SETS, PARTITIONS AND FUNCTIONS INTERPOLATIONS

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Abstract. This paper presents the use of the SKIZ transformation to interpolate sets, partitions and functions. This transformation generates a sequence of intermediary sets which can be used to define a distance between the interpolated sets. Other distances are also introduced. These distances could be useful to generalise the interpolation algorithm. Some examples of use are given.

1. Introduction

Set interpolation and partition interpolation are useful techniques in many domains of image analysis. It is the case for instance in morphing processes, where the design of the deformation grid could be automatised and even replaced by an automatic deformation of homologous regions in the initial and final images. These regions could be built using morphological segmentation tools as watersheds and marker-controlled watersheds. The watershed transform generating a partition of an image (the mosaic image), the problem consists therefore in finding tools to distort the partition corresponding to the initial image and to apply it to the final one. Another efficient use of image and partition interpolation is in image compression. Many morphological compression techniques use mosaic images and interpolation can dramatically reduce the bandwidth when transmitting image sequences. A third domain where interpolation can be applied is for quantifying the distortion or the displacement of objects or regions from a picture to another one. It is the case in particular in stereovision or in motion estimation.

This paper aims at introducing a simple interpolation technique based on SKIZ for sets, partitions and functions. Some extensions will be discussed, in particular the introduction of various distances which can be defined to control these interpolations or to combine displacements and deformations of sets. Some examples of use will also be given.

2. Set Interpolation

All the interpolations described in this paper are based on a sole morphological transform, the skeleton by influence zones or SKIZ [3]. Other approaches using more or less sophisticated geodesic transforms have been studied [4,5]. However, the SKIZ is easy to use, more general (it applies on sets and on functions) and the results it gives are better. We shall not prove in this paper that this transformation induces a geodesic distance and the theoretical aspects will be put aside in favour of a more intuitive and algorithmic approach.

2.1. The basic transformation: the SKIZ

Let, in a space E, X and Y be two sets $\in (E)$ such that $X \cap Y \neq \emptyset$ and $X \cup Y \neq E$. Consider the set W made of the two following components (Fig. 1.):

$$W = \{X \cap Y, (X \cup Y)^c\}$$

The SKIZ of W produces the influence zones (IZ_W) of the two components of W (note that these two components are not necessarily connected). In particular:

$$IZ_W (X \cap Y) = \{x \in E : d(x, X \cap Y) < d(x, (X \cup Y)^c)\}$$

where d is the Euclidean pseudo-distance of a point to a set.

Let us denote $M(X, Y)$ the influence zone of $X \cap Y$. This set is the median set of X and Y. This set is, in the geodesics generated by the SKIZ, at the same distance from X and from Y:

$$M(X, Y) = IZ(X \cap Y)$$

This transformation is auto-dual. We have, in fact:

$$M(X, Y)^c = M(X^c, Y^c)$$

This property provides a good robustness for the set complementation, which is not always fulfilled for any geodesics. It is not true, in particular, for the Haussdorf distance [5].

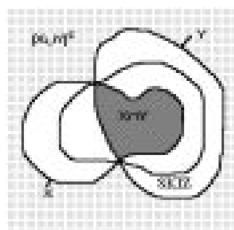

Fig. 1. SKIZ of $X \cap Y$ and $(X \cup Y)^c$

In return, this operator is not associative. If we take three sets X, Y and Z, the median set of these three sets cannot be built by generating $M(X, Y)$ followed by $M[Z, M(X, Y)]$. The result (assuming that it is non simple) will depend on the order of the operations.

The median of n sets X_i, with $\bigcap_i X_i \neq \emptyset$ and $\bigcup_i X_i \neq E$ (all the sets are sharing at least one point and do not cover the entire space) can be written:

$$M(X_i) = IZ_W\left(\bigcap_i X_i\right) \quad \text{with} \quad W = \left\{\bigcap_i X_i, \left[\bigcup_i X_i\right]^c\right\}$$

2.2. Sequences of deformations

By iterating this transformation from the sets X and Y, a sequence of $n+1$ sets representing the n deformation steps from X to Y can be built. We have:

$$K_0 = X; K_n = Y$$

Then: $K_{n/2} = M(K_0, K_n)$; $K_{n/4} = M\left(K_0, K_{n/2}\right)$; $K_{3n/4} = M\left(K_{n/2}, K_n\right)$
and after the $(i+1)$th step:

$$K_{n/2^i} = M\left(K_0, K_{n/2^{i-1}}\right); \ldots$$

The number of intermediary sets increasing by 2^{i-1} at each step i, n must be a power of 2 in order that this number corresponds to an integer number of iterations.

We get then a sequence of 2^{i+1} sets, two consecutive ones being at the distance $1/2^i$ if we assume that the distance between the two extreme sets is 1. When n is not a power of 2, intermediary sets can still be built with the same process and be labelled with $n-1$ integer values from 0 to n. However, the distance between two consecutive sets will not be rigorously equal to $1/n$.

The table 1 shows such a deformation sequence. Note that the initial and final sets do not belong to the same class of connectivity.

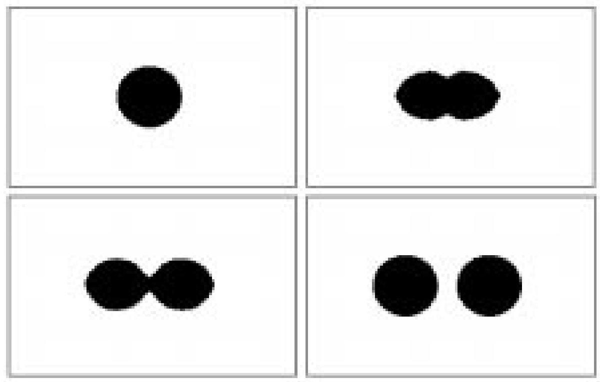

Table 1. Interpolation sequence between sets X (upper left) and Y (lower right).

2.3. The algorithm

Computing $M(X,Y)$ starting from X and Y is simple. Simple dilations are used, instead of thickenings as it is the case in the classical SKIZ. As a matter of fact, the union of $X\cap Y$ and of $(X\cup Y)^c$ can be connected, which prevents the use of thickenings. Moreover, the use of dilations produces a better and unbiased SKIZ because all the connected components grow at the same time. We have:

$$Z_0 = X\cap Y \quad \text{and} \quad W_0 = (X\cup Y)^c$$

We just have to iterate until idempotence the following operations (B is the elementary ball):

$$Z_i = [(Z_{i-1}\oplus B)/(W_{i-1}\oplus B)]\cup Z_{i-1}\ ;\ W_i = [(W_{i-1}\oplus B)/(Z_{i-1}\oplus B)]\cup W_{i-1}$$

Then, we have: $$M(X,Y) = Z_\infty$$

The union at each step with the previous result insures that, despite some parity problems on the digital grid, the operation is extensive. For the same reasons, we generally do not have the equality $Z_\infty = (W_\infty)^c$.

This method does not allow to obtain directly the set at a distance d_1 from X and d_2 from Y with $d_1+d_2=1$. However, this could be possible by modifying the growth speeds (that is the sizes of dilations) of the sets Z_i et W_i. If v_1 is the size of the elementary dilation of Z_i and v_2 the size of the dilation of W_i, and if k is a normation factor, we have:

$$d_1 = k\ v_1 \quad \text{and} \quad d_2 = k\ v_2 \quad \text{with} \quad d_1+d_2=1$$

which gives:

$$d_1 = \tfrac{v_1}{v_1+v_2};\ d_2 = \tfrac{v_2}{v_1+v_2} \quad \text{and} \quad v_1/v_2 = d_1/d_2$$

However, this approach has two drawbacks. Firstly, the induced metrics is different from the one generated by means of the dichotomic technique. Secondly, with dilations of respective sizes v_1 and v_2, the SKIZ thickness may be larger than 1. Therefore, a more sophisticated sequence of dilations must be used to fit with the real ratio v_1/v_2.

3. Interpolation of partitions

3.1. General case

Consider two partitions T and T':

$$T = \{C_i\}, T' = \{C'_i\}$$

We suppose that there exists a one-to-one correspondence between the cells C_i of the first partition and the cells C'_i of the second one. More precisely, we have (Fig. 2):

$$\forall\ C_i \in T,\ \exists\ C'_i \in T'\ :\ C_i \cap C'_i \neq \emptyset$$

Let $W = \{C_i \cap C'_i\}$ be the set made of all the components $C_i \cap C'_i$.

A median partition $M(T, T')$ can then be defined as the influence zones of the sets $(C_i \cap C'_i)$:

$$M(T, T') = \{IZ_W (C_i \cap C'_i)\}$$

The definition of a median partition is, to a certain extent, simpler than the definition of the median of two sets.

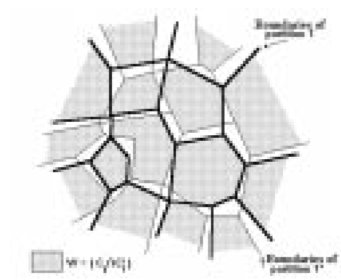

Fig. 2. Partitions with a one-to-one correspondence between cells

3.2. Algorithm

Two kinds of algorithms can be designed to build $M(T, T')$. the first one uses a simple SKIZ and produces a partition with one pixel boundaries. The second (and better) one produces a labelled partition without boundaries between the cells. The cell $C_i \in T$ and its corresponding cell $C'_i \in T'$ share the same label. The partitions T and T' can then be considered as multilevel staircase functions f and f':

$$f : E \rightarrow N_+ \quad x \rightarrow f(x) = i, x \in C_i \quad ; \quad f' : E \rightarrow N_+ \quad x \rightarrow f'(x) = i, x \in C'_i$$

$M(T, T')$ is built by means of simple dilations. A function g can be defined by:

$$g\ : E \rightarrow N_+ \quad x \rightarrow g(x) = i, x \in IZ_W (C_i \cap C'_i)$$

A double labelling of the partitions T and T' is used (they are supposed to contain n cells). The algorithm is as follows:

- g_0 is computed:

$$g_0 (x) = f (x) = f' (x) \text{ iff } f (x) = f' (x) \Longleftrightarrow x \in C_i \cap C'_i, \forall i\ ;\ g_0 (x) = 0 \text{ if not}$$

– then g'_0 is defined:

$$g'_0(x) = n - g_0(x) \quad \text{iff } g'_0(x) \neq 0 \; ; \; g'_0(x) = 0 \quad \text{if not}$$

This function is similar to g_0. The only difference is that the labels i of the sets $C_i \cap C'_i$ have been changed to $(n - i)$.

Following this initialisation phase, for each g_i and g'_i, we do:

– calculation of the function h_i:

$$h_i = (g_{ih} \oplus B) + (g'_i \oplus B) \quad (B\text{,elementary ball of } E)$$

– Then, the indicator function h of the points where h_i is less or equal to n is defined:

$$h = 1 \quad \text{iff} \quad h_i \leq n, \; h = 0 \quad \text{if not}$$

– The functions g_{i+1} and g'_{i+1} are calculated:

$$g_{i+1} = \sup\left[(g_i \oplus B) \times h, g_i\right]; g'_{i+1} = \sup\left[(g'_i \oplus B) \times h, g'_i\right]$$

The algorithm is iterated until idempotence. Finally we get $g = g_\infty$.

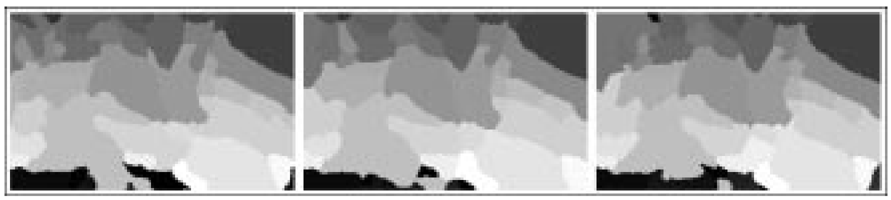

Table 2. Interpolation of partitions (stereo pair of cleavage facets in steel)

4. Interpolation of functions

4.1. General case

The notion of median set can be generalised to functions by means of the sets over and under the graph of these functions. Let f and g be two functions defined in the space E. Consider first the set under the graph D of $\inf(f, g)$:

$$D\left[\inf(f, g)\right] = D(f) \cap D(g)$$

and then the set over the graph U of $\sup(f, g)$:

$$U\left[\sup(f, g)\right] = \left[D(f) \cup D(g)\right]^c$$

Consider the set W made of the two following components:

$$W = \left\{D\left[\inf(f, g)\right], U\left[\sup(f, g)\right]\right\}$$

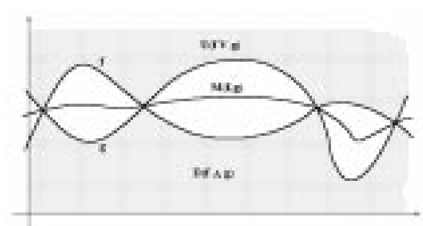

Fig. 3. Median of two functions by SKIZ of $D[Inf(f, g)]$ and $U[\sup(f, g)]$

The median function of f and g is the function $M(f, g)$, its set under the graph being the influence zone of $D[\inf(f, g)]$ in the 3D SKIZ of W (Fig. 3):

$$D\left[M(f, g)\right] = IZ_W\left[D\left[\inf(f, g)\right]\right]$$

4.2. Algorithm

As for sets, the algorithm uses 3D dilations of U and D. We have:

$$Z_0 = D\left[\inf(f,g)\right]; W_0 = U\left[\sup(f,g)\right]$$

We can write, with $z_0 = \inf(f,g)$ and $w_0 = \sup(f,g)$:

$$Z_1 = D\left[\sup\left(\inf\left(z_0 \oplus B, w_0 \ominus B\right), z_0\right)\right] \;\; ; \;\; W_1^c = D\left[\inf\left(\sup\left(w_0 \ominus B, z_0 \oplus B\right), w_0\right)\right]$$

Z_1 is the set under the graph of $z_1 = \sup\left(\inf\left(z_0 \oplus B, w_0 \ominus B\right), z_0\right)$ and W_1 the set over the graph of $w_1 = \inf\left(\sup\left(w_0 \ominus B, z_0 \oplus B\right), w_0\right)$.

This is true also at step i:

$$z_i = \sup\left(\inf\left(z_{i-1} \oplus B, w_{i-1} \ominus B\right), z_{i-1}\right); \;\; w_i = \inf\left(\sup(w_{i-1} \ominus B, z_{i-1} \oplus B), w_{i-1}\right)$$

Finally, we obtain: $\qquad M\left(f,g\right) = z_\infty = w_\infty$

The median of two functions is a function.

5. Distance functions

Let us come back to the interpolation of sets. Starting from two sets X and Y with $X \cap Y \neq \emptyset$ and $X \cup Y \neq E$, the intermediary sets $M_\lambda(X,Y)$ depending of the parameter $\lambda(0 \leq \lambda \leq 1)$ can be defined, with $M_0(X,Y) = X$ and $M_1(X,Y) = Y$. This transformation is not monotonous with respect to λ. However $M_\lambda(X,Y)$ can be split into two components, an extensive one and an anti-extensive one with respect to λ. Consider indeed the sets $X \cap Y$ and X^c. The SKIZ of these two disjoint sets can be performed. Let $M^-(X,Y)$ be the influence zone of $X \cap Y$. We see immediately that the obtained SKIZ corresponds to the part of the SKIZ associated to $M(X,Y)$ situated in X/Y. This operation can also be iterated to define a sequence of sets $M_\lambda^-(X,Y)$ with $M_0^-(X,Y) = X$ and $M_1^-(X,Y) = X \cap Y$. The sequence $M_\lambda^-(X,Y)$ is monotonously decreasing with respect to λ. A function d^- defined on X can be associated to this sequence, each M_λ^- corresponding to the threshold at λ of d^-:

$$M_\lambda^- = \{x \in X : d^-(x) \geq \lambda\}$$

In the same way, let us consider now the sets $X \cap Y$ and Y^c. The SKIZ of these two disjoint sets produces the set $M^+(X,Y)$ corresponding to the influence zone of $X \cap Y$. A monotonously increasing sequence of sets $M_\lambda^+(X,Y)$ can then be defined and a function d^+ defined on Y. We have:

$$M_\lambda^+ = \{x \in Y : d^+(x) \leq \lambda\}$$

The sets M_λ^- and M_λ^+ split the deformation of the sets X and Y into an anti-extensive component (an erosion) and an extensive one (a dilation).

The function $d = (d^+ + d^-)/2$ is defined on $X \cup Y$ and is constant on $X \cap Y$ (Fig. 4b).

If the intersection of X and Y is empty, d is not defined. But another distance can be defined on $X \cap Y$ and on the complementary set of their union by means of another intermediary set between X and Y^c. Let us consider two sets X and Y. We can see that, whatever X and Y, the sets $X \cap Y$, X/Y, Y/X and $(X \cup Y)^c$ constitute a partition of the space E. The definition of $M(X,Y)$ is based on the SKIZ of $X \cap Y$ and $(X \cup Y)^c$. But two other sets in the partition could be chosen. Among the possible choices, only the sets X/Y and Y/X are of some interest. Let us define the set $M^*(X,Y)$ as the influence zone of X/Y in the SKIZ of X/Y and Y/X:

$$W = \{X/Y, Y/X\} \; ; \; M^*(X,Y) = IZ_W(X/Y)$$

We see immediately that $M^*(X,Y)$ is equal to $M(X,Y^c)$.

The set $M^*(X,Y)$ is therefore the median set of X and Y^c. By iterating the transformation, the sequence M^*_λ of deformations from X to Y^c can be generated. This sequence is not monotonous. However, as for the sequence M_λ, it can be split into two parts, an extensive one and an anti-extensive one. With these two sequences, two new distance functions d^{*-} et d^{*+} can be defined. These two functions are merged to give the function d^* (Fig. 4c):

$$M^{*-}_\lambda = \{x \in X : d^{*-}(x) \geq \lambda\} \ ; \ M^{*+}_\lambda = \{x \in Y^c : d^{*+}(x) \leq \lambda\}$$

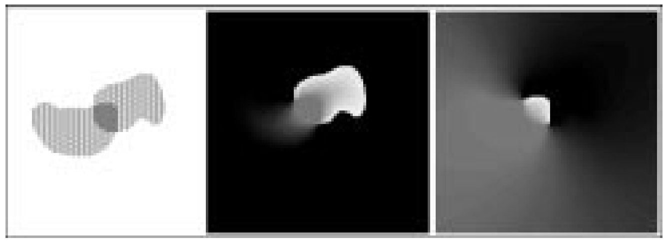

Fig. 4. The sets X and Y, the distances d and d^*

The interest of these functions comes from the fact that they can be used to define deformation fields from a set X to a set Y. The simplest way to achieve this is to derive the functions d and d^*. The gradient vectors represent this deformation field. The use of d and d^* ascertain the existence of at least one such field allowing to go from X to Y or from X to Y^c even when X and Y have no common points. It is also possible to use the field generated by the distance d^* as a displacement field whereas the distance d (when it exists) controls the deformation field.

6. Interpolations in compression of sequences. Use of mosaic images

An interesting application of image interpolation lies in image compression in a moving sequence. Instead of transmitting the entire sequence, one picture over n can be sent, the missing pictures been interpolated. A compression rate of $n:1$ is then easily obtained. Unfortunately, the interpolation of functions deeply smoothes the intermediary images in the sequence. This is why, instead of using the initial sequence, mosaic images have been built by a watershed transformation [1,2]. This approach reduces the image damaging because, on the one hand, a mosaic image does not present too thin regions and, on the other hand, the calculation of the intermediary mosaic image by means of an interpolation of partition preserves the contrast between the adjacent cells. In the result given in table 3, one picture over 4 is sent, the three others are interpolated. We can see that the general motion in the sequence is well preserved. This amazing result comes mainly from the fact that, as shown before, when two cells in the initial and final mosaic images have some points in common, the interpolation produces a displacement of the cell. So, if there exist a sufficient number of associated cells in the two images, a visual displacement effect will dominate.

Table 3. Sequence interpolations. The 2nd, 3rd and 4th images are interpolated

7. Conclusion

Image interpolation by SKIZ is an efficient and easy to implement tool, in particular when it is associated with segmentation transforms as watersheds and mosaic images. When the direct display of a sequence is the unique goal (videophone, telemonitoring, etc.), we obtain, by these means, good compression rates at low cost.

The use of these interpolations in motion analysis is more complex. The major drawback of the original approach is the necessity for the sets or partitions to be interpolated to share common points (non empty intersection). However, when the motion is important, this correspondence may not exist. But the use of hierarchical segmentations, in the one hand, and the combination, in the other hand, of the distances d and d^* introduced above, which split the displacement and deformation parts of the interpolations, give interesting starting ideas for a better solution of these problems.

8. References

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