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# SHAPES AND PATTERNS OF MICROSTRUCTURES CONSIDERED AS GREY-TONE FUNCTIONS

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## ABSTRACT

Mathematical morphology, which is a set theory, provides means for studying shapes and patterns of grey tone functions. Indeed, with every function $f(x)$, $x \in \mathbb{R}^n$, one associates its umbra, i.e. the set of those points $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ such that $t \leq f(x)$. Here the upper semi-continuous functions play a role similar to that of the closed sets in the Euclidean space. The basic operations are the sup and the inf, corresponding to the set union and intersection respectively.

After having established a few fundamental results, we first by emphasize the increasing operations, and especially the two basic ones (erosion and opening). An interpretation of the openings in terms of non linear filtering is developed. Then, we study the extensive transformations whose the thinnings and the thickenings are the prototype.

Finally, the notion of homotopy originally defined for sets, is generalized to functions.

## INTRODUCTION

Basically, Mathematical Morphology is a set theory. In order to extend it to grey tone images, it suffices to consider the functions defined in $\mathbb{R}^2$ as a particular class of sets of $\mathbb{R}^3$ (fig. 1), via their reliefs. Then all the notions developed in set morphology reappar as tools for handling functions. Note that to inververt the priority between sets and functions leads us to emphasize the non-linear operations of sup and inf to the detriment of addition and subtraction.

Mathematical morphology for grey tone functions has to be investigated from three different points of view. Each transformation acts on the functions themselves, and also on their horizontal cross sections, and finally has a geometrical interpretation. Therefore a transformation is completely known when we are able to express it in terms of each approach, and to derive the corresponding properties. We shall fulfill such a program for all the notions presented below. After having established a few general results we treat separately the two main classes of morphological operations, i.e. the increasing ones and the extensive ones (plus the derivative notions).

Except G. Matheron's pioneer work (1969) on the compacity of the umbrae, the literature on the subject dates principally from the second part of the 70's and thinned of its redundancies, reduces to the following major contributions. J. Serra (1975) started by extending the Hit or Miss transformation and the size distributions from sets to functions by using flat structuring elements. The description of a function by its watersheds, and the associated algorithms, are due to Ch. Lantuéjoul (1977). Later S. Beucher and Ch. Lantuéjoul (1979) formalized the concept by using geodesic distances. In the meantime F. Meyer (1977) introduced the top-hat transformation and fruitfully implemented it in quantitative cytology. A new wave of generalizations was due to S. Sternberg (1978) who systematically considered the functions as set in $R^3$. A first survey of the question appears in V. Goetcharian's thesis (1979), where the author contributes some original notions such as the lower skeleton, and convexity criteria. In 1981, J. Serra synthetised these various results in a unique formalism and introduced the concepts of homotopy for function, of lower and upper thinnings and of a new type of random functions.

## GENERAL RESULTS

### Semi-continuous functions and their thresholdings

The notion of an upper semi continuousfunction (u.s.c.) defined on the plane corresponds to that of a closed set. Remember that function $f(x)$, $x \in R^2$ is u.s.c. when for every $x$ and every $t > f(x)$ there exists a neighborhood $V_x$ of $x$ such that $f(y) > t$ for every $y \in V_x$. According to a classical topological result (Choquet, 1966), $f(x)$ is u.s.c. iff its horizontal cross sections are closed sets :

$$f(x) \text{ u.s.c. } \Leftrightarrow X_t(f) = \{x : f(x) > t\} \text{ a closed set } \quad (-\infty < t < +\infty)$$

(fig. 1-a). When the context is not ambiguous, the u.s.c. functions are simply called "function" ; their class is denoted $\mathcal{F}_u$. We call a <u>picture</u> a positive u.s.c. function $f(x)$ bounded by the value m :

$$f(x) \text{ picture } \Leftrightarrow 0 \leq f(x) \leq m \tag{1}$$

The class of pictures is denoted $\mathcal{P}_i$. Photographs and images are modeled by pictures, but we also need functions to perform a subtraction, or for certain parametrizations (for example, build a function from the set $X \in \mathcal{F}(R^2)$ by adding its quench function to the negative of the quench function of $X^C$).

When the point x spans the Euclidean plane $\mathbb{R}^2$, the support of function f(x) is the set of points x where f(x) ≠ 0. The class of functions (resp. pictures) with compact support is denoted by $\mathcal{K}_u$ (resp. $\mathcal{K}_i$).

By thresholding a function f(x) at successive levels, we can associate a family of sets with the function. But by acting on those sets using methods of mathematical morphology, we can generate a new set family. Under what conditions to these new sets represent a function ? The following theorem characterizes legitimate set families.

Theorem 1 - Let f be a u.s.c. function defined in $\mathbb{R}^2$, and

(2) $$X_t(f) = \{ x : f(x) \geq t \} \qquad -\infty \leq t \leq +\infty$$

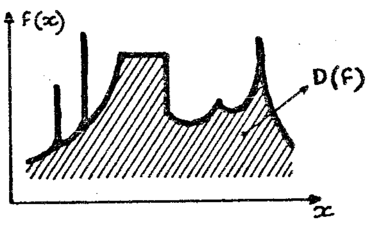
is the set family generated by thresholding f at level t. Then, the $X_t$'s are closed and monotonically decreasing, i.e.

(3) $$t' < t \quad X_{t'} \supset X_t \text{ and } X_t = \lim_{t' \uparrow t} \downarrow X_{t'} = \bigcap_{t' < t} X_{t'}$$

and we have

(4) $$f(x) = \text{Sup} \{ t : x \in X_t \}$$

Conversely, a family $X_t$ $(-\infty \leq t \leq +\infty)$ of closed sets generates a u.s.c. function f(x) if and only if conditions (3) are satisfied ; f(x) is then defined by (4) and satisfies (2).



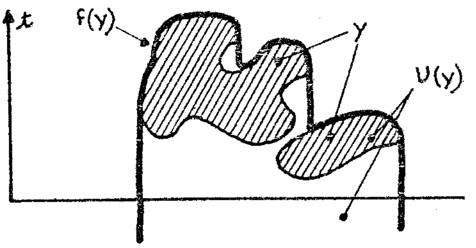




Figure 1-a - umbra of a u.s.c. function (it looks like paint that trickled upwards).

b - umbra U(Y) associated with the set Y.

Umbrae

The notion of an umbra (Sternberg 1978) is the link between the functions $f \in \mathcal{F}_u(\mathbb{R}^2)$ and the closed set $Y \in \mathcal{F}(\mathbb{R}^3)$ of the space. Points of $\mathbb{R}^3$ are parametrized by their projection x on $\mathbb{R}^2$ and their altitude t on an axis perpendicular to $\mathbb{R}^2$. The umbra U(Y), $Y \in \mathcal{F}(\mathbb{R}^3)$ is the dilate of Y by the positive axis $[0, +\infty]$ of the t's, i.e. since $(0, -\infty)$ is the transposed of $[0, +\infty]$ :

(5) $$U(Y) = Y \oplus [0, -\infty] = \{(x,t') : (x,t) \in Y \; ; \; t' \leq t \}$$

Note that U(Y) is morphologically open by the negative t-axis $[0, -\infty]$

and closed by the positive t-axis $[0, +\infty]$. The class $\mathcal{U}$ of the umbrae of $\mathcal{F}(\mathbb{R}^3)$ plays a fundamental role in the morphological study of the functions $\mathcal{F}_u$. Each umbra U induces a unique function f(U), whose value at point x is the sup of the t's such that $(x,t) \in U$. Conversly, given a function $f \in \mathcal{F}_u$, the set :

(6) $$U(f) = \{(x,t) : f(x) \geq t \quad x \in R^2, \ t \in R$$

is by construction an umbra. Finally we have the following implications :

(7) $$X \in \mathcal{F}(R^3) \Rightarrow U(X) \in \mathcal{U}(R^2 \times R) \rightleftarrows f(U) \in \mathcal{F}_u.$$

There is a unique umbra and a unique u.s.c. function associated with each closed set X of $\mathbb{R}^3$ ; conversly, each u.s.c. function f corresponds to a unique umbra U(f), but to the infinity to the closed sets of $\mathbb{R}^3$ which possess the same umbra U(f).

The simplest way to provide functions (and pictures) with a topology suitable for the morphological treatment is to use the topology induced by the Hit or Miss topology on their umbrae. Indeed the class $\mathcal{U}$ of the umbrae in $\mathbb{R}^3$ is a compact part of the closed sets $\mathcal{F}(\mathbb{R}^3)$. Therefore, one can define on it limits, continuity, semi-continuity, probabilities, etc... Here, we will only quote, as an example, the following criterion of convergence.

Theorem 2

A sequence $\{f_i\}$, $f_i \in \mathcal{F}_u$, tends toward $f \in \mathcal{F}_u$, iff :

a) for every $x \in \mathbb{R}^2$, there exists a sequence $\{x_i\} \to x$ such that $f_i(x_i) \to f(x)$

b) and if the set sequence $\{x_{i_k}\} \to x$, there $\{f_{i_k}(x_{i_k})\}$ satisfies $\overline{\lim}\{f_{i_k}(x_{i_k})\} \leq f(x)$

Note that, w.r. to this topology, the sup is a continuous operation, but the addition is only semi-continuous.

Elementary transformations

a - Sup. and inf. Denote the sup (resp. the inf) of f(x) and g(x) by $f \vee g$ (resp. $f \wedge g$) (see fig. 2 a,b). $f \vee g$ and $f \wedge g$ turn out to be the immediate generalization to functions of the set notions of on union and an intersection. Indeed, we have :

(8) $$\begin{cases} X_t(f \vee g) = \{x : f(x) \text{ or } g(x) \geq t\} = X_t(f) \cup X_t(g) \\ X_t(f \wedge g) = \{x : f(x) \text{ and } g(x) \geq t\} = X_t(f) \cap X_t(g) \end{cases}$$

b - Complementation

- for functions : defined by symmetry with respect to the plane t = 0.

(9) $$[X_t^-(-f)]^c = \{x : -f \leq t\} = X_{-t}(f)$$

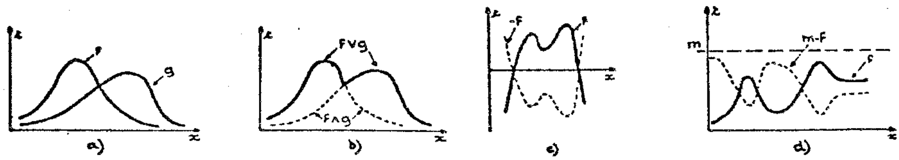


Figure 2 a,b - sup $f \vee g$ and inf $f \wedge g$ of functions f and g

c,d - the two modes of complementation for functions (b) and for pictures (c).

- for pictures : defined by symmetry with respect to the median plane $\frac{m}{2}$ :

(10) $[X_t^-(m-f)]^c = \{x : m-f \leq t\} = \{x : f \geq m-t\} = X_{m-t}(f)$ .

In photography for example, this latter operation corresponds to taking the negative of a photograph. Note that in both cases, the sets $\{X_t\}$ associated with the l.s.c. functions-f and pictures m-f, are open.

c - duality. Two function (resp. picture) transformations $\psi$ and $\psi^*$ are said to be dual of each other w.r. to the complementation when :

(11) $\psi(-f) = -\psi^*(f)$ for functions

or

(12) $\psi(m-f) = m - \psi^*(f)$ for pictures

DILATIONS, EROSIONS AND DERIVED NOTIONS

Dilation and erosion

Let us briefly recall the Minkowski operations for sets. The Minkowski sum $X \oplus B$ and difference $X \ominus B$ of X by B are defined by the relations :

$$X \oplus B = \bigcup_{b \in B} X_b \qquad X \ominus B = \bigcap_{b \in B} X_b$$

Denote by $\check{B} = \bigcup_{b \in B} (-b)$ the transposed set of B and by $B_x$ the translate of B by vector $\vec{Ox}$. Then $X \oplus B$ is also the dilate of X by $\check{B}$, i.e. locus of centres x of $\check{B}_x$ which hit X and $X \ominus B$ is also the eroded of X by $\check{B}$, i.e. the locus of the centers x of $\check{B}_x$ which are included in X.

When dealing with umbrae the special role played by the third dimension (t-axis) leads us to introduce the symmetry by reflexion. Given a set Y, the reflected set $\check{Y} = \{(x,t) ; (x,-t) \in Y\}$ is symmetrical to Y with respect to the horizontal plane of coordinates (Fig. 3). This new transformation interacts with the elementary operations of transposition $Y \rightarrow \check{Y}$, of complementation $Y \rightarrow Y^c$ and of umbrization $Y \rightarrow U(Y)$, according to a few algebraic rules, such that :

(13) $(\hat{B})^{\wedge} = B$ ; $(\hat{B})^{\vee} = (\check{B})^{\wedge}$ ; $[\hat{U}(B)]^{\vee} = U[(\hat{B})^{\vee}]$ ; $\hat{U}(f) = [U(-f)]^c$

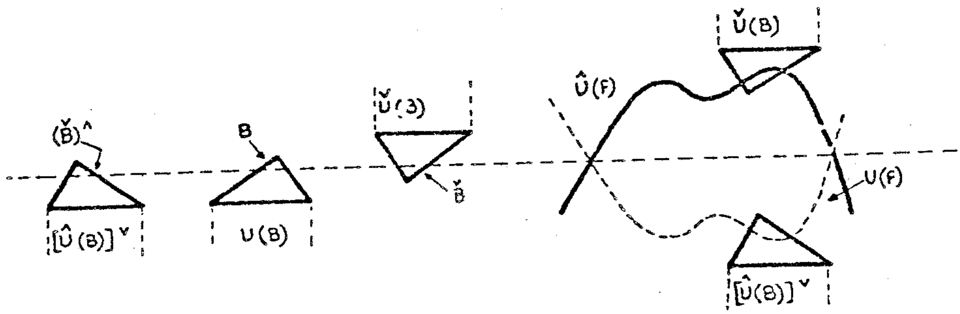



Figure 3 : Set B, its reflected $\hat{B}$, is transposed $\check{B}$, and the corresponding umbrae.

Concerning the dilation and the erosion, the equivalences :

$$\check{B}_{x,t} \Uparrow U(Y) \Leftrightarrow \check{U}(B_{x,t}) \Uparrow U(Y)$$

and $(\hat{B}_{x,t})^{\vee} \subset U(Y) \Leftrightarrow \hat{U}(B_{x,t})^{\vee} \subseteq U(Y)$

imply that

(14) $U(Y) \oplus B = U(Y) \oplus U(B)$ ; $U(Y) \ominus \hat{B} = U(Y) \ominus \hat{U}(B)$.

Suppose now that Y itself is an umbra ; therefore it characterizes the function f given by the relation $U(f) = Y$. Then we define the Minkowski sum $f \oplus B$ of function f by the structuring element $B \in \mathcal{K}(R^2 \times R)$ via the umbrae, by putting :

(15) $U(f \oplus B) = U(f) \oplus B \qquad F \in \mathcal{F}_u,\ B \in \mathcal{K}(R^3)$

It is not possible to introduce the Minkowski subtraction simply by replacing $\oplus$ by $\ominus$ in relation (15) since $U(f) \ominus U(B)$ is reduced to the point at $-\infty$, whatever f and B are. Preferably, we will derive it by duality, by starting from algorithm (11). This algorithm, applied to the present case, implies that

$$\hat{U}(F \ominus B) = [U(-f) \ominus B]^c \ .$$

The right hand side of this expression delineates the set of point (x,t) such that $\check{B}_{x,t}$ misses $U(-f)$, by reflection the condition becomes $(\check{B}_{x,t})^{\wedge} \subset U(f)$ and finally

(16) $U(f \ominus B) = U(f) \ominus \hat{B} \qquad F \in \mathcal{F}_u,\ B \in \mathcal{K}(R^3)$.

The functions $f \oplus B$ and $f \ominus B$ belong to $\mathcal{F}_u$.

The two definitions (15) and (16) can immediately be extended to operations for functions. By taking for B = U(g) the umbra of function f, we obtain :

$$(17)\quad \begin{cases} f \oplus g \rightarrow U(f \oplus g) = U(f) \oplus B = U(f) \oplus U(B) = U(f) \oplus U(g) \\ f \ominus g \rightarrow U(f \ominus g) = U(f) \ominus \hat{B} = U(f) \ominus \hat{U}(B) = U(f) \ominus \hat{U}(g) \\ \text{with } -(f \ominus g) = (-f) \oplus g \end{cases}$$

For f, g $\in$ $_u$, and g with a compact support, $f \oplus g$ and $f \ominus g$ belong to $\mathcal{F}_u$. The dilation is continuous, increasing with respect to f and to g and extensive with respect to f when g contains the origin. The erosion is u.s.c., increasing with respect to f and decreasing with respect to g (i.e. $f \leq f' \Rightarrow f \ominus g \leq f' \ominus g$ ; $g \leq g' \Rightarrow f \ominus g \geq f \ominus g'$) and extensive with respect to f when g contains 0. All the classical properties of distributivity and iterativity for sets are still valid for the Minkowski operations on functions.

Relations (17) provide not only the definitions of $f \oplus g$ and $f \ominus g$, but also the geometrical interpretations in terms of umbrae ; their translation in terms of sup and inf results and gives :

$$(18)\quad \begin{cases} (f \oplus g)_x = \operatorname*{Sup}_{y \in R^2} [f(y) + g(x-y)] \\ (f \ominus g)_x = \operatorname*{Inf}_{y \in R^2} [f(y) - g(x-y)] \end{cases}$$

with $f(x) = g(x) = -\infty$ for $x \notin$ support of f (resp. of g)

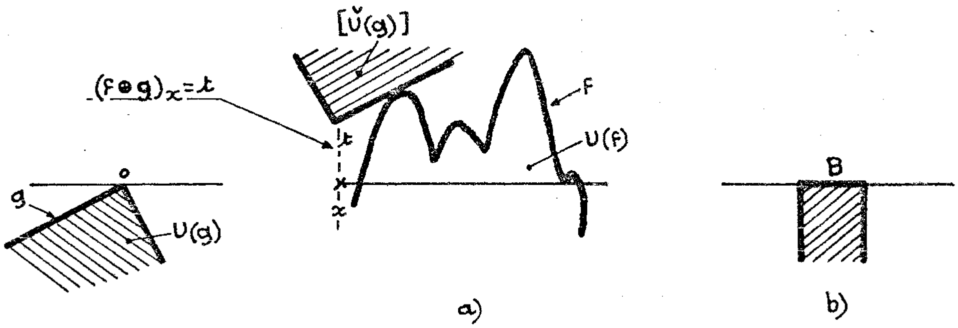



Figure 4-a : Dilations of umbrae
-b : Flat structuring element and its umbra

The expression of $X_{t_0}(f \ominus g)$ as a function of the $X_t(f)$'s and $X_t(g)$'s derives from rel (18), and we can write :

(19) $$X_{t_o}(f \oplus g) = \bigcup_t \left[ X_t(f) \oplus X_{t_o - t}(g) \right]$$

By duality, we get for the erosion :

(20) $$X_{t_o}(f \ominus g) = \bigcap_t \left[ X_t(f) \ominus X_{t - t_o}(g) \right]$$

We now are able to construct openings and closings by composition of erosions and dilations. The symbol $\check{g}$ (transposed of g) denotes the function $\check{g}(x) = g(-x)$. We define the opening $f_g$ of f by g from their umbrae as follows :

(21) $$U(f_g) = \left[ U(f) \ominus \check{U}(g) \right] \oplus U(g)$$

The umbra of $f_g$ is the part of the domain of the umbra U(f) spanned by all the translates of U(g) which are included in U(f). By noticing that $\check{U}(g) = \hat{U}(\check{g})$, we can interpret rel. (21) in terms of functions, and :

$$f_g = (f \ominus \check{g}) \oplus g \; .$$

We shall introduce the notion of a closing $f_g$ by duality by writing

$$f^g = -(-f)_g \Leftrightarrow f^g = (f \oplus \check{g}) \ominus g$$

The relation between umbrae derives immediately, and we have :

(22) $$U(f^g) = \left[ U(f) \oplus U(\check{g}) \right] \ominus \check{U}(\check{g})$$

The complement of the umbra of $U(f^g)$ is the zone spanned by all the translates of $\hat{U}(g)$ which are included in $[U(f)]^c$. The theory of size distribution for sets extends integrally to functions via their umbrae and relations (21) and (22). From the same basis, we could also generalize gradients, ultimate erosions, conditional bisectors, etc... Finally the approach led to a very comprehensive class of morphological operations for functions (and pictures).

Flats structuring elements

As a particular, but important case, we now take for B a 2-D set lying in the horizontal plane t = 0. Then, the umbra U(B) is just the half cylinder of top B (Fig. 4,b). The basic rel. (18) (19) and (20) become :

$$\begin{cases} (f \oplus B)_x = \sup \; \{ f(y) \; ; \; y \in \check{B}_x \} \\ (f \ominus B)_x = \inf \; \{ f(y) \; ; \; y \in \check{B}_x \} \end{cases}$$

and

(23) $$X_t(f \oplus B) = X_t(f) \oplus B \; ; \; X_t(f \ominus B) = X_t(f) \ominus B$$

Each level of $f \oplus B$, and of $f \ominus B$ is obtained by processing uniquely the same level of f. This simplification makes the computation of $f \oplus B$, and $f \ominus B$ extremely easy by means of set image analysers such as the Texture Analyser. As we can see on fig. (5), $f \ominus B$ reduces the peaks and enlarges the valleys, and vice versa for $f \oplus B$.

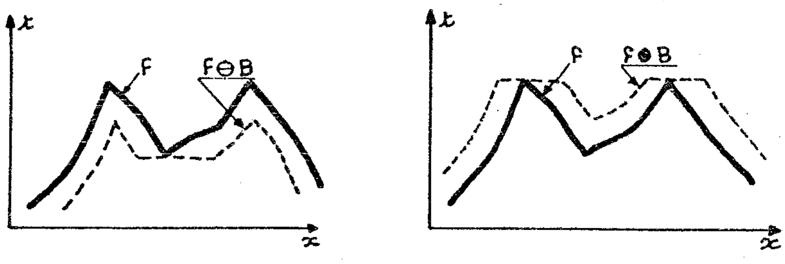



Figure 5 : Erosion and dilation of f by the horizontal segment B.

For B a compact convex set, the family $(f \ominus \lambda\check{B}) \oplus \lambda B = f_{\lambda B}$ is the size distribution of f by openings (for functions and pictures). The geometrical interpretation of the opening $f_{\lambda B}$ is given by the two reciprocal formulae

$$(24)\quad \begin{cases} X_t(f_{\lambda B}) = \{x : \exists B \ni x \text{ , and } \forall z \in B \; f(z) \geq t\} \\ f_{\lambda B}(x) = \mathrm{Sup} \; \{\inf \{f(z), z \in B_y\} \; : y \in \check{B}_x\} \end{cases}$$

At the point x, $f_{\lambda B}(x)$ has the highest value of the infimums of f taken over all the B's containing x. Equivalently, $x \in X_t(f_{\lambda B})$ if and only if D(f) can be penetrated by a vertical cylinder $C(\lambda B)$ of section $\lambda B$, which hits the point x, and such that all the altitudes of f within $C(\lambda B)$ are higher than t (fig. 6 a,b).

When dealing with a picture, one often changes its scale of greys, by analogic or digital means (contrast of a photograph, correction of a T.V. camera, cut-off of high values). The most usual transformations of this type are $f(x) \rightarrow a\, f(x) + b$ $(a, b > 0)$ ; Log f(x) ; $[f(x)]^2$ ; $\sqrt{f(x)}$ ; f(x) for $f(x) \leq \lambda$, $\lambda$ for $f(x) \geq \lambda$, and their various combinations. It is essential that we know the interferences between these grey scale changes and the morphological operations. Define an anamorphosis $\Psi(f)$ to be an increasing and continuous mapping of $\mathcal{F}_u$ (or $\mathcal{F}_i$) onto $\mathcal{F}_u$. We have, for every set Z,

$$: \mathrm{Sup} \; \{\Psi(f(x)) \; ; \; x \in Z\} = \Psi(\sup \{f(x) \; ; \; x \in Z\}),$$

therefore :

$$(25)\quad \Psi(f \oplus B) = \Psi(f) \oplus B \text{ and } \Psi(f \ominus B) = \Psi(f) \ominus B \qquad \forall B \in R^2$$

In words : when the structuring element B is flat, the Minkowski operations commute with the anamorphoses.

Example

· Erosions and openings often intervene via differences of functions, as can be shown by the following examples.

a - Gradient

Assume that f is differentiable everywhere in its domain of definition, except on a set of regular curves S of $\mathbb{R}^2$ ; but each partial derivative has a limit on both sides of S (vertical cliffs). Then the gradient of f is vectorial measure whose module satisfies the relationship (S. Beucher, 1978).

(26) $$\rho(dx) = \lim_{\lambda \to 0} \frac{(f \oplus \lambda B) - (f \ominus \lambda B)}{2\lambda}$$ (B closed unit disk)

which can easily be digitalized. Similarly, the rose of directions of the gradient can be derived from the total diameters of the horizontal sections (J. Serra, 1981).

b - Rolling ball and top hat transformation

B is a convex, but not necessary, flat structuring element. The rolling ball transformation is defined by the difference $f - f_B$ (Sternberg, 1979). In the case of B a flat set, the same concept $f - f_B$ was already studied by F. Meyer (1977) who called it "top-hat transformation" (fig. 6,c). Rolling ball and top hat transform extract peaks and ridges of the function, independently of their altitude,

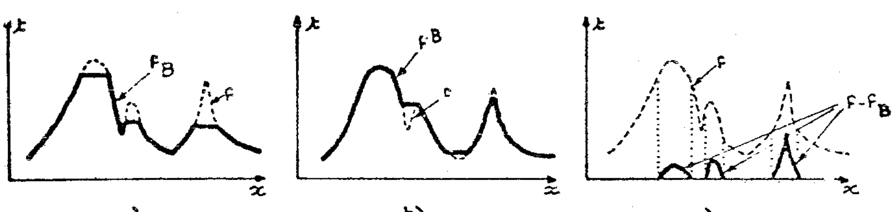



Figure 6 a,b - Opening and closing of a function by a compact convex set

c - Difference $f - f_B$ between f and its opening

but only according to their degree of "sharpness". Not only do they lead to size distributions involving the contrast of the image (by using sequences of similar B's) but they also provide one of the best types of algorithms for segmenting the images, since they are insensitive to the low variations of grey tones (fig. 7).

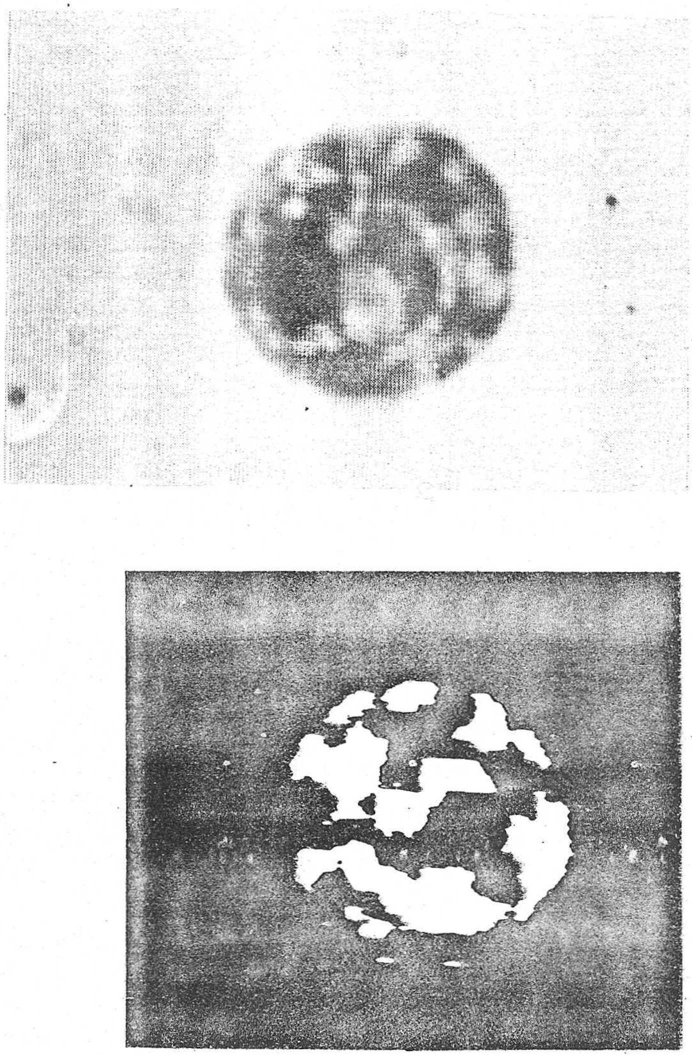


Figure 7 : Top-hat transformation used for extracting the zone Z of coarse chromatine (a) in a nucleus (Afterwards, one can measure the DNA content in Z, or study the size and the shape of the coarse chromatine by performing openings and skeletons on Z and on its complement.

Indeed opening can be interpreted as a 2-D filtering process, but in a rather special sense. Consider the Fourier expansion of a function f (in one dimension for simplicity's sake). Let $\{a_i\}$ $(a_i \geq 0)$ be the energies associated with the frequencies. In Fourier space, a filter $\{\varphi_i\}$ $(0 \leq \varphi_i \leq 1)$ acts by replacing $\{a_i\}$ with $\{\varphi_i\, a_i\}$. If we iterate the filtering we obtain $\{\varphi_i^2\, a_i\}$ , which is identical to $\{\varphi_i\, a_i\}$ if and only if $\varphi_i$ = 1 or 0. If so, we can see that the sizing axioms are satisfied for $\{a_i\}$ , i.e. satisfied in the Fourier plane. Such is the case, for example, for the low pass filters $(\varphi_i = 0$ for $i \geq i_0, \varphi_i = 1$ for $i < i_0)$. On the other hand, the opening acts in the plane of definition of f itself. It happens for example that f is always above $f_{\lambda B}$ (first axiom), but can be below its filtered version obtained by a low pass filter. The second difference between the two techniques is that the opening is not linear, but only satisfies the relationship $f \leq g \Leftrightarrow f_B \leq g_B \; \forall B \in \mathcal{H}$. The third, and main difference is not logical but morphological. The residual $f-f_B$ extracts the sharp peaks, ridges and saddles of f. Then a further morphological treatment can easily separate them from each other. Nothing equivalent exists with Fourier transformation. Finally the opening is essentially a digital technique, whose numerical implementation in $\mathbb{R}^2$ is simpler than the fast Fourier transform. All the mappings presented in this section C are digitalizable using the covering representation, defined on the umbra U(f), and therefore are particularly robust. The digital algorithms are formally identical to the corresponding Euclidean ones, and transforms exhibit the same number of grey levels as the original pictures.

## EXTENSIVE TRANSFORMATIONS AND THEIR DERIVATIONS

### Thinnings and Thickenings for functions

Thinnings and thickenings are, for sets, very powerful transformations. The definitions of these transformations are the following: Consider the set $X \in \mathcal{P}(\mathbb{R}^2)$ and the hit-or-miss transform X * T by the couple $T = (T_1, T_2)$

(27) $X * T = (X \ominus \check{T}_1)/(X \oplus \check{T}_2)$ (where/denotes the set differences)

We thin X by T when we substract X T from X, and we thicken X by T when we add X * T to X. We can write :

(28)
$X \bigcirc T = X/(X * T)$ , Thinning

$X \odot T = X \cup (X * T)$ , Thickening

These two operations are dual each other when related to the complementation :

(29) $(X \odot T)^c = X \bigcirc T^*$ , where $T^* = (T_2, T_1)$

Now, we can extend this definition to the functions. Let f be a function defined on $\mathbb{R}^2$, and $(T_1, T_2)$ two structuring elements belonging to $\mathcal{P}(R^3)$ (don't worry about the topological status of f,

$T_1$ and $T_2$. This problem will be examined later). We can apply the previous definitions to the umbra U(f) of f, and define :

$$f \bigcirc T \quad \text{such that } U(f \bigcirc T) = U(f)/U(f) * T)$$

and, in the same way :

$$f \odot T \quad \text{such that } U(f \odot T) = U(f) \bigcup \left[ U(f) * T \right]$$

The hit-or-miss transformation can be written as following :

$$U(f) * T = (U(f) \ominus \check{T}_1) \bigcap (U(f) \oplus \check{T}_2)^c$$

U(f) * T is the set of points of $\mathbb{R}^3$ such that $T_1 \subset U(f)$ and $T_2 \subset U(f)^c$. See figure n°

Using the formula of reflection, and the notion of umbra, we have :

$$U(f) \ominus \check{T}_1 = U(f) \ominus \check{U}(T_1)$$

and

$$(U(f) \oplus \check{T}_2)^c = \left[ U(f)^c \ominus \check{T}_2 \right] = \left[ U(f)^c \ominus U(\check{T}_2) \right] = U^c(f) \ominus \left[ \hat{U}(\hat{T}_2) \right]^v$$

Let us try to explain these formulae (figure 8)

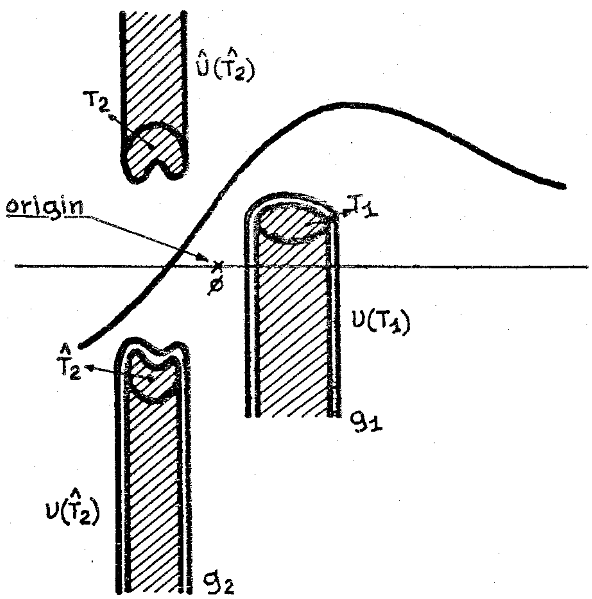



Figure 8 : processing of a thinning

The points of f * T are those such that $U(T_1)$ is below f and $\hat{U}(\hat{T}_2)$ above f.

The umbrae $U(\hat{T}_2)$ et $U(T_1)$ characterize two fonctions $g_1$ and $g_2$. So, we can define the thinning and the thickening of functions by a doublet $(g_1,g_2)$ of functions :

(30)
$$U(f \bigcirc g) : U(f)/ [U(f \ominus g_1)/U(f \oplus g_2)]$$

$$U(f \odot g) = [U(f)] \cup [U(f \ominus g_1)/U(f \oplus g_2)]$$

with $g = (g_1,g_2)$

We did not yet make any assumption concerning the topological status of f, $g_1$ and $g_2$. We can show that if f, $g_1$ and $g_2 \in \mathcal{F}_u$, $g_1$ and $g_2$ with compact supports, then $f \bigcirc g$ and $f \odot g$ belong to $\mathcal{F}_u$.

In order to transpose the geometrical definition (30) in terms of algebraic algorithm, we will apply rel.(19) and (20). This results in the following rules :

when $\sup_{z \in T_{2_x}} [f(z) + g_2(z-x)] < f(x) \leq \inf_{y \in T_{1_x}} [f(y) - g_1(y-x)]$ , then

(33)
$$(f \odot g)(x) = \inf_{y \in T_{1_x}} [f(y) - g_1(y-x)]$$

when not : $(f \odot g)(x) = f(x)$

Flat structuring elements

When $T_1$ and $T_2$ are embedded in $\mathbb{R}^2$, we simply have :

$$g_1(x) = 0, \forall x \in T_1 \ ; \text{ and } g_2(x) = 0, \forall x \in T_2$$

The formulae giving $X_{t_o}(f \odot g)$ and $X_{t_o}(f \bigcirc g)$ are instructive.

We find :

$$X_{t_o}(f \odot g) = \bigcap_{t \in t_o} [X_t(f) \odot T] \quad \text{and}$$

$$X_{t_o}(f \bigcirc g) = \bigcup_{t \geq t_o} (X_t(f) \bigcirc T), \text{ with } T = (T_1,T_2)$$

In that particular case, the relations on sections (or thresholds) show that we must take into account the intersection of the thickenings of every section $X_t(f)$ below $t_o$, to obtain the corresponding section of the thickened function (for the thinning, we perform the union). This is due to the fact that the sequences $\{X_t(f) \bigcirc T\}$ and $\{X_t(f) \odot T\}$ of sets are not monotone decreasing (see above). Taking the intersection (or the union) forces the property (figure 9).

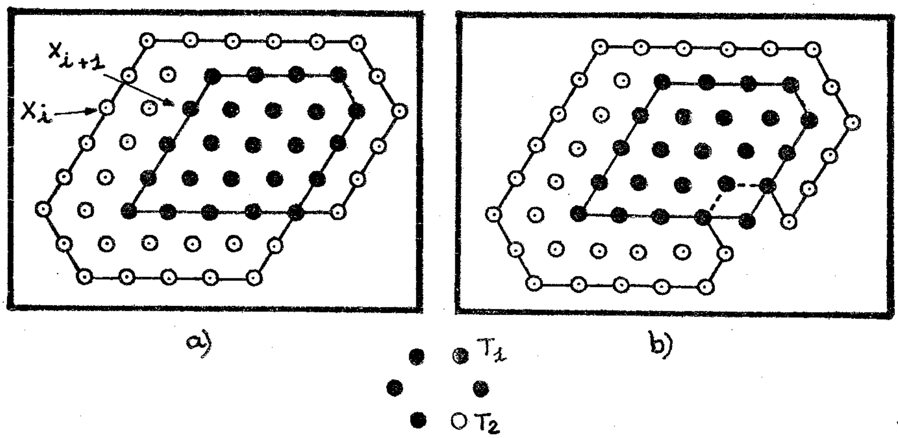



Figure 9 : The sequence $(X_t(f) \bigcirc T)$ does not fullfil the inclusion rule counter-example : (a) initial
(b) thinning

Sequential extensive transformations (digital case)

This section is restricted to digital pictures. We shall restrict our study to the functions which are indicator functions of two-dimensionnal sets $T_1$ and $T_2$. Let us come back to the general case. In $\mathbb{R}^n$, we know that if $T_1$ and $T_2$ are not disjoint, then $f \bigcirc T = f \odot T = f$, $\forall f$. The result is obvious : If the two structuring elements are not disjoint, it is difficult to find points of $\mathbb{R}^3$ such that $T_1 \subset U(f)$ and $T_2 \subset U^c(f)$ ! But there exists a more sophisticated condition based on the relative position of $T_1$ and $T_2$ in $\mathbb{R}^3$, which does not appear in $\mathbb{R}^2$. That is :

$$\text{if} \quad \hat{U}(\hat{T}_2) \cap U(T_1) \neq \emptyset \quad \text{then } f * T = \emptyset, \ \forall f$$

Or, in terms of functions :

$$\text{if} \quad U(g_1) \cap \hat{U}(g_2) \neq \emptyset \quad \text{then } f * g = \emptyset \ \text{(figure 10)}$$

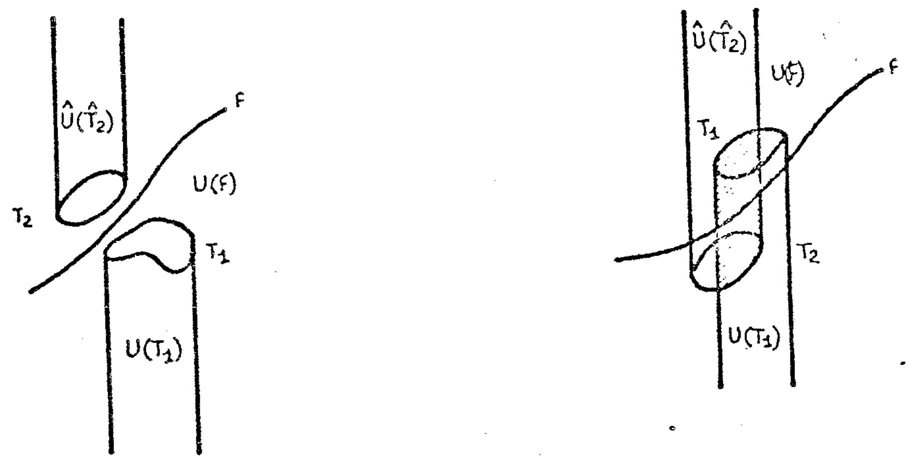



Figure 10

Figure 10 illustrates the condition. To extract points with the configuration in b, U(f) should present some over-hangs which is impossible since f is a function. This condition is not a must, it simply avoids the choice of ineffective doublets.

Given the structuring element $T = (T_1, T_2) \in \mathbb{R}^2$, each section $X_i(f)$ is thinned and the ith section of $f \bigcirc T$ $(f \in \mathcal{P}_i)$ is defined by :

(34) $$X_i(f \bigcirc T) = \bigcup_{j=i}^{m} [X_j(f) \bigcirc T] \text{ , (m, maximum)}$$

and similarly, for the thickening :

(35) $$X_i(f \odot T) = \bigcap_{j=0}^{i} [X_j(f) \odot T]$$

The efficiency of the thinning and thickening operations lies in the fact that we can iterate them. Given a sequence of structuring elements $\{T^i\} = \{T^1, T^2, \dots, T^n\}$, we can define :

$$f \bigcirc \{T^i\} = (((\dots(f \bigcirc T^1) \bigcirc T^2) \quad \dots\dots) \bigcirc T^n)$$

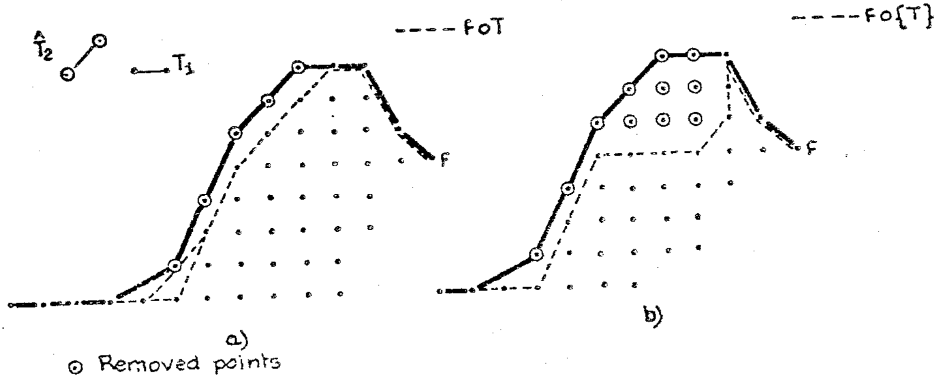



Figure 11 : example of 3-D thinning (a) one step
(b) infinity of steps

For instance, if the digitalizisation grid is hexagonal, a sequence can be generated by taking the various rotations of one structuring element :

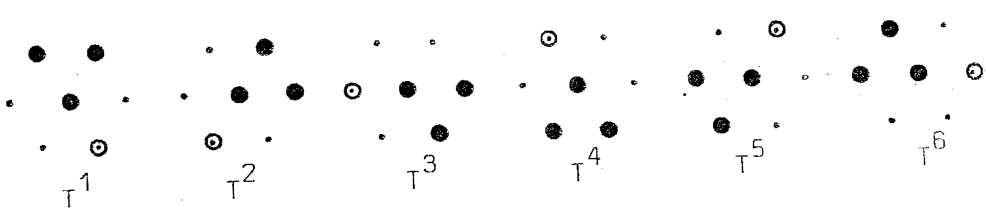
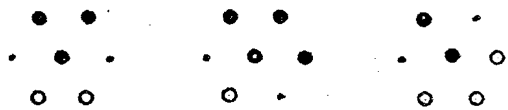



Figure 12 : example of sequence

Homotopic thinnings

We shall say that a morphological transformation $\Psi$ preserves the homotopy of functionf when $X_t(f)$ and $X_t(\Psi(f))$ have the same homotopy, $\forall$ t. (by so doing, we restrict a more general definition, see J. Serra, 1981). It is well known that for the hexagonal grid, the basic 2-D structuring elements which preserve the homotopy are the following :



L M D

As an example, consider the action of $\{L\}$ on function f represented in fig.(13,a), and in particular the two level $X_i$ and $X_j$ $(i < j)$.

Since the thinnings $(X_i \bigcirc T)$ and $(X_j \bigcirc T)$ preserve the homotopy for sets, the result of the transformation produces one connected component for $X_i \bigcirc T$, and two components in the same relative position for $X_j \bigcirc T$. If we denote by c(x) and d(x) the supports of the channels and the divides respectively, the supports c'(x) and d'(x) of the thinned function $f \bigcirc T$ are jointly homotopic to both supports c(x) and d(x).

A crater with a hill in it in the relief f corresponds to a crater with a hill in it in $f \bigcirc \{L\}$, and so on... regardless of the exact location of the hill in the crater or of their shapes.

Examples

The invariance of the homotopy characterization makes thinnings and thickenings suitable tools for constructing "skeletons" of pictures. For instance the transformation $f \bigcirc \{L\}$ leads to the skeleton of f (figure 14).

Thinnings and thickenings provide also efficient algorithms in watershed detection problems (S. Beucher, Ch. Lantuéjoul, 1979). Figure 15 illustrates the use of these transformations in a contour detection problem : The homotopic thinning of the gradient function emphasizes the edges of the original picture.

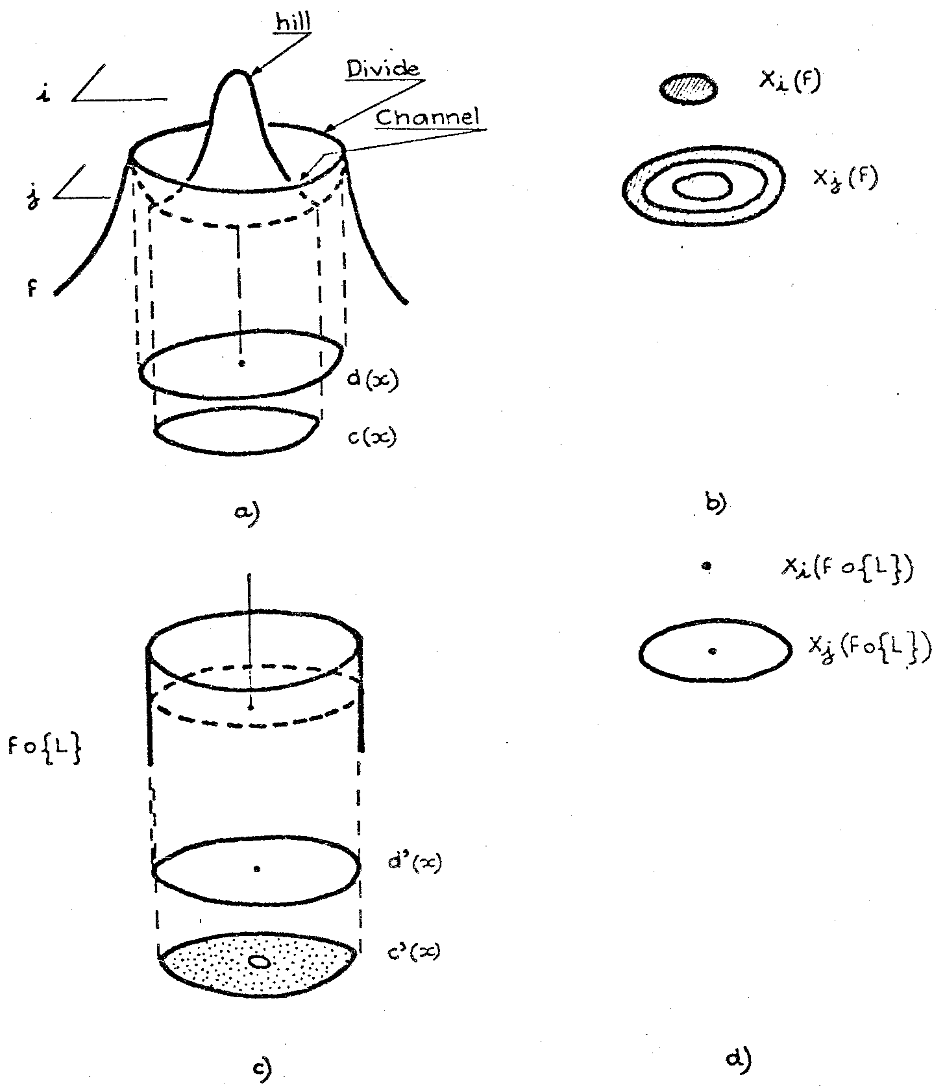



Figure 13 : homotopy for functions
a) function
b) two thresholds of the function
c) thinned function
d) thresholds after thinning

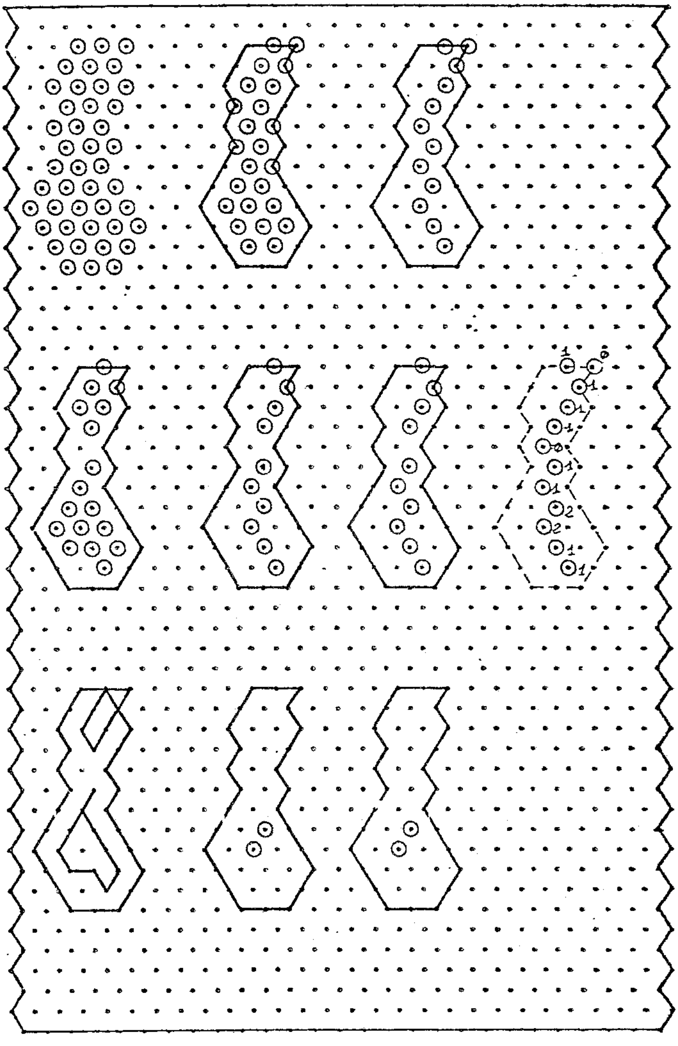
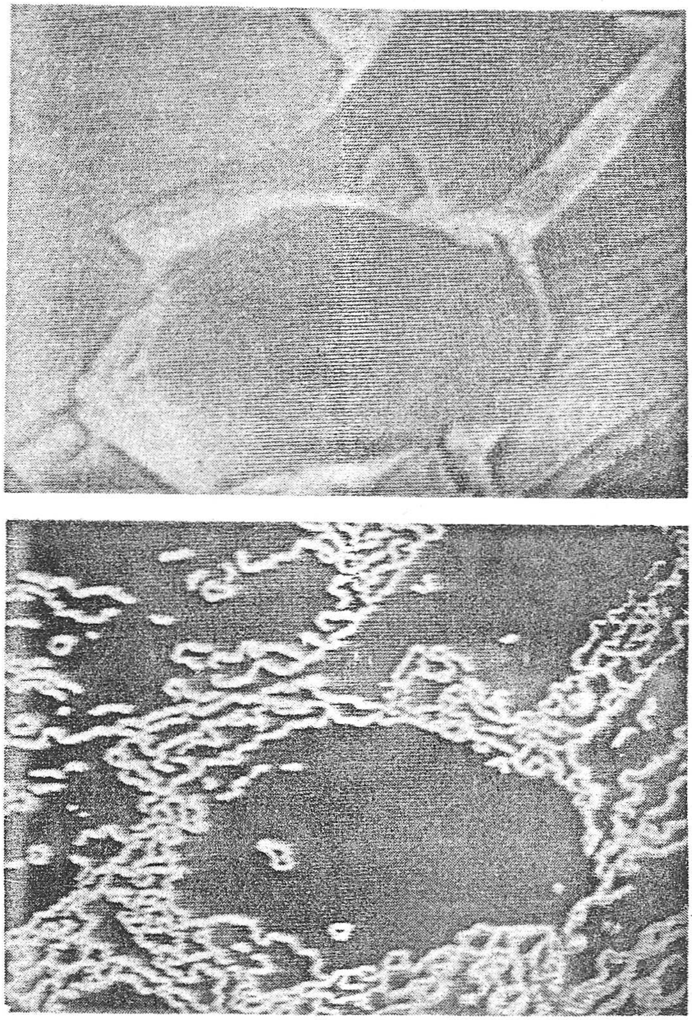


Figure 14 : a) picture f with three levels
b) f ○ L
c) levels of the skeleton
d) skeleton function f ○ {L}



a)

b)

Figure 15 : a) original picture
b) edge detection by thinnings $( |\overrightarrow{grad}\, f| \bigcirc \{L\} )$

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