FROM NON CONNECTED TO HOMOTOPIC SKELETONS IN MULTIDIMENSIONAL DIGITAL SPACES

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Abstract

The aim of this paper is to bridge the gap between two classes of skeleton algorithms widely used in Mathematical Morphology: the skeleton by maximal balls on the one hand and the skeleton by thinnings on the other hand.

First, we show that the first type of skeleton, usually resulting from morphological openings according to Lantuéjoul's formula, can also be obtained by non homotopic thinnings. Moreover, these thinnings are not sequential but based on the intersection of elementary thinnings.

In the second step, a restrictive selection among the previous structuring elements of those which preserve homotopy leads to the definition of a connected skeleton containing the skeleton by maximal balls. This algorithm is given in the 2D space both for the hexagonal and square grids. This skeleton combines the advantages of the two classes of skeletons and avoids the main drawbacks involved by the use of rotating thinnings for building connected skeletons.

The extension of this definition for 3D sets is discussed in the third part.

Introduction

There exist currently two kinds of morphological tools used for building skeletons: the residues of openings on the one hand, and the thinning transformations on the other hand. In fact, the former are directly derived from the definition of the skeleton of a set in terms of maximal balls. Unfortunately, the transforms they provide do not exactly correspond to the intuitive notion of skeleton, mainly because they are not homotopic. The result of such a skeleton transform is generally not connected. Although it is not proved that the skeleton of a connected set must be connected, the fulfillment of this condition leads many people to try to find new algorithms for a skeletonization able to produce connected skeletons. Among them, the morphological transforms called thinnings are widely used, due to the fact that they are the only morphological operators which may be homotopic. However, these skeletons are not very handy: they are not unique because of the multiple ways for defining the thinning sequences to build them. Moreover, their relationship to the non connected skeleton is not obvious [3].

The aim of this paper is to briefly describe a methodology for bridging the gap between these two kinds of skeletons. We shall see that this approach leads to the definition of a connected skeleton containing the maximal balls skeleton. We shall also see that this technique is not linked to the dimension of the working space. So, it is possible to define, by this means, 3D skeletons, geodesic skeletons or skeletons for functions.

1 Maximal balls skeleton and skeletons by thinnings

All the notions described here are given for the digital case. Moreover, the whole set of results have not been thoroughly demonstrated. The reader will find complete proofs in [1,2].

1.1 Definitions

1.1.1 Maximal balls skeleton

Let X be a digital set. Let B(x,i) a ball of radius i centered at point x and included in X. This ball is a maximal ball if and only if there exists no other ball $B(y,j) \subset X$ such that $B(x,i) \subset B(y,j)$.

The skeleton S(X) of X is the set of all the centers of the maximal balls included in X.

The residue R(X) of X is the set made of those points of X which do not belong to its opening by an elementary ball:

$$R(X) = X / (X)_B$$

where $(X)_B$ is the opening of X by the elementary ball B.

According to Lantuejoul's formula, one can show that the skeleton is also the set of all the residues of the successive erosions of X:

$$S(X) = \bigcup_i (X \ominus iB)/(X \ominus iB)_B$$

$$S(X) = \bigcup_i R(X \ominus iB)$$

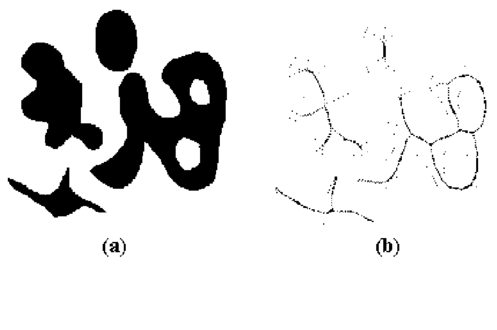

Figure 1: skeleton (b) of a set (a) defined with maximal balls

Although this skeleton has interesting properties (such as the capability of entirely reconstruct the initial set from its set of maximal balls), it does not preserve the connectivity of the set X (Figure 1).

1.1.2 Skeletons by thinnings

Let $T = (T_1, T_2)$ be a two-phase structuring element (T_1 and T_2 are two structuring elements with a common origin). A thinning of X by T is defined by:

$$X \bullet T = X / (X * T)$$

where:

$$X * T = (X \ominus T_1) \cap (X \ominus T_2)$$

is the Hit-or-Miss transform.

Let $\mathsf{T} = \{T^1, T^2, \ldots, T^n\}$ be a family of structuring elements. The thinning of a set X by the family T is defined by:

$$X \bullet \mathsf{T} = X / (X * \mathsf{T})$$

where $\quad X * \mathsf{T} = \bigcup_i (X * T^i)$.

This definition produces a thinning which does not depend on the order of use of the various T^i.

It is not however this kind of thinning which is ordinarily used for building connected skeletons, but sequential thinnings [4].

By applying successive thinnings with homotopic elements to a set X:

$$(((X \bullet T_a) \bullet T_b) \bullet \ldots)$$

where T_a, T_b, are structuring elements preserving homotopy, one can produce homotopic transforms called "connected skeletons".

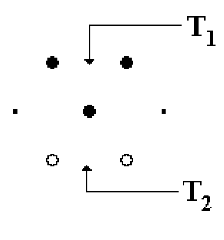

Figure 2: The L structuring element

The choice of the sequences used leads to different kind of "skeletons". The most classical on the hexagonal grid is the L-skeleton obtained with the sequence (L^1, L^2, ..., L^6) where the L^i are the successive rotations of the L structuring element (Figure 2).

Although these skeletons are indeed connected (Figure 3), they are not unique. The result of the transformation strongly depends upon the order of the structuring elements in the sequence. The quality of the result also leaves much to be desired.

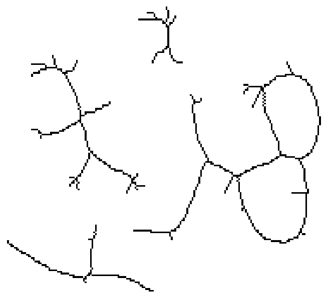

Figure 3: L-skeleton of the previous set

2 From non connected to connected skeletons

2.1 Methological approach

To bridge the gap between these two types of skeletons, we follow the approach below:

- We try to express the maximal balls skeleton in terms of thinnings. These thinnings will not use a sequence of structuring elements but a simple family (without order) of them in order to avoid the non-uniqueness of the result.
- Then, we sort these structuring elements to eliminate those which may break the connectivity.

2.2 Maximal balls skeleton by thinnings

Starting from the formula:

$$S(X) = \bigcup_i [(X \ominus iB)/(X \ominus iB)_B]$$

let us define:

$$S_n(X) = \bigcup_{i=0}^{n} [(X \ominus iB) / (X \ominus iB)_B]$$

One can prove that the following iterative transform:

$$Z_n = (Z_{n-1} \ominus B) \cup R(Z_{n-1})$$

with $Z_0 = X$, is equal to:

$$Z_n = (X \ominus nB) \cup S_{n-1}(X)$$

So:

$$S(X) = \lim_{n\to\infty} Z_n$$

We can also write:

$$Z_n = Z_{n-1} \cap [(Z_{n-1} \ominus B)^c \cap (Z_{n-1})_B]^c$$

but $(Z_{n-1} \ominus B)^c \cap (Z_{n-1})_B$ is equal to:

$$\bigcup_{a\in B, b\in B} [(Z_{n-1} \ominus B_a) \cap (Z^c_{n-1} \ominus L_b)]$$

where B_a is the ball B translated in direction a and L_b the structuring element corresponding to the translation of a point in direction b (a and b are all the possible direction vectors centered at the origin which can be defined in the elementary ball B).

Finally, we have:

$$Z_n = Z_{n-1} / \bigcup_{a,b} (Z_{n-1} * T_{a,b})$$

$$Z_n = Z_{n-1} \bullet \mathsf{T}$$

with $\mathsf{T} = \{T_{a,b} = (B_a, L_b) \forall a, b \in B\}$.

$$Z_n = Z_{n-1} \bullet \mathsf{T} = (Z_{n-2} \bullet \mathsf{T}) \bullet \mathsf{T}$$
$$= (X \bullet \mathsf{T}) \bullet \ldots) \bullet \mathsf{T} \text{ (n thinnings).}$$

and the maximal balls skeleton can be written:

$$S(X) = \lim_{n\to\infty} Z_n = (X \bullet \mathsf{T}) \bullet \ldots \bullet \mathsf{T} \bullet \ldots$$

that is, the successive iterations of union thinnings. Q.E.D.

No assumption has been made in the previous proof concerning the dimension of the working space. So, this formula holds for the 2D skeleton as well as for the 3D one.

Let us emphasize the T family for the 2D skeleton on the hexagonal grid. T is made of the following structuring elements (Figure 4) with all their possible rotations (notice the position of the origin of these structuring elements).

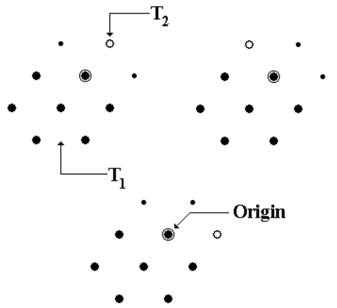

Figure 4: The T family of thinnings used for skeletonizing

2.3 From the non-connected to the connected skeleton

To build a connected skeleton from the previous thinnings, two methods are available. The first one consists in analyzing the various structuring elements of the T family in order to keep only those which preserve homotopy. However, this procedure may be tedious especially when dealing with the 3D space. The second method is based on the analysis of what is to be done to connect the maximal balls skeleton. Two kinds of connections must be made:

- the connection of the residues obtained at step i with the eroded set of size i+1.

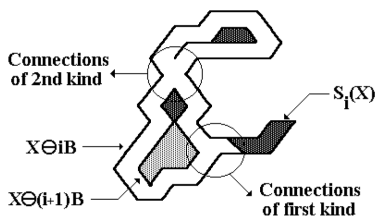

Figure 5: The two kinds of connections in the connected skeleton

- the connection of the connected components of the eroded set of size i+1, when the disconnection happens between the erosions of sizes i and i+1. These residues correspond in fact to the centers of elementary balls which touch each other (Figure 5).

For preserving this second kind of connection, we simply have to check all the possible configurations for two elementary balls to touch each other and suppress those which may break the connections. Figure 6 displays all these configurations in the case of the hexagonal grid.

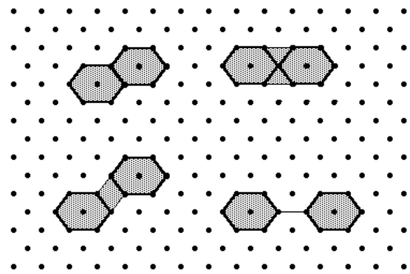

Figure 6: The second kind of connections on the hexagonal 2D grid

So, the structuring elements which preserve this second kind of connections are the following (Figure 7):

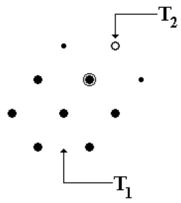

Figure 7: Structuring element preserving the second kind of connections

The first kind of connections can be obtained by adding to the residues at step i the points which connect these residues and the eroded set after step i. Doing so, we define a new iterative transform:

$$Z_{i+1} = (Z_i \ominus B) \cup [(R(Z_i) \oplus B) \cap Z_i]$$

with $Z_0 = X$.

This can be written:

$$Z_{i+1} = Z_i \cap A^c$$

with:

$$A = (Z_i)_B \cap (Z_i \ominus B)^c \cap (R^c(Z_i) \ominus B)$$

The interpretation of the above equation is the following: Z_{i+1} can be obtained by thinnings using the T family of structuring elements (two first terms of A) provided that there is no residue $R(Z_i)$ adjacent to the origin of the structuring element (third term of A).

3 Application to 2D and 3D skeletons

3.1 2D skeleton

Let us compute the structuring elements which preserve the connectivity in the case of an hexagonal grid when B is an elementary hexagon. For preserving the connections of the first kind, these structuring elements must be (Figure 8):

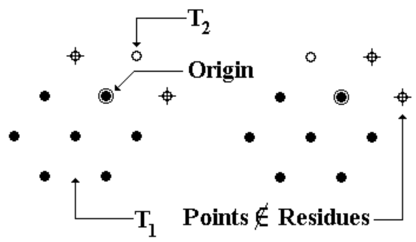

Figure 8: First group of structuring elements

and all their rotations and symmetries, where the coded numbers mean that:

- (1) the point must belong to the set to be thinned.
- ($\bar{2}$) the point must not be a residue point.
- (0) the point must belong to the complementary set.

But, for preserving the second kind of connections, the second configuration must be suppressed. In fact, this configuration is irrelevant for preserving the first kind connections because we can easily show that when it happens, the first configuration also exists. Finally, the structuring elements used for producing connected skeletons by thinnings are given at Figure 9a (and all their rotations).

When we work on a 8-connectivity square grid, the same analysis leads to the structuring elements given at Figure 9b.

Figure 10 gives an example of such a connected skeleton. This skeleton contains the maximal balls skeleton.

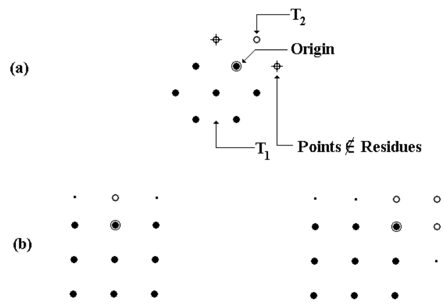

Figure 9: Structuring element for the 2D connected skeleton (hexagonal grid (a) and square grid (b))

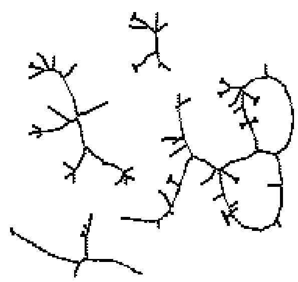

Figure 10: Connected skeleton by union thinnings

3.2 3D skeleton

Let us define the corresponding structuring elements in the case of a digital 3D space. The elementary ball is a cuboctahedron and the digital grid is cubic. The second kind of connections correspond to the following touching cuboctahedrons (Figure 11):

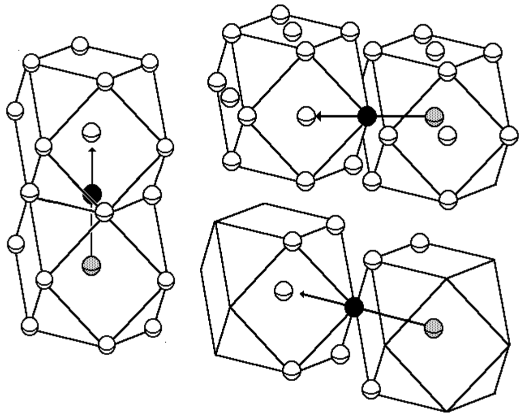

Figure 11: Second kind of connections in the 3D cubic space

To preserve these connexions, the structuring element must be (Figure 12):

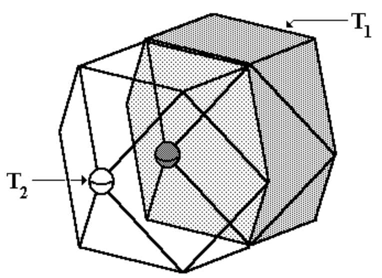

Figure 12: Structuring element preserving the second kind of connections in the 3D skeleton

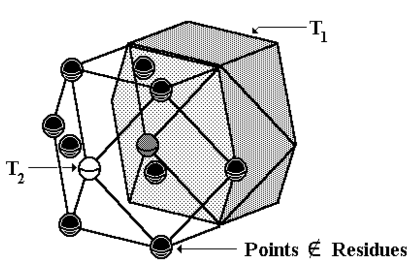

Figure 13: Structuring element for the 3D connected skeleton

The first kind of connections are preserved by checking that no residue point is adjacent to the origin. Finally, it can be shown (as for the 2D hexagonal case) that the configuration given at Figure 13 and all its rotations produce by union thinnings a 3D connected skeleton.

Conclusion

We have provided a general methodology for finding the configurations of structuring elements producing connected skeletons by thinnings.These skeletons are connected and moreover, they contain the maximal balls skeleton. Many extensions of these transformations are possible: geodesic skeletons, skeletons for functions and so on. They also enable the building of watersheds without errors [1]. Finally, further reductions in the family T of structuring elements lead to the definition of new types of homotopic transformations called "smoothed skeletons" [2].

Bibliography

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