transformations of mathematical morphology are transformations on sets. For that reason, the texture analysers now existing are designed to work with sets (they have, for instance, binary memories, and a hit-or-miss logic). On the contrary, most of the studied objects are recognized by means of visual sensors which provide images. In mathematical terms, an image is a function or, more precisely, a grey-tone function. This facts induces us to try to define morphological transformations related to functions.

We shall see in the next sections the great utility of these transformations. Therefore, the manner in which these transformations were built allows us to extend to functions many results concerning sets.

## I - BASIC TRANSFORMATIONS

### I.1 - Thresholding.

The Ariadne's Thread which may guide us in our work is well known : it is thresholding.

For simplicity's sake, in what follows we only consider functions defined in $\mathbb{R}^1$ and $\mathbb{R}^2$, or in subsets of these spaces.

Let f be a function defined in $\mathbb{R}^n$ $(n = 1,2)$. Thresholding this function at level $\lambda$ produces two sets :

$$X_\lambda(f) = \{x : f(x) \geq \lambda\}$$

$$Y_\lambda(f) = \{x : f(x) > \lambda\}$$

A function f can be defined by its explicit formulation or by knowledge of the sequence $\{X_\lambda(f)\}$. Conversely, this definition allows us to generate functions starting from sequences of sets.

No topological assumption was made concerning the sets $X_\lambda(f)$ and $Y_\lambda(f)$. Classical results in topology show that if and only if f is continuous, $X_\lambda(f)$ is a closed set and $Y_\lambda(f)$ an open set. When dropping one of these two conditions, we define semi-continuous functions (upper or lower semi-continuity). On the other hand, given a sequence $\{X_\lambda\}$ of sets, we must verify that this sequence defines a function. If for example, $\{X_\lambda\}$ is not a decreasing sequence when $\lambda$ is increasing, this sequence will not define a function. So, we must have :

$$\forall\ \lambda_1 \geq \lambda_2 \quad , \quad X_{\lambda_1} \subset X_{\lambda_2}$$

The question is : is it the only condition that must be fulfilled by the $\{X_\lambda\}$ ? The answer is no, and we can give a simple example to illustrate this point.:

Let $\{X_\lambda\}$ be a $\mathbb{R}^2$ sets sequence defined as follows :

$X_\lambda = \mathbb{R}^2$ when $\lambda \leq 0$

$X_\lambda = \bar{B}(0,1)$ when $0 < \lambda < 1$

$X_\lambda = \emptyset$ when $\lambda \geq 1$

$\bar{B}(0,1)$ is a closed ball of radius 1 centered at the origin. All the $X_\lambda$'s are closed. But they do not define a u.s.c. function. More precisely, they do not define a function ! Indeed, consider any point x of $\mathbb{R}^2$ belonging to $\mathring{B}(0,1)$. What is the value of f(x) ? We can only say that :

$$f(x) < 1 \quad \text{and for any } \lambda < 1\ ,\ f(x) \geq \lambda\ .$$

This is obviously a very strange definition. These considerations lead us to the following fundamental theorem :

<u>THEOREM</u> - Let f be an upper semi-continuous function defined on $\mathbb{R}^n$ $(n = 1,2)$ and : $X_\lambda(f) = \{x : f(x) \geq \lambda\}$ be its associated set sequence, by thresholding. Then, the $X_\lambda$'s are closed and monotone decreasing :

$$\lambda' \leq \lambda \ , \ X_\lambda \subset X_{\lambda'} \quad \text{and} \quad X_\lambda = \lim_{\lambda' \uparrow \lambda} X_{\lambda'} = \bigcap_{\lambda' < \lambda} X_{\lambda'}$$

Conversely, a monotone decreasing sequence $\{X_\lambda\}$ of closed sets generates a u.s.c. function f, and we have

$$f(x) = \text{Sup } \{\lambda : x \in X_\lambda\}$$

In fact, a topological status for $X_\lambda$ is not compulsory. But in this case the formulation of the monotone decreasing convergence is more complex.

Actually, we do not have to worry about these topological considerations : in practical work, the functions we can handle are digitalized functions, and the only condition which concerns the sequence $\{X_i\}$ , $i \in \mathbb{Z}$ is the inclusion condition.

In what follows, we shall assume that the digitalized functions are the representation of functions defined in $\mathbb{R}^2$ which fulfill the proper conditions mentioned above. In the same way, the digital representation of the structuring elements will be considered as the representation of compact sets (closed and bounded) of $\mathbb{R}^2$.

## I.2 - <u>Erosion and dilation of a function by sets</u>.

Let f be a function defined in $\mathbb{R}^n$. No condition is made for that function (it can be bounded or not, positive or negative). $\{X_\lambda(f)\}$ is its associated sequence of sets.

### a) <u>Dilation.</u>

Let B be a structuring element, and for each $X_\lambda(f)$ perform the dilation of $X_\lambda(f)$ by B. We obtain a new sequence of sets which defines a new function called dilation of f by B and denoted $g = f \oplus \check{B}$. We have :

$$X_\lambda(g) = \{x : g(x) \geq \lambda\} = X_\lambda(f) \oplus \check{B} \; .$$

Develop this formula :

$$X_\lambda(g) = X_\lambda(f) \oplus \check{B} = \{x : B_x \cap X_\lambda(f) \neq \emptyset\}$$

$$= \{x : \exists\, y \in B_x \text{ such that } f(y) \geq \lambda\}$$

$$= \{x : \sup_{y \in B_x} f(y) \geq \lambda\}$$

Hence, we can write :

$$g(x) = [f \oplus \check{B}](x) = [\mathrm{Sup}_B\, f](x)$$

The value of g at point x is nothing other than the upper bound of the values of f in the set B centered at x.

b) Erosion.

In the same way, we now associate with each $X_\lambda(f)$ its erosion by B. We generate a new function called erosion of f by B, and denoted $h = f \ominus \check{B}$.

$$X_\lambda(h) = \{x : h(x) \geq \lambda\} = X_\lambda(f) \ominus \check{B}$$

We can write :

$$X_\lambda(h) = X_\lambda(f) \ominus \check{B} = \{x : B_x \subset X_\lambda(f)\}$$

$$= \{x : \forall\, y \in B_x\, ,\; f(y) \geq \lambda\}$$

$$= \{x : \inf_{y \in B_x} f(y) \geq \lambda\}$$

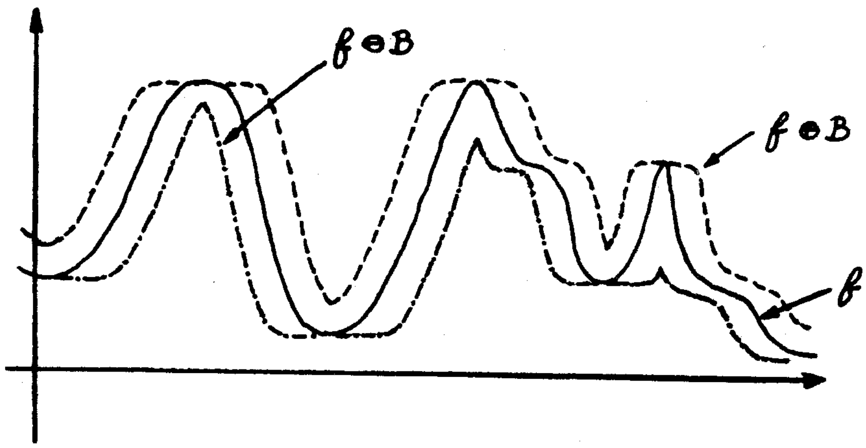
and :

$$h(x) = [f \ominus \check{B}](x) = [\inf_B f](x)$$

The value taken by h at point x is the lower bound of the values of f in the set B centered at x.

c) Algebraic properties of the erosion and dilation of functions.

First, consider the behaviour of a function after an erosion or a dilation (see figure) :




The erosion reduces the tops of the function, and enlarges the valleys, and vice-versa for the dilation.

Let us give some of the algebraic properties of these transformations, without proof. These properties can easily be deduced from those established for sets.

- $(f \ominus B_1) \ominus B_2 = f \ominus (B_1 \oplus B_2)$

- $B \subset B'$ implies that for every $f$ :

$$f \oplus B \leq f \oplus B'$$

$$f \ominus B \leq f \ominus B'$$

## I.3 - Boolean algebra and operators on functions.

We know the formula for sets :

$$X \oplus (B_1 \cup B_2) = (X \oplus B_1) \cup (X \oplus B_2)$$

What is the meaning of this transform, when we consider functions ? To answer this question, we must know which operations on functions are related to the Boolean operations $\cup$ and $\cap$.

Let f and f' be two functions and $\{X_\lambda(f)\}$, $\{X_\lambda(f')\}$ be their a.s.s. The union $X_\lambda(f') \cup X_\lambda(f)$ defines a new sequence and a new function g :

$$X_\lambda(g) = X_\lambda(f') \cup X_\lambda(f) = \{x : f'(x) \geq \lambda \quad \text{or} \quad f(x) \geq \lambda\}$$

$$= \{x : \mathrm{Sup}(f(x), f'(x)) \geq \lambda\}$$

g is the upper envelope of the two functions f and f'. We denote : $g = f \vee f'$. Mutatis mutandis, we have :

$$X_\lambda(h) = X_\lambda(f) \cap X_\lambda(f') = \{x : \mathrm{Inf}(f(x),\ f'(x)) \geq \lambda\}$$

h, denoted $h = f \wedge f'$ is the lower envelope of f and f'. Therefore, we can write :

$$f \oplus (B_1 \cup B_2) = (f \oplus B_1) \vee (f \oplus B_2)$$

The third operation in Boolean algebra is complementation. What happens if we define the following mapping on the $X_\lambda(f)$'s :

$$X_\lambda(f) \longrightarrow X^c_\lambda(f)$$

We do not define a function, because we have the condition :

$$\forall\ \lambda' \geq \lambda \qquad X^c_{\lambda'}(f) \supset X^c_\lambda(f)$$

Now, if we express explicitly $X^c_\lambda(f)$, we find that :

$$X^c_\lambda(f) = \{x : f(x) < \lambda\} = \{x : -f(x) > -\lambda\} \ .$$

So, we can write :

$$X^c_\lambda(f) = \underset{-\lambda}{Y}(-f)$$

In other words, by complementation, we associate with the sequence $\{X_\lambda(f)\}$ of the function f, the sequence $\{Y_\lambda(-f)\}$ of the function -f. If the plane $\mathbb{R}^2$ is regarded as a mirror, complementation looks like a reflection of the function f.

As an application, let us prove the following equation :

$(-f) \oplus B = -(f \ominus B)$ , which is for functions, what the formula corresponding to equation $X^c \oplus B = (X \ominus B)^c$ is for sets ; we have :

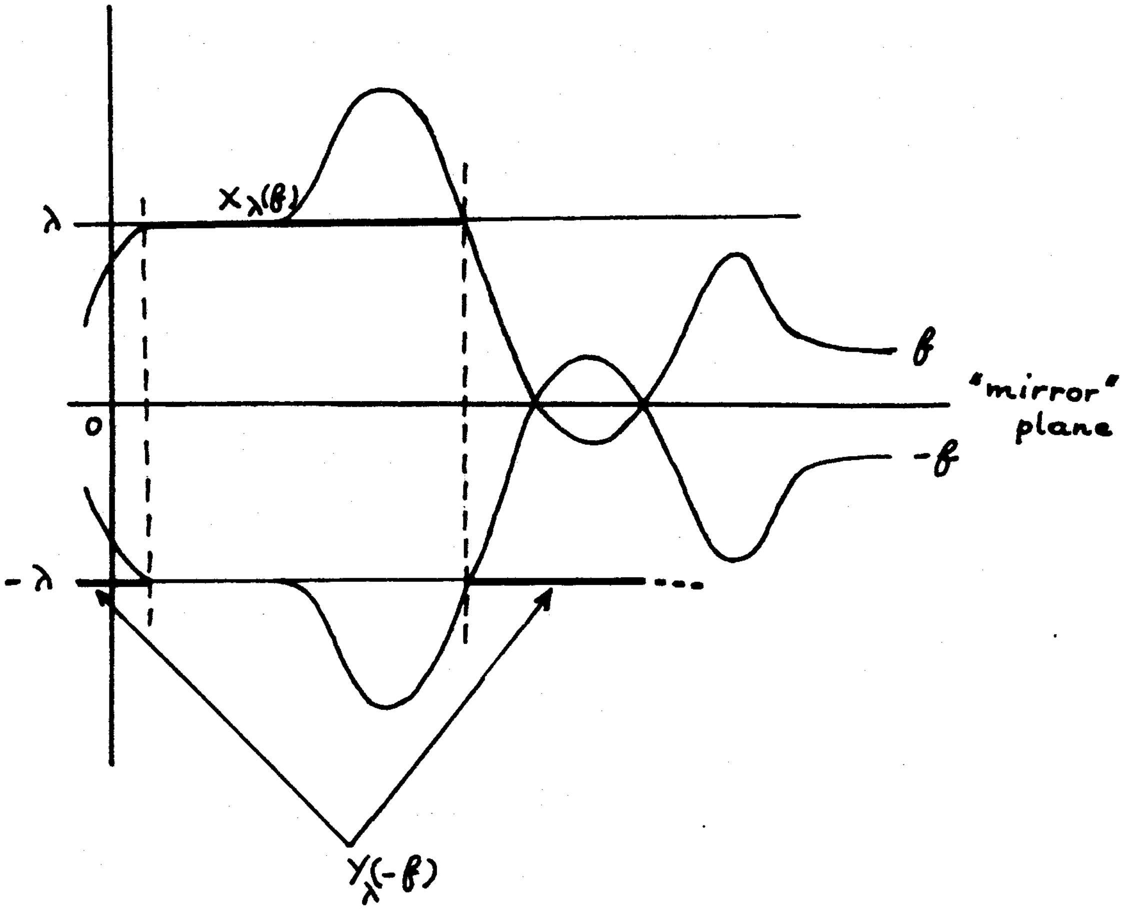
$$X_\lambda((-f) \oplus B)) = X_\lambda(-f) \oplus B = [X^c_\lambda(-f) \ominus B]^c = [Y_{-\lambda}(f) \ominus B]^c$$

but $[Y_{-\lambda}(f) \ominus B]^c = Y^c_{-\lambda}(f \ominus B) = X_\lambda(-(f \ominus B))$ Q.E.D.

More generally, we can show that :

$$a - (f \ominus B) = (a-f) \oplus B \qquad (a : \text{given constant})$$

(we set the "mirror" plane at the ordinate a, instead of the ordinate 0).



## I.4 - Addition, subtraction, product of functions.

We know how to express Boolean operations in terms of relations between functions. Conversely, given two functions f and f' and their associated sequences, we want to formulate the various sequences $\{X_\lambda(f+f')\}$ , $\{X_\lambda(f-f')\}$ , $\{X_\lambda(f.f')\}$ of Boolean functions of the $X_\lambda(f)$'s and the $X_\lambda(f')$'s.

We shall see that these operations involve two different sections of the functions f and f'.

### a) Sum of two functions.

Let f and f' be two functions, and $\{X_\lambda(f)\}$ , $\{X_\lambda(f')\}$ be their a.s.s. We have :

$$X_\lambda(f+f') = \{x : f(x) + f'(x) \geq \lambda\}$$

Suppose that $f'(x) \geq \mu$, a constant. $f(x) + f'(x)$ will be greater than or equal to $\lambda$ if and only if $f(x) \geq \lambda-\mu$. So, the previous formula becomes , for $f'(x) \geq \mu$ :

$$X_\lambda(f+f') = \{x : f'(x) \geq \mu \text{ and } f(x) \geq \lambda-\mu\}$$

We now let $\mu$ vary and take all possible values on $\mathbb{R}$. We can write :

$$X_\lambda(f+f') =\{x : \exists\, \mu,\ f'(x) \geq \mu \text{ and } f(x) \geq \lambda-\mu\} ,$$

which is equivalent to :

$$X_\lambda(f+f') = \bigcup_{\mu\in\mathbb{R}} [X_{\lambda-\mu}(f) \cap X_\mu(f')]$$

This formula is very interesting : it shows that to get the threshold at level $\lambda$ of the sum, we must take into account, by pairs, all the thresholds of f and f' which are separated by a constant $\lambda$.

This formula shows an analogy with the formula of the convolution product of two functions (replace $\cap$ by multiplication and $\cup$ by the sum).

We can very easily prove that :

$$Y_\lambda(f+f') = \bigcap_{\mu\in\mathbb{R}} [Y_{\mu-\lambda}(f) \cup Y_\mu(f')]$$

b) Difference between two functions.

The formula for differences is quite simple. We have :

$$f - f' = f + (-f').$$

So :

$$X_\lambda(f-f') = \bigcup_{\mu\in\mathbb{R}} [X_{\lambda-\mu}(f) \cap X_\mu(-f')]$$

but :

$$X_\mu(-f') = Y^c_{-\mu}(f') .$$

Hence :

$$X_\lambda(f-f') = \bigcup_{\mu\in\mathbb{R}} [X_{\lambda+\mu}(f) \cap Y^c_\mu(f')]$$

The set difference $X_{\lambda+\mu}(f) - Y_\mu(f')$ appears in the formula. But this difference has nothing to do with the difference of functions. In fact, in most cases, the set difference does not produce a monotone sequence of sets. The union over the entire space $\mathbb{R}$ forces the sequence to be monotone.

c) Multiplication.

In this section, f and f' are assumed to be strictly positive functions. We have :

$$L\,n(f.f') = L\,n(f) + L\,n(f') \quad .$$

We transform the multiplication operation into an addition operation. We have :

$$X_\lambda(L\,n\,f) = \{x : L\,n\,f(x) \geq \lambda\} = \{x : f(x) \geq e^\lambda\}$$

$$X_\lambda(L\,n\,f) = X_{e^\lambda}(f)$$

and :

$$X_\lambda(L\,n(f\,f')) = X_{e^\lambda}(f\,f') = \bigcup_\mu [X_{e^\lambda}(f) \cap X_{e^{\lambda-\mu}}(f')]$$

The multiplication and the addition are defined by the same formulae, up to a change in labelling of the cross-sections for the multiplication.

There exist many other operations on functions which can be expressed in terms of set transformations. Some examples are given below without proof.

- $X_\lambda(f^{-h}) = X_\lambda^{-h}(f)$

$f^{-h}$ is the translated function $f^{-h}(x) = f(x+h)$.

- If $g = f(\alpha x)$ ($\alpha$, constant) : $X_\lambda(g) = \alpha X_\lambda(f)$ .

- $X_\lambda(\alpha f) = X_{\lambda/\alpha}(f)$ , ($\alpha$, positive) .

## II - MORPHOLOGICAL TRANSFORMATIONS ON FUNCTIONS - SEQUENTIAL OPERATIONS.

The combinations of the basic transformations provide many operations which are, for functions, the reflection of the classical set transformations. For instance, we can define the opening and the closing of a function (and consequently size distribution on functions), reconstruction of functions, and various other sequential algorithms.

### II.1 - Opening, closing and size distributions.

The function $f_B$ defined by : $f_B = (f \ominus \check{B}) \oplus B$ is called opening of f. The function $f^B$ defined by : $f^B = (f \oplus \check{B}) \ominus B$ is called closing of f. These functions are interesting for three main reasons :

- Firstly, they provide good operators in noise cleaning problems and in regularisation problems (see exercises).

- Secondly, if $\lambda$ B denote the homothetic of a convex B, the open and closed functions fulfill the granulometric axioms given by G. MATHERON for sets. For every $\mu \geq \lambda$, we have the following inequalities :

$$f_{\mu B} \leq f_{\lambda B} \leq f \leq f^{\lambda B} \leq f^{\mu B}$$

Openings and closings of functions are idempotent transformations.

If f is assumed to be summable (i.e. $\int_{\mathbb{R}^2} |f(x)|\, dx < +\infty$), we can calculate the quantities :

$$\int f_{\mu B}\, dx \quad \text{and} \quad \int f^{\mu B}\, dx \qquad \text{for various sizes } \mu \text{ ,}$$

and obtain a granulometric representation of f.

Notice that the size distribution of f according to openings and closings is not the only size distribution we can define on a function. For example, thresholding f defines a size distribution :

$$\text{The set} \quad \psi_\ell(X_\lambda) = X_\lambda \quad \text{if} \quad \lambda \leq \ell$$

$$\psi_\ell(X_\lambda) = \emptyset \quad \text{if not} \quad .$$

The transformation $\psi_\ell$ depends upon parameter $\ell$, and satisfies the granulometric axioms. The size distributions by opening/closing on one hand, and by thresholding on the other hand, are not independent. The higher is the grey level at point x, the greater is its probability of disappearance by opening. But because of their correlation, the study of the conditional laws provides various informations about the shape of the function.

- Thirdly, openings and closings of functions are very helpful in pattern recognition problems. The transformation involved in these problems is called "top hat" Transformation. This transformation is the following :

$$f - f_{\rho B}$$

This function is made up of the residuals of the openings of f by a ball B of radius $\rho$. This transformation extracts peaks, crest lines, of the picture. Its usefulness was clearly demonstrated by F. MEYER who applied it on cancerous cells detection problems. We can write :

$$X_\lambda(f-f_{\rho B}) = \bigcup_\mu [X_{\lambda+\mu}(f) \cap Y^c_\mu(f_B)] \qquad \text{(see exercises).}$$

## II.2 - Gradient algorithms.

These operators are in fact contrast detectors. Many versions of gradient operators exist, and their efficiency depends upon the problem to be solved. One of the most commonly used operators is described below :

Let f be a function defined in $\mathbb{R}^2$. We call oscillation of f in a bounded set B the function :

$$\mathrm{Sup}_B\, f(x) - \mathrm{Inf}_B\, f(x)$$

Let B be a ball of radius $\rho$, centered at x. The oscillation of f at point x is :

$$\omega(f,x) = \lim_{\rho \to 0} \left[\sup_{y \in B} f(y) - \inf_{y \in B} f(y)\right]$$

If f is a continuous function $\omega(f,x) = 0$ , $\forall$ x.

Let us now define the variation of f at point x.

$$v_f(x) = \lim_{\rho \to 0} \left[ \frac{\sup_{y \in B} f(y) - \inf_{y \in B} f(y)}{2\,\rho} \right]$$

We can prove that if f(x) is continuous, $v_f(x)$ is nothing other than the gradient module of f at point x :

$$v_f(x) = |\vec{\mathrm{grad}}\, f(x)| = \left[\left(\frac{\partial f}{\delta x_1}\right)^2 + \left(\frac{\partial f}{\delta x_2}\right)^2\right]^{1/2}$$

If f is not continuous, $v_f(x)$ is the sum of a continuous component and a discontinuous one called jump of the derivative at point x. The algorithm of the gradient is given by :

$$|\vec{\mathrm{grad}}\, f(x)| = \lim_{\rho \to 0} \left( \frac{[f \oplus B](x) - [f \ominus B](x)}{2\,\rho} \right)$$

The digital version of this algorithm is given by the formula :

$$g(x) = \frac{[f \oplus B](x) - [f \ominus B](x)}{2}$$

where B is a hexagon of size 1.

We have :

$$X_\lambda(2g) = \bigcup_\mu [X_{\lambda+\mu}(f \oplus B) \cap \ Y^c_\mu(f \ominus B)]$$

$$X_{\lambda/2}(g) = \bigcup_\mu \Big[[X_{\lambda+\mu}(f) \oplus B] \cap [Y^c_\mu(f) \oplus B]\Big]$$

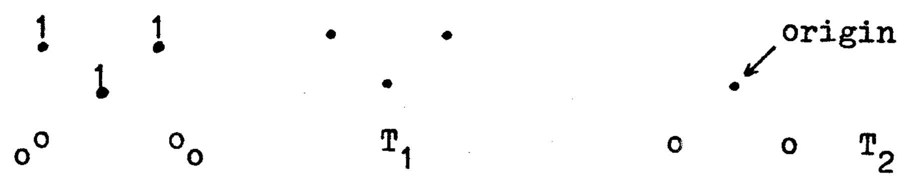
There exist other gradient operators. We can look for the gradient module in one direction $\alpha$, i.e. the gradient module of the curve obtained by section of the graph D of f (D is a subset of $R^3$ whose points are $(x, f(x))$ by a vertical plane in direction $\alpha$. We use the same operator, but now the structuring element B is a doublet of points in direction $\alpha$. This gradient leads to very interesting notions as, for instance, the rose of directions of a function, which is the angular distribution of the gradient measure.

## III - GOLAY LOGIC AND FUNCTIONS.

In this section, we limit our study to digitalized functions. These functions are defined on a hexagonal grid.

### III.1 - Review of set morphology.

A Golay configuration is a two-phase structuring element denoted $T = (T_1, T_2)$. The classical representation consists in setting at 1 the points of $T_1$, and at 0 the points of $T_2$ .




Example of a Golay configuration

Let X be a set. Performing a Golay operation on the set is to find and extract all the points of X which are surrounded by the configuration $T = (T_1, T_2)$. Calling Z the set of the extracted points, we show that

$$Z = (X \ominus T_1) \cap (X^c \ominus T_2)$$

Unfortunately, the mapping $X \longrightarrow Z$ is not monotone (counter-examples are numerous) ; in addition this transformation is not anti-extensive :

$$X \longrightarrow Z = \psi_T(X) \not\subset X \quad \text{and}$$

$$X \longrightarrow Z \not\supset X$$

But it is possible to define two transformations $\psi_1$ and $\psi_2$ derived from the Golay transform $\psi_T$, which verify $X \supset \psi_1(X)$ for the first one, and $X \subset \psi_2(X)$ for the second one. They are called Thinning and Thickening respectively.

a) Thinning and thickening.

Thinning is defined and denoted by :

$$X \bigcirc T = X \cap Z^c = X \cap [(X \ominus T_1) \cap (X^c \ominus T_2)]^c$$

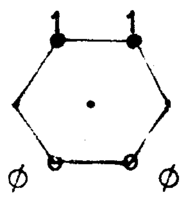
Thickening is defined and denoted by :

$$X \odot T = X \cup Z = X \cup [(X \ominus T_1) \cap (X^c \ominus T_2)]$$

Notice that if the center point of T is 1 (resp. 0), the thickening $X \odot T$ (resp. the thinning $X \bigcirc T$) is equal to X. For that reason, it is superfluous to give the status of the center point (the Golay logic for thickening and thinning is a "ring" (or contour) logic).

b) Ring configurations.

The Golay configurations T are now supposed to be ring configurations, defined on the boundary of the hexagon of size 1 :

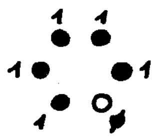



Example of Ring configuration

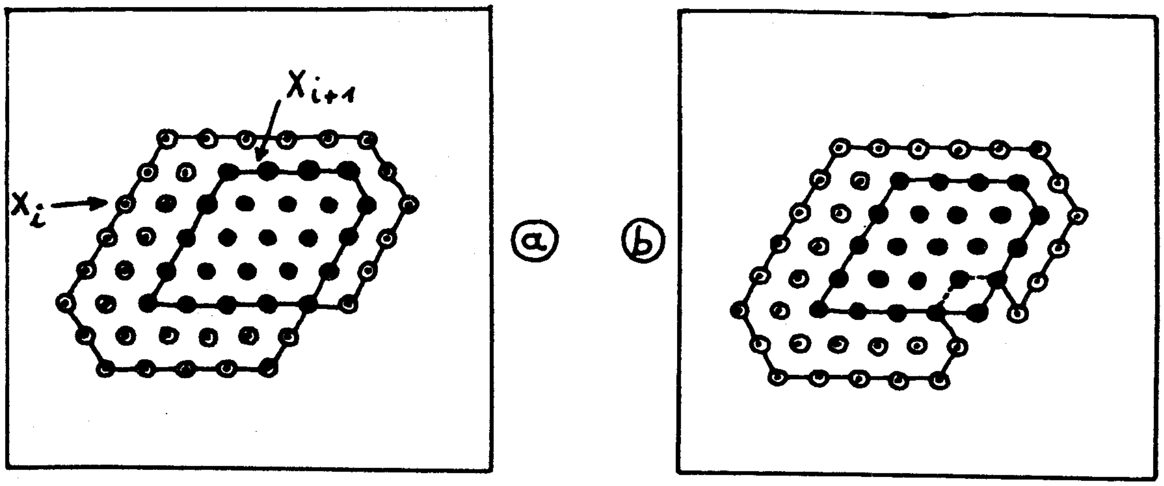
## III.2 - Thinning and Thickening of digitalized functions.

Let f be a digitalized function and $\{X_i(f)\}$ , $i \in \mathbb{Z}$ its associated sequence of sets.

If we perform thinning or thickening on each $X_i(f)$, the sequences $\{X_i(f) \circ T\}$ and $\{X_i(f) \odot T\}$ do not generate functions. To be convinced of that fact, consider the following counter-example :



Ring Configuration




a) initial sequence

b) result of thinning

In order to eliminate this major drawback, the solution is to force the inclusion rule upon the sequence of sets (remember the formula for the subtraction of functions : this intersection of $X_\mu(f)$ and $Y^c_{\mu+\lambda}(f')$ does not satisfy the inclusion rule, but the union will force inclusion upon the associated sequences. Obviously, in this case it is a matter of fact, since the subtraction of two functions is a function !). Therefore, we set :

$$X_i(g) = \bigcup_{j \geq i} (X_j(f) \circ T)$$

and

$$X_i(h) = \bigcap_{j \leq i} (X_j(f) \odot T)$$

The sequences $\{X_i(g)\}$ and $\{X_i(h)\}$ satisfy the inclusion rule and generate two functions called respectively thinning and thickening of f, and denoted

$$g = f \bigcirc T$$

$$h = f \odot T .$$

- Generation of thinnings and thickenings.

To be generated, the preceding sequences $\{X_i(g)\}$ and $\{X_i(h)\}$ need the knowledge of all the thinnings $X_j(f) \bigcirc T$ for $j \geq i$ and all the thickenings $X_j(f) \odot T$ for $j \leq i$ respectively. Therefore, the implementation of these transformations on texture analysers may take a long time. Hence, we shall now proceed in the inverse way and try to give the explicit form of the thinning and thickening operations, i.e., an algorithm to generate them directly on the digitalized function f, without using its sections.

Consider one section $X_j(f) \bigcirc T$. We can write :

$$X_j(f) \bigcirc T = X_j(f) \cap [(X_j(f) \ominus T_1) \cap (X_j^c(f) \ominus T_2)]^c$$

We transform the formula and find :

$$X_j(f) \bigcirc T = [(X_j^c(f) \oplus T_1) \cup (X_j(f) \oplus T_2)] \cap X_j(f)$$

Let x be a point of the hexagonal grid. x belongs to $X_j(f) \bigcirc T$ if and only if :

i) $T_{1_x} \cap X_j^c(f) \neq \emptyset$ or $T_{2_x} \cap X_j(f) \neq \emptyset$

ii) x belongs to $X_j(f)$

The point x belongs to $X_i(f \bigcirc T)$ if and only if there exists at least one value $j \geq i$ for which the above conditions are satisfied.

$$x \in X_i(f \bigcirc T) \Leftrightarrow \begin{cases} \exists\, j \geq i \quad \text{such that :} \\ \text{i) } f(x) \geq j \\ \text{ii) } \inf_{T_1} f(x) < j \quad \text{or} \quad \operatorname{Sup}_{T_2} f(x) \geq j \end{cases}$$

We can write (by complementation) :

$$x \notin X_i(f \bigcirc T) \Leftrightarrow \begin{cases} \text{i) } f(x) < i \quad \text{or} \\ \text{ii) } \inf_{T_1} f(x) \geq i \quad \text{and} \quad \mathrm{Sup}_{T_2} f(x) < i \end{cases}$$

Suppose now that $f(x) \leq \underset{T_2}{\mathrm{Sup}}\ \rho(x)$ .

For every i such that $f(x) < i$ , $x \notin X_i(f \bigcirc T)$ .

For every i such that $f(x) \geq i$ , we have :

$$i \leq f(x) \leq \underset{T_2}{\mathrm{Sup}}\ f(x) \quad , \quad \text{then} \quad x \in X_i(f \bigcirc T)$$

Hence, if $f(x) \leq \underset{T_2}{\mathrm{Sup}}\ f(x)$ , we have $(f \bigcirc T)(x) = f(x)$. In the same way, if $f(x) > \mathrm{Inf}_{T_1}^{2}\ f(x)$, the thinning function leaves the value at point x unchanged. If now we have both :

$\underset{T_2}{\mathrm{Sup}}\ f(x) < f(x) \leq \underset{T_1}{\mathrm{Inf}}\ f(x)$ , for every $i > \underset{T_2}{\mathrm{Sup}}\ f(x)$, then : $x \notin X_i(f \bigcirc T)$. And for every $i \leq \underset{T_2}{\mathrm{Sup}}\ f(x)$ , $x \in X_i(f \bigcirc T)$.

Let us summarize this tedious investigation by giving the algorithm for thinning :

- Calculate $f(x)$, $\underset{T_2}{\mathrm{Sup}}\ f(x)$ , $\underset{T_1}{\mathrm{Inf}}\ f(x)$
- If $\underset{T_2}{\mathrm{Sup}}\ f(x) < f(x) \leq \underset{T_1}{\mathrm{Inf}}\ f(x)$, then $(f \bigcirc T)(x) = \underset{T_2}{\mathrm{Sup}}\ f(x)$
- If not $(f \bigcirc T)(x) = f(x)$

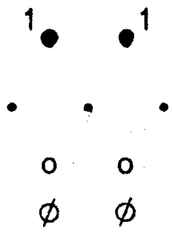
The proof for thickening is similar, and we find the following algorithm :

- Calculate $f(x)$ , $\underset{T_1}{\mathrm{Inf}}\ f(x)$ , $\underset{T_2}{\mathrm{Sup}}\ f(x)$
- If $\underset{T_2}{\mathrm{Sup}}\ f(x) \leq f(x) < \underset{T_1}{\mathrm{Inf}}\ f(x)$ , then $(f \odot T)(x) = \underset{T_1}{\mathrm{Inf}}\ f(x)$
- if not $(f \bigcirc T)(x) = f(x)$ .

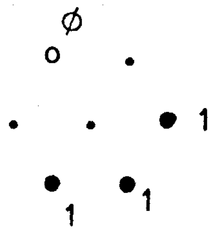
## III.3 - Applications of Thinnings and Thickenings of functions.

The most important application is the skeleton of a function (Goetcharian, 1980).

The ring patterns used to skeletonize a function in a hexagonal grid are as follows :



or



We must rotate the configuration. Call $\{T\}$ the sequence of the six rotated configurations :

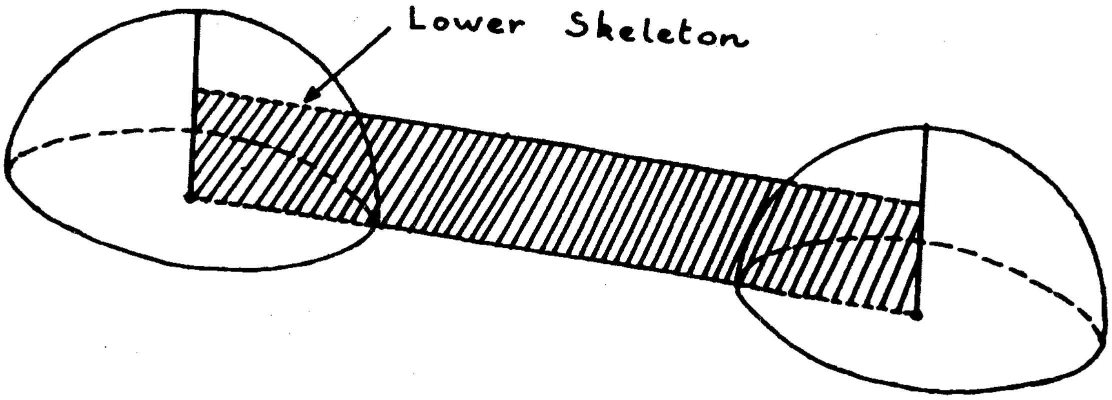
$$\{T\} = \{T^1, T^2, \ldots, T^6\} \quad .$$

The skeleton of f is given by :

$$\text{ske}(f) = \lim_{n \to \infty} g^{(n)} \quad , \text{ where } g \text{ is}$$

$$g = (((f \circ T^1) \circ T^2) \circ T^3) \ldots) \circ T^6)$$

This skeleton is also called a lower skeleton (Serra, 1980). As an example of a skeleton function, consider the function f drawn below :



The graph of f is composed of two hemispheres connected by a wall. The skeleton function penetrates under the graph of f. It is the reason why we say that it is a lower skeleton (this is quite normal : it is the consequence of the anti-extensivity of thinning).

Other applications exist, in various domains of pattern recognition. Notice that the thinning and thickening operations are not the only Golay operations which permit to define functions (as a counter-example, see exercise on median filtering).

Another operation on functions uses Golay configurations, and is quite useful in picture segmentation problems : it is catchment basins and watersheds detection.

### III.4 - Catchment basins, watersheds, application to segmentation.

The reader is requested to refer to the following articles, and to the chapter of this book where segmentation problems are discussed.

# USE OF WATERSHEDS IN CONTOUR DETECTION

BEUCHER S. and LANTUEJOUL C.

Centre de Géostatistique et de Morphologie Mathématique

FONTAINEBLEAU - France -

ABSTRACT

A non-parametric method is developped for contour extraction in a grey image. This method relies in defining the contours as the watersheds of the variation function (gradient modulus) of the light function (considered as a relief surface). Two application examples are described : bubble detection in a radiographic plate, and facet detection in fractures in steel.

## I - INTRODUCTION

Two traditional methods are generally used in contours detection : The first one consists of detecting the strong values of the gradient in an image. This method requires the choice of a threshold value of the gradient modulus. Depending upon this value, the contours are, either thin but not closed, or on the contrary, well closed but too thick, therefore lacking in precision. The second method lies in image segmentation starting from the grey levels histogram. It is based upon the idea that the phases of interest correspond to the most frequent grey values. Unfortunately, this method requires a more or less important smoothing of the histogram and is particularly inoperative when dealing with a great number of phases.

In this paper, we propose a non-parametric contour detection method (the advantage being that no threshold value is used). Giving to its principle, this method gives closed contours. Two examples will illustrate it : bubbles detection in a radiographic plate, and display of facets in a metallic fracture.

## II - DESCRIPTION OF THE METHOD

### II-1) The tools

#### II-1-1) Gradient modulus

Let f be the grey function of an image, supposed to be continuous.

We shall denote the variation of f at point x(u,v) of $R^2$ by the function f defined by :

$$g(x) = \lim_{\varepsilon \to 0} \frac{Sup_{B(x,\varepsilon)}[f] - Inf_{B(x,\varepsilon)}[f]}{2\varepsilon}$$

with

$Sup_{B(x,\varepsilon)}[f]$ maximum value of the function f in the ball of radius $\varepsilon$ centered in x.

$Inf_{B(x,\varepsilon)}[f]$ minimum value of f in $B(x,\varepsilon)$

If f is continously differentiable, it is easy to show that the variation of f is nothing other than the gradient modulus :

$$g(x) = |\text{grad } f(x)| = \left[\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right]^{1/2}$$

#### II-1-2) Thresholds

Thresholding f at level$\lambda$ defines two sets :
the set, denoted $X_\lambda$, of all points x of $R^2$ such that f(x) is less than or equal to $\lambda$ :

$$X_\lambda = \{x \in R^2 : f(x) \leq \lambda\}$$

and the set, denoted $Y_\lambda$, of all points x of $R^2$ such that f(x) is strictly less than $\lambda$.

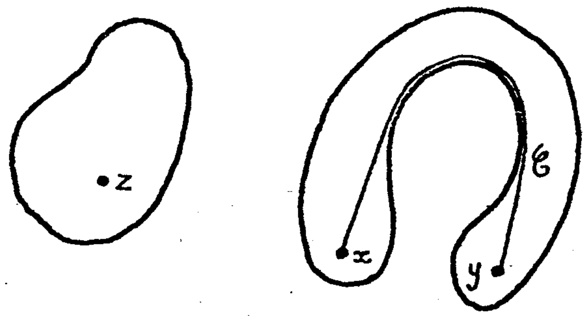
$$Y_\lambda = \{x \in R^2 : f(x) < \lambda\}$$

Notice that the family $\{X_\lambda\}$ for $0 \leq \lambda$ perfectly defines the function f;indeed, we have :

$$\forall\, x \in R^2, \quad f(x) = Inf(\lambda | x \in X_\lambda)$$

II-1-3) Zones of influence

Let X be a part of $R^2$, not necessarily connected. It is possible to define the distance between two points x and y of X as the smallest length of the arcs, if they exist, enclosed in X and joining x to y. If there exists no such arc, the distance is conventionally equal to infinity (Figure 1)




$d_X(y,x) = |\mathcal{C}|$

$d_X(x,z) = +\infty$

Figure 1 : Geodesic distance

This distance is called the geodesic distance. Given a point x of X and a subset Y of X, the geodesic distance between x and Y is :

$$d_X(x,Y) = \operatorname*{Inf}_{y \in Y} d_X(x,y)$$

Let Y be a subset of X consisting of n set $K_1,\dots,K_n$, and disjoined pairwise:

$$Y = \bigcup_{p=1}^{n} K_p \quad \text{with} \quad \forall\, p \neq q,\ K_p \cap K_q = \emptyset$$

A zone of influence $I_p(Y;X)$, consisting of the set of all points of X at a finite distance from $K_p$ and closer to $K_p$ than to any other $K_q$ (with respect to the geodesic distance), can be associated to every $K_p$ (Figure 2).

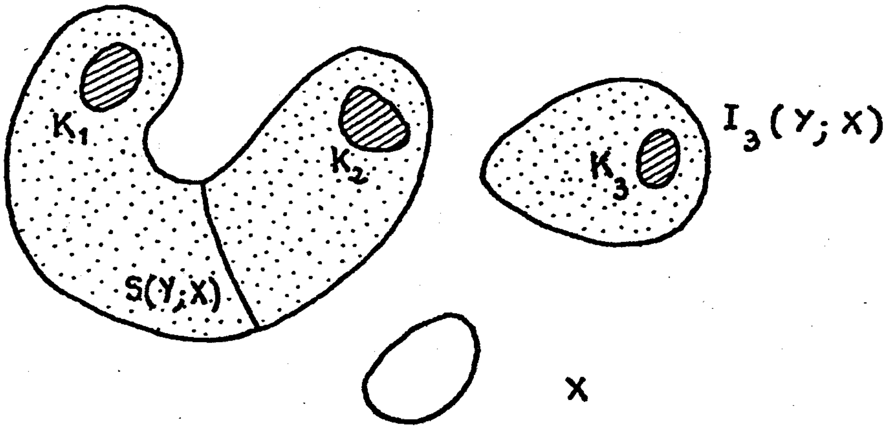



Figure 2 : Zones of influence

The points of X which do not belong to any zone of influence are either points of X at an infinite distance from Y or points equidistant from two different connected components of Y. The set of these latter points is called the "Skeleton by zone of influence" of Y with respect to X. It is denoted by S(Y ; X), and it is possible to prove that it is locally a finite union of simple arcs.

## II-2) Use of the tools

### II-2-1) Minima of a function

Let f be a function defined in $R^2$ and $\{X_\lambda\}$ be its corresponding family of sets. The function f is said to have at point x a minimum of height $\lambda$ (with $\lambda = f(x)$) if :

$$d_{X_\lambda}(x, Y_\lambda) = +\infty$$

(see Figure 3)

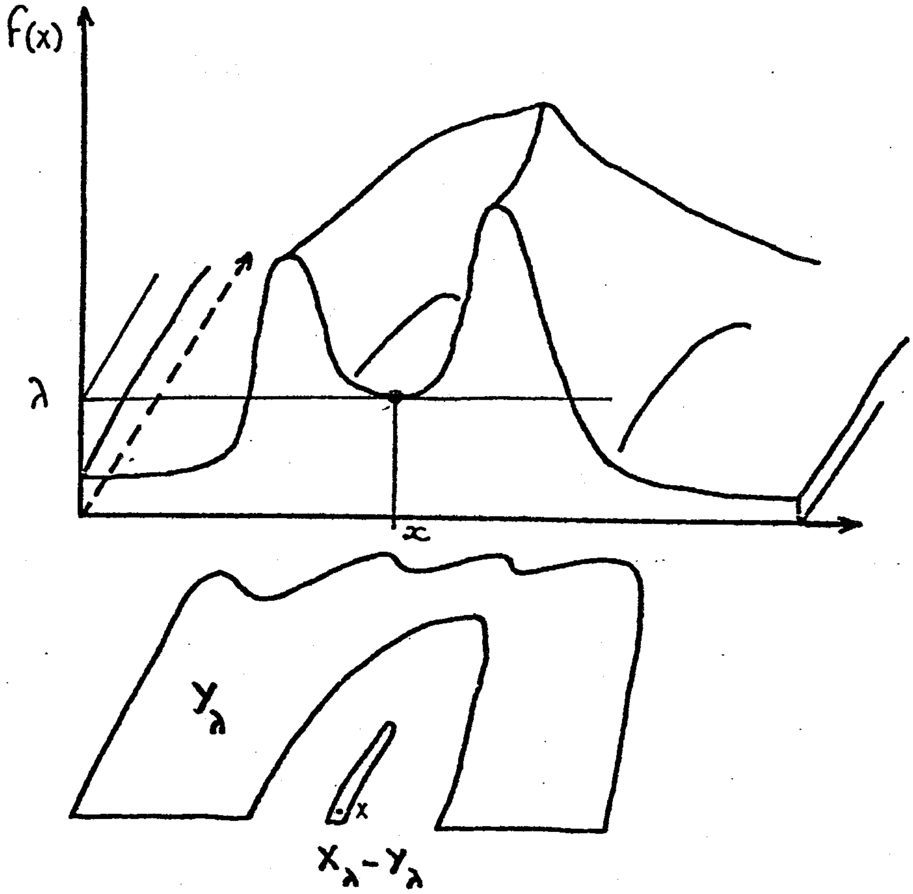



Figure 3 : Minima of a function

We denote by M(f) the set of the minima of the function f.

## II-2-2) Notions of catchment basins and watersheds

These two terms are derived from geography. For better understanding, we shall use geographic vocabulary in this section.

The graph of a function f can be regarded as a topographic surface. f(x) is the height at point x.

Let us consider a drop of water on this topographic surface. The water streams down, reaches a minimum of height and stops there. The set of all points of the surface which the drops of water reaching this minimum can come from can be associated with each minimum. Such a set of points is a catchment basin of the surface. Notice that several catchment basins can overlap: Their common points form the watersheds.

In a more formal way, let Z be the set of the watersheds, and $Z_\lambda$ the subset of Z of those points which are at height λ. Obviously, we have :

$$Z = \bigcup_\lambda Z_\lambda$$

Suppose that we know $Z_\mu$, for every μ strictly less than λ. We shall try to determine $Z_\lambda$.

$(Y_\lambda - \bigcup_{\mu<\lambda} Z_\mu)$ is the set of points whose height is less than λ, and belonging to one, and only one, catchment basin. Let x be a point at height λ. If $d_{X_\lambda}(x, Y_\lambda - \bigcup_{\mu<\lambda} Z_\mu) < +\infty$, and if this distance is the same for two different connected components of $(Y_\lambda - \bigcup_{\mu<\lambda} Z_\mu)$, x appears to be equidistant from two different catchment basins, and consequently, must be considered as a point of Z. In other words, we define :

$$Z_\lambda = S(Y_\lambda - \bigcup_{\mu<\lambda} Z_\mu \; ; X_\lambda)$$

This definition gives a mode of operation for building catchment basins and watersheds.

## III - APPLICATION TO CONTOUR DETECTION

In a picture, we define as contours the watersheds of the variation function g.

This definition seems to be somewhat arbitrary. It has, however, three major advantages :

- it gives a rigorous mathematical definition of the objects under study.
- it furnishes a mode of operation
- the result is visually satisfactory.

As a matter of fact, every object in a picture corresponds to a minimum of the grey variation function.

## IV) EXAMPLES OF USE

### IV-1) Contour detection in a micrography of fractures in steel

The fractures under study are cleavage fractures. The facets of cleavage (see Figure 4) do not present homogeneous grey values.

Figure 4 : Cleavage fractures in steel

Figure 5 shows the result of this procedure of contour detection applied to the previous picture.

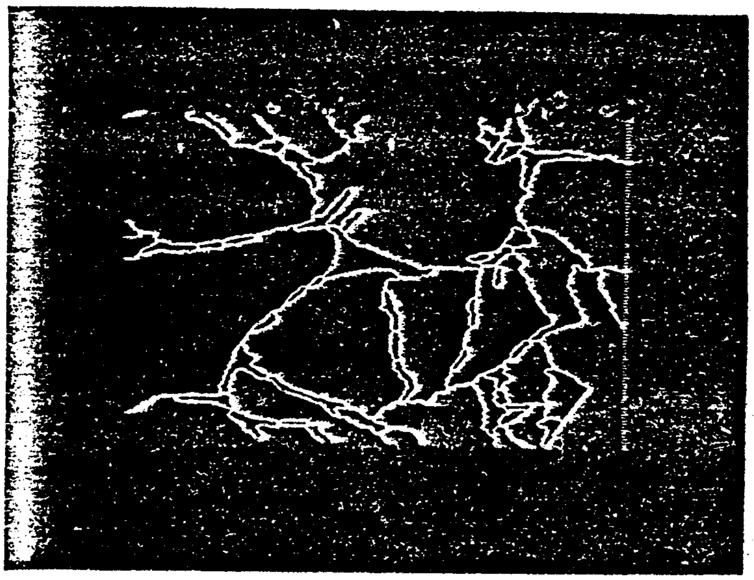


Figure 5 : Result of contour detection

IV-2) Bubbles detection in a radiography

The difficulty here lies in the fact that the bubbles are surrounded by diffraction halos which make them appear larger than they are, and therefore show very fuzzy contours (Figure 6)

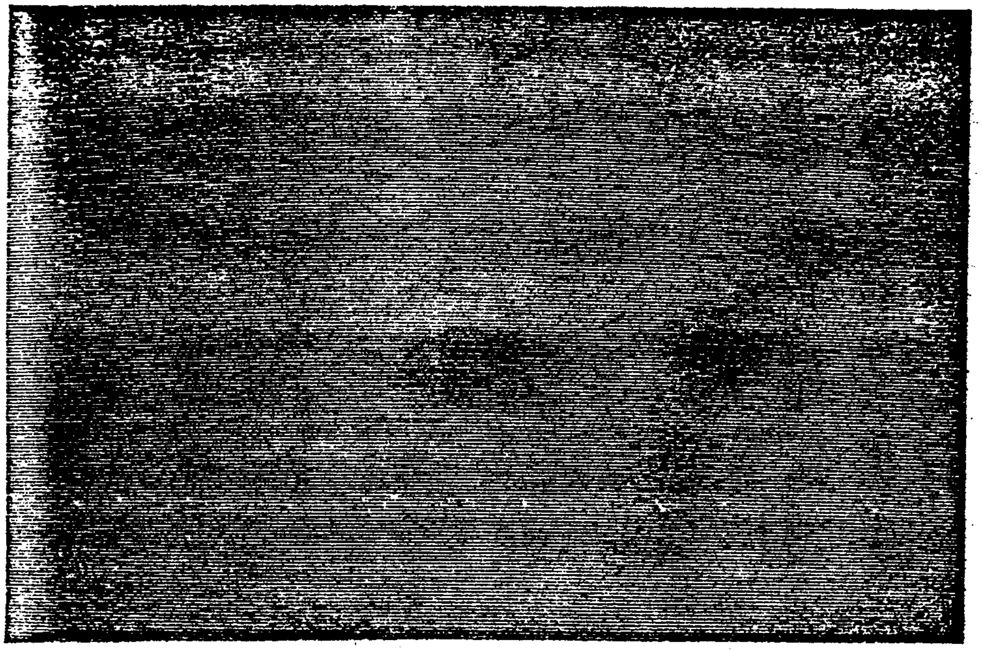


Figure 6 : Bubbles in a radiographic plate

The set Z of contours is constructed using the method previously described (Figure 7).

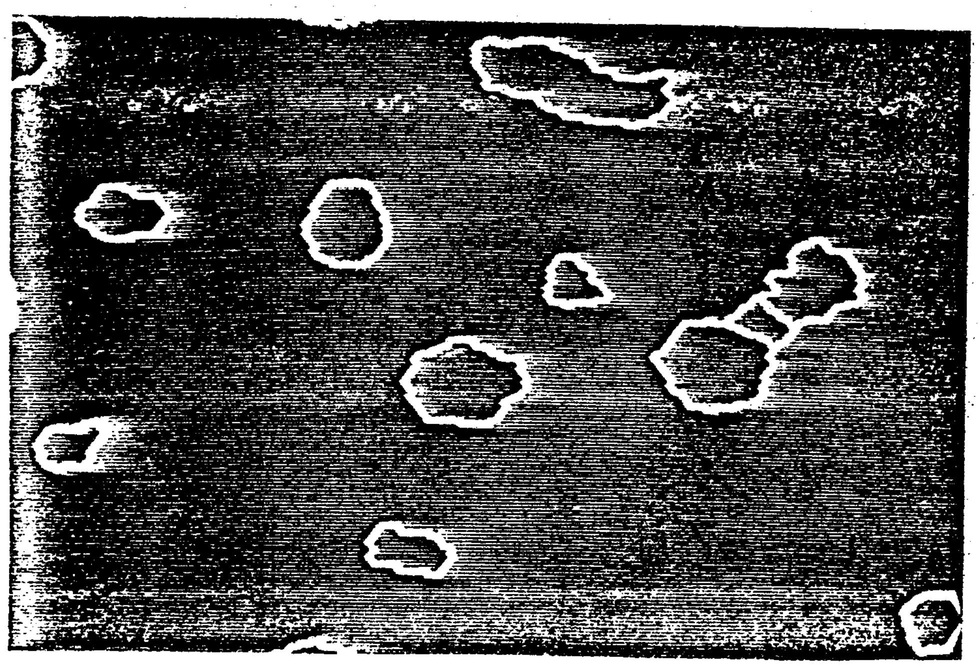


Figure 7 : Result of segmentation

As can be seen, the image is over-segmented. It is easy to suppress the unnecessary contours. For that, we call a bubble every connected component of the partition defined by the contours which contains a minimum of the light function f.

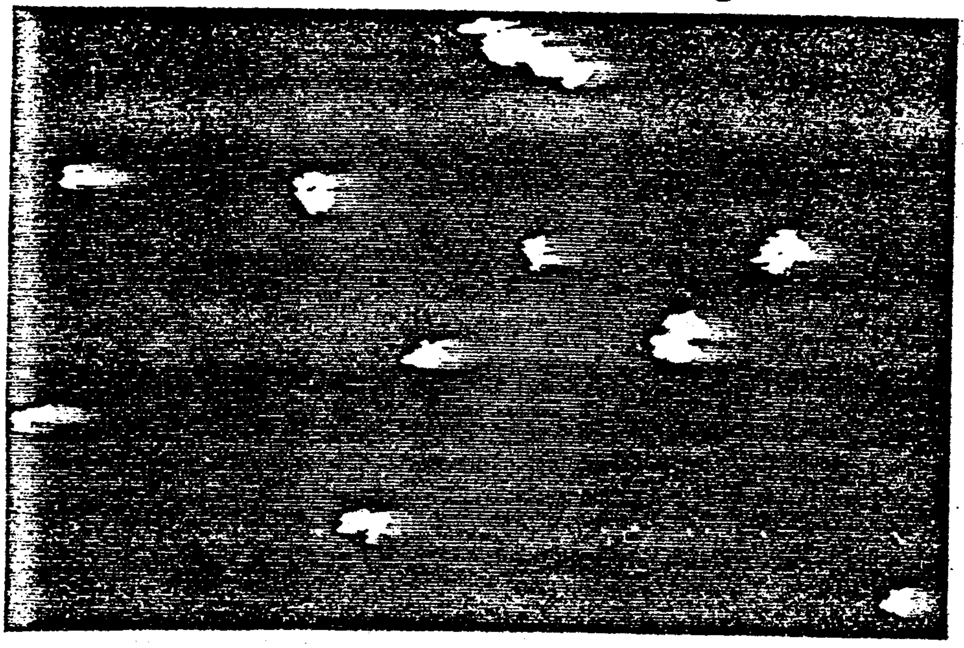


Figure 8 : Minima of the function f.

Hence, we keep only those points x of the contours picture for which :

$$d_{R^2-Z}(x,M) < +\infty$$

Where Z is the contours set, and M, the minima of the function f (Figure 8).

Figure 9 show the final result.

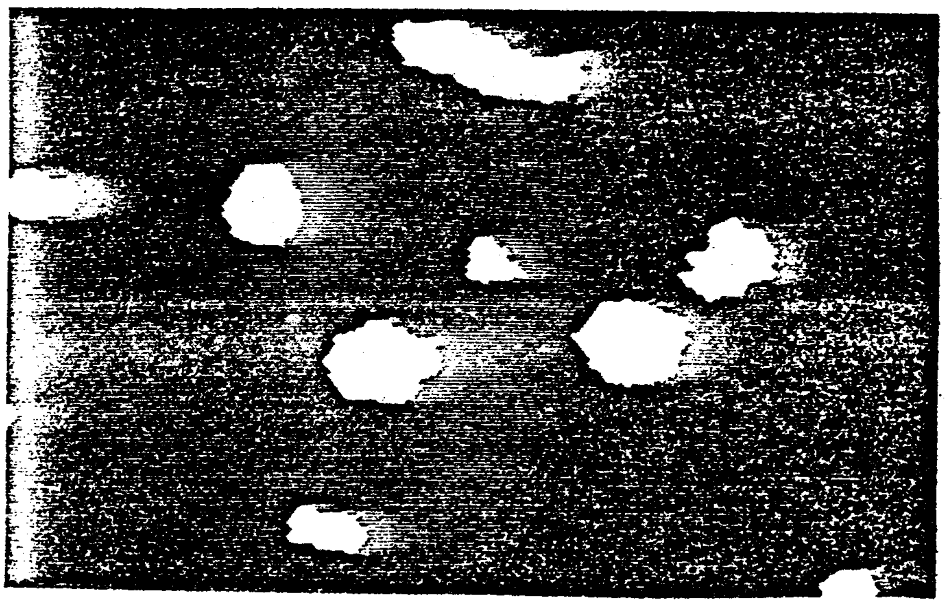


Figure 9 : Final result of the detection of bubbles

Acknoledgments

The authors wish to thank J. PIRET for his kind help during the preparation of this paper.

# BIBLIOGRAPHY

LANTUEJOUL C., BEUCHER S. : On the importance of the field in image analysis. Proceedings of the ICSS, Salzbourg 1979

BEUCHER S., LANTUEJOUL C. : On the change of space in image analysis. Proceeding of the ICSS, Salzbourg 1979

LANTUEJOUL C. : Détection de bulles sur un cliché micrographique par élimination des halos de diffraction qui les grossissent. Internal report, 1979.

BEUCHER S., HERSANT T. : Analyse quantitative de surfaces non planes. Application à la description de faciès de rupture fragile par clivage. Rapport final D.G.R.S.T. (Aide n° 76.7.1209,1210), 1979.