Connectivity on Complete Lattices

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Abstract

Classically, connectivity is a topological notion for sets, often introduced by means of arcs. A non topological axiomatics has been proposed by Matheron and Serra. The present paper extends it to complete sup-generated lattices. A connection turns out to be characterized by a family of openings labelled by the sup-generators, which partition each element of the lattice into maximal terms, of zero infima. When combined with partition closings, these openings generate strong sequential alternating filters. Starting from a first connection several others may be designed by acting on some dilations or symmetrical operators.When applying this theory to function lattices, one interprets the so-called connected operators in terms of actual connections, as well as the watershed mappings. But the theory encompasses the numerical functions and extends, among others, to multivariate lattices.

Keywords : connectivity, complete lattices, mathematical morphology, connected operators, filters by reconstruction, watershed.

1 The connectivity concepts

1.1 State of the art before 1988

In mathematics, the concept of connectivity is formalized in the framework of the topological spaces and is introduced in two different ways. First, a set is said to be connected when one cannot partition it into two non empty closed (or open) sets [1]. This definition makes precise the intuitive idea that $[0,1] \cup [2,3]$ consists of two pieces, while $[0,1]$ consists of only one. But this first approach, extremely general, does not derive any advantage from the possible regularity of some spaces, such as the Euclidean ones. In such cases, the notion of *arcwise connectivity* turns out to be more convenient. According to it, a set A is connected when, for every $a, b \in A$, there exists a continuous mapping ψ from $[0,1]$ into A such that $\psi(0) = a$ and $\psi(1) = b$. Arcwise connectivity is more restrictive than the general one ; however, in $\mathbb{R}^d$, any open set which is connected in the general sense is also arcwise connected.

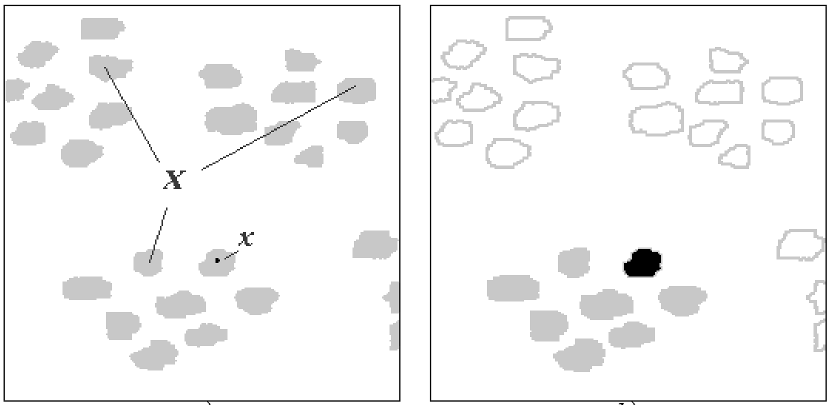
In image analysis, the digital connectivities transpose the arcwise corresponding notion of the Euclidean case, by introducing some elementary arcs between neighboring pixels. This results in the classical 4- and 8-square connectivities, as well as the hexagonal one, or the cuboctahedric one in 3-D spaces. During the seventies, these connectivities have been extensively used to design thinning and thickening operations [2][3][4].

In the same decade arose another development of digital connectivity, which seemed apparently similar. The initial algorithm, due to J-C. Klein [5], concerned the partial reconstruction of a set A from an inside point marker x. The technique consists in iterating the dilation of x by the elementary disc B (i.e. the 6-pixel hexagon, or the 9-pixel square), but restricted to A at each step:

$$\gamma_x(A) = (..(\{x\} \oplus B) \cap A...)_{n\, times} \tag{1}$$

For n large enough, the expansion of x inside A stops (see fig. 1b), and $\gamma_x(A)$ extracts the connected component of A containing x.

In rel. (1), the successive steps (and not only their limit) are instructive by themselves. They indicate the progression of a wave front which emanates from A. Their study was formalized by Lantuejoul and Beucher [6] under the name of "geodesic" methods.



a) *b)*

Figure 1 a) set X and marker x;
b) In black and in grey, the connected component marked by point x according two different connections

1.2 State of the art since 1988

As the basic disc B changes, does rel. (1) cover all arcwise digital connectivities ? The answer is yes, in the sense that each of the latter is associated with a specific unit disc. But rel. (1) encompasses more. Replace, for example, B by $10B$ in rel. (1), then $\gamma_x(A)$ extracts *clusters* of particles for 9 pixels apart (see fig. 1b). To which extent may these clusters be considered "connected units" ?

The question will make sense if there exist image transformations which involve connectivity, and whose good properties remain true when the notion of connectivity is more than "a one piece object". (Here, we may think of the watershed transformation, in particular, which involves minima, i.e. connected components, and, above all, of morphological filtering).

But, first of all, what could be an appropriate definition for such purposes ? When G. Matheron and J. Serra proposed a new one, in 1988, they used to aim at strong filters [7][8]. However, their definition is rather general and stated as follows.

Definition 1 *Let E be an arbitrary space. We call connected class $\mathcal{C}$ a family in $\mathcal{P}(E)$ such that*
(i) $\emptyset \in \mathcal{C}$ and for all $x \in E$, $\{x\} \in \mathcal{C}$
(ii) for each family $\{C_i\}$ in $\mathcal{C}$, $\cap\, C_i \neq \emptyset$ implies $\cup C_i \in \mathcal{C}$.

As we can see, the topological background has been deliberately thrown out. The classical notions (e.g. connectivity based on digital or Euclidean arcs) are certainly particular cases, but the emphasis is put on another aspect, that allows to cover also cases such as fig. 1a (the clusters). And this emphasis is clarified by the following theorem [9, Chap. 2] :

Theorem 1 *The datum of a connected class $\mathcal{C}$ on $\mathcal{P}(E)$ is equivalent to the family $\{\gamma_x\, , x \in E\}$ of openings such that*

(iii) for all $x \in E$, we have $\gamma_x(x) = \{x\}$

(iv) for all $A \subseteq E$, $x, y \in E$, $\gamma_x(A)$ and $\gamma_y(A)$ are equal or disjoint

(v) for all $A \subseteq E$, and all $x \in E$, we have $x \notin A \Rightarrow \gamma_x(A) = \emptyset$.

The theorem provides a theoretical status which is perfectly convenient to the reconstruction algorithm (1). But historically speaking it did much more, and the number of applications or of theoretical developments which was suggested (and permitted) by this theorem is considerable. The references [8] to [27] show that it has opened the way to an object-oriented approach for the compression and comprehension of still and moving images. They also show how fruitful are the exceptional properties of the connected filters (see in particular [8] [10] [11] and [26], to which are associated the names of G. Matheron, J. Serra, Ph. Salembier, J. Crespo, and R.W. Schafer), and their uses in segmentation, as shown in particular by S. Beucher, F. Meyer, B. Marcotegui and C. Vachier [12], [19], [21], [23].

1.3 Purpose of the present study

There is a paradox in that. On the one hand connected filtering has proved its efficiency in pyramidal segmentation [13] [15] [23], object based coding [17] [28], motion prediction [25] [22][21], and sequences enhancement [18], namely in applications which always hold on grey or color images. But on the other hand, the underlying axiomatics for connectivity (i.e. definition 1 above), is strictly binary. An alternative (and equivalent) axiomatics has been proposed by Ch. Ronse [20] ; it contains, as a particular case, another one by R.M. Haralick and L.G. Shapiro [28] ; however, both approaches are still set-oriented.

How could we express connectivity for lattices, in general ? More specifically, can we equip the function lattices with connected classes ? If so, what are the interesting ones ?

Here are the questions this paper seeks to solve.

2 A few basic notions and notations

2.1 Reminder on lattices, atoms, co-primes and sup-generators

In this paper, the term "lattice" always means "complete lattice" and is denoted by the generic symbol L. The elements of L are given small letters, such as x, y, a, b... , whereas the capital letters such as X, Y, A, B, ... , correspond to families in L, i.e. to the elements of $\mathcal{P}(L)$ (Birkhoff's notation [29]). The minimum and the maximum element of L are indicated by 0 and m respectively. Symbols M^a and M_a stand for the lower bounds and for the upper bounds of element a, i.e.:

$$M^a = \{x : x \in L\,,\, x \leq a\} \quad ; \quad M_a = \{x : x \in L\,,\, x \geq a\}$$

The notation $\bigvee A$ (resp. $\bigwedge A$) designates the supremum (resp. the infinum) of family A :

$$\bigvee A = \bigvee \{a : a \in A\} \quad ; \quad \bigwedge A = \bigwedge \{a : a \in A\} \qquad A \in \mathcal{P}(L)$$

(Matheron's notation [30]). Birkhoff-Matheron notation, very partical in the general case, is less suitable for $\mathcal{P}(E)$ lattices. In these cases, the two levels of E and $\mathcal{P}(E)$ are usually distinguished by affecting small letteres to the elements of E and capital ones to sets in $\mathcal{P}(E)$. Therefore, in this case only, we shall introduce letter $\mathcal{C}$ to stand for connected classes, i.e. subparts of $\mathcal{P}$.

A non zero element a of lattice L is an *atom* if $x \leq a$ implies $x = 0$ or $x = a$. For example, when L is of the type $\mathcal{P}(E)$, the points of E are atoms in $\mathcal{P}(E)$. An element $a \in L$, $a \neq 0$ is said to be co-prime when $a \leq x \bigvee y$ implies $a \leq x$ or $a \leq y$, in a non exclusive manner. We will complete these two classical definitions (see Heijmans [31], or Gierz et al. [32]) with a third one, from Matheron [30], according to which $a \in L$ is *strongly* co-prime when for any family B in L (finite or not), $a \leq \bigvee B$ implies the existence of a $b \in B$ with $a \leq b$.

Definition 2 *Let L be a lattice and $X \subset L$ a family in L. The class X is a sup generator when every element $a \in L$ is the supremum of the elements of X that it majorates:*

$$a = \bigvee (X \cap M^a) = \bigvee \{x \in X,\, x \leq a\} \quad .$$

Lattice L is said to be atomic (resp. co-prime, strongly co-prime) when it is generated by a class of atoms (res. co-prime, strong co-primes).

Clearly, every atom and every strong co-prime belong to every sup-generating family. Here, two results are worth mentioning.The first one, due to G. Matheron [30, p. 179] combines the notions we have just introduced with that of complementation.

Theorem 2 *For a lattice L, the four statements are equivalent:*
a/ L is co-prime and complemented
b/ L is atomic and strongly co-prime
c/ If Q, Q_a and Q_f denote the classes of co-primes, atoms and strong co-primes respectively, then

$$Q = Q_a = Q_f$$

and T is isomorphic to lattice $\mathcal{P}(Q)$.

The second one (Ch. Ronse, verbal communication), is stated as follows.

Proposition 1 *Any strong co-prime lattice is isomorphic to a complete sublattice of $\mathcal{P}(S)$, where S is the sup-generating family of co-primes (N.B. The isomorphism is given by the map $a \to S \cap M^a$).*

These two results show how demanding is the assumption of strong coprimarity, which in fact restricts the approach to the set-oriented case.

In this reminder, and in the study which follows, the emphasis is put on the supremum. But it is clear that each of the above notions admits a dual form. It suffices to consider the dual lattice L^* of L (where inequalities, and sup and inf are inverted). Atoms, co-prime, strong co-prime and sup-generators on L^* define, on L, dual atoms (also called anti-atoms), prime, strong prime and inf-generators respectively.

2.1.1 Distributivity

Several useful properties involve distributivity, or rather, distributivities. Remember that a lattice L is *distributive* if

$$\begin{aligned} x\bigwedge\left(y\bigvee z\right) &= \left(x\bigwedge y\right)\bigvee\left(x\bigwedge z\right) \\ x\bigvee\left(y\bigwedge z\right) &= \left(x\bigvee y\right)\bigwedge\left(x\bigvee z\right) \end{aligned}$$

for all x, y, $z \in L$. The two equalities are equivalent. When the collection of elements between parentheses is allowed to extend to infinity, i.e. when

$$\begin{array}{ll} x\bigwedge\left(\bigvee y_i \ , i\in I\right) = \bigvee\left\{\left(x\bigwedge y_i\right), i\in I\right\} & \text{(infinite } \bigvee\text{-distributivity)} \\ x\bigvee\left(\bigwedge y_i \ , i\in I\right) = \bigwedge\left\{\left(x\bigvee y_i\right), i\in I\right\} & \text{(infinite } \bigwedge\text{-distributivity)} \end{array}$$

for any collection $\{y_i\} \in L$ and for $x \in L$, then lattice L is *infinite distributive*. There exist more severe distributivities, and in particular the *complete distributivity* [29], which has been discussed by G. Matheron [30, p. 77] under the name of *total distributivity*. It governs the function models we shall use in part 4. If $\mathcal{A}$ stands for a family of subsets of an arbitrary set E, we denote $h(\mathcal{A})$ the class of those parts H of E which are obtained by taking one point in each $A \in \mathcal{A}$. Then a lattice L is *totally distributive* when

$$\bigwedge_{B\in\mathcal{B}}\bigvee B = \bigvee_{H\in h(\mathcal{B})}\bigwedge H \qquad (B\in\mathcal{P}(L))$$

or equivalently

$$\bigvee_{B\in\mathcal{B}}\bigwedge B = \bigwedge_{H\in h(\mathcal{B})}\bigvee H \qquad (B\in\mathcal{P}(L))$$

Total distributivity implies the two other ones, and can be identified with the very strong property of *monoseparation* [30, p. 121]. In particular, every class of functions which is closed under numerical sup and inf forms a totally distributive lattice.

Coprimarity and distributivity in lattices are related to each other. Half of Matheron's monography [30] is devoted to this matter. More modestly, we shall restrict ourselves to the three following results.

a/ Any co-prime lattice is infinite $\bigwedge$-distributive [30, th. 8-11] b/ In a distributive lattice, every atom is co-prime [31, prop. 2-37].

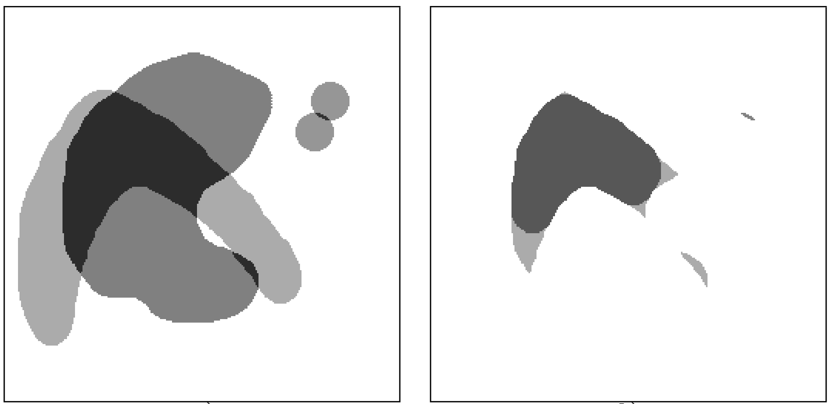
c/ In an infinite $\bigvee$-distributive lattice, every atom is strong co-prime.

Examples

There are numerous lattices associated with image processing. The reader will find in Ronse's paper [19], for example, a comprehensive list. The ones we quote below are instructive because they illustrate differently atoms, co-primes and distributivity. They will be completed, in sect. 4-1, by some others about function lattices.

(1) Lattice of the open sets in the Euclidean space ($\mathbb{R}^2$ for example). It does not admit atoms, but the complements of the points are dual atoms. They are not co-prime, but form a inf-generating family, and indeed this lattice is not infinite $\bigwedge$-distributive (property a).

(2) Lattice $\mathcal{P}(E)$ of the subsets of an arbitrary set E. The points of E are atoms, strong co-primes and sup-generators of $\mathcal{P}(E)$; $\mathcal{P}(E)$ is also complemented, hence strongly co-prime, and totally distributive : it accumulates all nice features.



a) *b)*

Figure 2 a) Two large particles and two atoms in the lattice of the dilates by a disc,

b) In light grey, the corresponding intersections, and in dark grey, the inf in the dilates lattice sense. Since the two small discs have an empty inf, they are seen as **disjoint** *particles in the dilate lattice, whereas they intersect with each other.*

(3) Lattice of the Minkowski dilates by a disc B, in $\mathbb{R}^2$, i.e.

$$L = \left\{X \oplus B, X \in \mathcal{P}(\mathbb{R}^2)\right\}$$

Here, the sup coincides with the usual union, but the inf is the opening by B of the intersection (Fig. 2a). The discs B_x, $x \in \mathbb{R}^2$, are sup-generating atoms, but not co-primes.Again, property b/ appears (in a contraposed form) since this lattice does not satisfy any distributivity.

(4) Lattice of all functions $f : E \to L$ from an arbitrary space E into a lattice T. The pulse functions:

$$\begin{cases} i_{x,t}(y) = t & if \quad y = x \\ i_{x,t}(y) = 0 & if \quad x \neq y \end{cases} \tag{2}$$

associated with each $x \in E$ and $t \in L$ are sup-generating co-primes but not atoms (except when $L = \{0, 1\}$) and generally not strong co-primes. However, when L is discrete (L finite, or $L = \overline{\mathbb{Z}}$, etc.), then the pulses are strong co-primes; this lattice is totally distributive, but not complemented.

(5) Lattice of the upper semi-continuous functions $f : \mathbb{R}^2 \to [0, 1]$. Its pulses are sup-generating co-primes, but not strong ones ; in this lattice, the inf of a family $\{f_i\}$ is the function which admits, at each point $x \in \mathbb{R}^2$, the numerical inf of the $f_i(x)'$s, but the sup is the function whose umbra is the topological closure of the union of the umbrae of the f_i's. This lattice is distributive, and infinite $\bigwedge$-distributive, but not infinite $\bigvee$-distributive.

The comparison of the five models is instructive. Some of them admit co-prime but not atoms (n^{os} 4-5) or *vice-versa* (n^{o} 3). Sup-generating families exist, which do not consist of atoms or co-primes (n^{o} 1). Distributivity is also a deep distinction between lattices. We will see in the following that a lattice L may be equipped with connections when it is sup-generated, and that these connections satisfy nice properties when L is infinite $\bigvee$-distributive (or $\bigvee$-distributive for finite L's).

2.2 Connected class and characteristic openings

This section generalizes def. 2.7. and theorem 2.8 in [9], which are devoted to the case of $\mathcal{P}(E)$. The approach follows similar steps, but the distinction between points and sets is replaced by the introduction of sup-generators.

Definition 3 *Connection. Let L be a lattice. A class $C \subseteq L$ is said to be connected, or to define a connection, when*

(o) $0 \in C$

(i) C is sup-generating: $\forall\, a \in L, \quad a = \bigvee (C \cap M^a)$

(ii) C is conditionally closed under supremum

$$X \subseteq C \quad , \quad \bigwedge X \neq 0 \quad \Rightarrow \quad \bigvee X \in C \tag{3}$$

Class C is generally neither closed under $\bigvee$or $\bigwedge$. However, if C_x stands for the subclass of C that contains the upper bounds of a given $x \in C \backslash \{0\}$, i.e.

$$C_x = \{c : x \leq c, \ c \in C\} = C \cap M_x$$

then, the supremum of each non empty family of elements of C_x is again in C because of *(ii)*. In other words, the class $C_x \cup \{0\}$, closed under supremum, characterizes the invariant sets of a unique opening γ_x, which is called the *connected opening* of origin x. For all $x \in C \backslash \{0\}$, we have

$$\gamma_x(a) = \bigvee \{c : c \in C, \ x \leq c \leq a\} \qquad a \in L. \tag{4}$$

We then say that $\gamma_x(a)$ is the connected component of a marked by x, and that x (which is itself a connected component) is called a *marker*. Clearly for $x, y \in C \backslash \{0\}$ such that $x \leq y$, we have $C_x \supseteq C_y$, hence $\gamma_x \geq \gamma_y$. We will summarize these results in proposition 1 :

Proposition 2 *Let C be a connection on a lattice L. For every $x \in C \backslash \{0\}$, the mapping $\gamma_x : L \to L$ defined by*

$$\gamma_x(a) = \bigvee (C \cap M_x \cap M^a) \qquad a \in L$$

is the opening of invariant elements $B_x = (M_x \cap C) \cup \{0\}$, and of marker x. Moreover, if $x, y \in C \backslash \{0\}$ with $x \leq y$ then $\gamma_x \geq \gamma_y$.

Proposition 3 *For all $x \in C \backslash \{0\}$,and for all $a \in L$, the following equivalences are satisfied:*

$$\gamma_x(a) \neq 0 \quad \Leftrightarrow \quad x \leq a \quad \Leftrightarrow \quad x \leq \gamma_x(a) \in C \tag{5}$$

Proof. Relation (4) implies that $\gamma_x(a) = 0$ when $x \not\leq a$. On the other hand, when $x \leq a$ we always have, from *(i)*, $x = \gamma_x(x) \leq \gamma_x(a)$ hence $\gamma_x(a) \neq 0$, and from (ii), we get $\gamma_x(a) \in C$. Q.E.D.

Indeed, the family of openings $\{\gamma_x \ ; \ x \in C \backslash \{0\}\}$, induced by the connected class C *characterizes* it, as shown by the following central theorem

Theorem 3 *Let C be a sup-generator in a lattice L. Class C is a connection if and only if it coincides with the invariant elements of family of openings $\{\gamma_x , x \in C\setminus\{0\}\}$ such that*
(iii) for all $x \in C\setminus\{0\}$, we have $\gamma_x(x) = x$,
(iv) for all $a \in L$, $x, y \in C\setminus\{0\}$, $\gamma_x(a)$ and $\gamma_y(a)$ are equal or disjoint, i.e.
$\gamma_x(a) \bigwedge \gamma_y(a) \neq 0 \Rightarrow \gamma_x(a) = \gamma_y(a)$,
(v) for all $a \in L$ and for all $x \in C\setminus\{0\}$, we have $x \nleq a \Rightarrow \gamma_x(a) = 0$.

Proof. Let C be a connection. Points *(iii)* and *(v)* have been proved in proposition 3. From *(ii)* we observe that $\gamma_x(a) \bigwedge \gamma_y(a) \neq 0$ implies $c = \gamma_x(a) \bigvee \gamma_y(a) \in C$. On the other hand,
$x \leq \gamma_x(a) \Rightarrow x \leq c \Rightarrow c \in C_x \Rightarrow c \leq \gamma_x(a) \Rightarrow \gamma_y(a) \leq \gamma_x(a)$.

We show the reverse inequality, and thus equality, in the same way. Hence *(iv)* follows. It is clear that C is the family of all $\gamma_x(a)$ for $x \in C$ and $a \in L$.

Conversely, suppose that C is a family of elements of L, and that C is also the family of the invariant elements of family $\{\gamma_x , x \in C\}$ of openings that satisfy axioms *(iii)* to *(v)*, i.e.

$$C = \{\gamma_x(a) \ , \ x \in C \ , \ a \in L\}$$

For $a = 0$, we find by *(v)* that $\gamma_x(0) = 0 \in C$. C is sup-generating by hypothesis, i.e. axiom *(i)* is satisfied. Now let c_i be a family in C with non empty inf, and $x \in C \setminus \{0\}$ with $x \leq \bigwedge c_i$. As $c_i \in C$ we can find a marker y_i for each i such that $c_i = \gamma_{y_i}(c_i)$. But $x \leq c_i$; therefore, from *(iii)* $x \leq \gamma_x(x) \leq \gamma_x(c_i)$. Thus $\gamma_{y_i}(c_i)$ and $\gamma_x(c_i)$ majorate x, and from *(iv)* we have $c_i = \gamma_{y_i}(c_i) = \gamma_x(c_i) \leq \gamma_x(\bigvee c_i)$, hence $\bigvee c_i = \gamma_x(\bigvee c_i)$. Thus $\bigvee c_i$ belongs to $C\setminus\{0\}$ and *(ii)* is satisfied.

We still have to prove that the connected openings γ'_x associated with class C coincide with the γ_x themselves; a necessary and sufficient condition is that for every $x \in C\setminus\{0\}$, γ_x and γ'_x have the same invariant sets, i.e.

$$\begin{array}{lcll} C_x & = & \{\gamma_x(a) : \gamma_x(a) \neq 0 , a \in L\} & x \in C\setminus\{0\} \\ C'_x & = & \{\gamma_y(a) , y \in C , a \in L , x \leq \gamma_y(a)\} & x \in C\setminus\{0\} \end{array}$$

Let $\gamma_y(a) \in C'_x$. From *(iii)* $x \leq \gamma_y(a) \leq a$ implies $x \leq \gamma_x(x) \leq \gamma_x(a)$, thus, from *(iv)*, $\gamma_y(a) = \gamma_x(a)$. Hence $C'_x \subseteq C_x$. Conversely, if $\gamma_x(a) \in C_x$, then, from *(v)*, $\gamma_x(a) \geq x$, so that $\gamma_x(a) \in C'_x$, thus $C_x \subseteq C'_x$, which achieves the proof. Q.E.D.

Corollary 1 *For all $x, y \in C \setminus \{0\}$ and for all $a \in L$, we have*

$$y \leq \gamma_x(a) \quad \Leftrightarrow \quad \gamma_x(a) = \gamma_y(a) \neq 0 \quad \Leftrightarrow \quad x \leq \gamma_y(a)$$

Proof. If $y \leq \gamma_x(a)$ then by *(v)* we also have $y \leq a$, therefore $y \leq \gamma_y(a)$. From *(iv)*, we draw $\gamma_y(a) = \gamma_x(a) \neq 0$. Conversely if $\gamma_x(a) = \gamma_y(a) \neq 0$, then we have $y \leq \gamma_y(a) = \gamma_x(a)$. Q.E.D.

Proposition 4 *Let S be a* sup*-generating family on L. To every part P of L containing S corresponds the connections C containing P ; we write $C(P)$. The set of connections containing S is closed under intersection; it is thus a complete lattice; in this lattice the infimum and supremum of a family of connections C_i , $(i \in I)$ are given respectively bu their intersection $\cap C_i$ and by the connection generated by their union $C\left(\cup C_i, i \in I\right)$.*

Proof. Consider a family $\{C_i, i \in I\}$ of connection which contain S. Their intersection obviously satisfy axioms (*0*) and (*ii*). Axiom (*i*) is also satisfied because class S is a sup-generator included in $\cap C$. The family $\{C_i\}$ is thus closed under intersection. Consequently, the family of all connections containing S is an inf semi-lattice, whose minimum element is their intersection. But this family also admits a maximum element, namely lattice L itself. Thus, it is a lattice.

In this lattice, consider now the family of all connections that contain a given part P of L, with $P \supseteq S$. The intersection of these connections is said to be the connection generated by P, and is written $C(P)$. In particular, if we take for P the union $\cup C_i, i \in I$, of an arbitrary family in the lattice, then the least connection $C\left(\cup C_i\right)$ is nothing but the supremum of the C_i's. Q.E.D.

This said, for anyone who wants to generate connected classes, proposition 4 is more formal than practical. We shall propose other ways, based on dilations and more operational, in section 3.

Proposition 5 *All atoms and all strong co-prime of L belong to any connection C.*

Proof. If element $a \neq 0$ is an atom, then $M^a = \{0, a\}$, and $a = \bigvee \{C \cap M^a\}$ implies $a \in C$. If q is a strong co-prime, then $q = \bigvee \{C \cap M^q\}$ implies $q \in M^{C \cap M^q}$, i.e. there exists a connected element $q' \geq q$ and such that $q' \in M^q$ hence $q = q'$. Q.E.D.

This proposition follows from the fact that every atom and every strong co-prime belong to every sup-generating family. One may notice also that if the co-prime q is not strong, then it does not necessarily belong to C.

2.3 Canonic markers

The generalization we have just developed avoids to favour any sup-generating class of co-primes in lattice L. In this sense, it differs from the set case where the points are supposed to belong to all connections. Now the major application we have in mind here concerns function lattices (grey tone, color, equi-continuous, etc.), where one can assume that a basic sup-generator class belongs to all connections. It will be, for example, the class of the pulses for the numerical functions, or that of the cones for the Lipschitz functions (see fig. 3).

This circumstance suggests to slightly reduce the generality of the previous approach by chosing a sup-generating class S in L, with $0 \notin S$, and by replacing assumption (i) in definition 3 by the following one:

assumption (vi): *C contains the sup-generator S.*

By so doing, we restrict the possible connections on L to those that contain S only. The sup-generator S, which is not itself a connection, will be called *canonic*. The interesting fact is that the canonic markers S are sufficient to label all openings, for all connections on L. Indeed we can state.

Proposition 6 : *When a canonic sup-generator S has been chosen in lattice L, then every connection C on L is identified to the invariant elements of the openings γ_s , $s \in S$, such that*

$$\gamma_s(a) = \bigvee \{x \in C, s \leq x \leq a\} \qquad s \in S \qquad a \in L$$

Proof. Consider a given, but arbitrary, connection C, and let $a \in L$ and $x \in C$ such that $\gamma_x(a) \neq 0$. From prop. 3, we have

$$x \leq a \qquad \text{and} \qquad x \leq \gamma_x \ (a) \in C \ .$$

On the other hand, there exists at least one $s \in S$ smaller that x, since class S is sup-generating. Now, by assumption (vi), $s \in C$, hence

$$s \leq x \leq a \qquad \Rightarrow \qquad s \leq \gamma_s \ (a) \ .$$

Both $\gamma_s \ (a)$ and $\gamma_x \ (a)$ containing $s \neq 0$, we draw, from *(iv)* in theorem 3, that $\gamma_s \ (a) = \gamma_x \ (a)$. We have supposed $\gamma_x \ (a) \neq 0$ for some $x \in C \backslash \{0\}$. If not, we have $\gamma_x \ (a) = \gamma_s (a) = 0$ for any $s \in S$, which achieves the proof. Q.E.D.

Conceptually speaking, we realize an economy since the bijection now holds on the connections C and on the families $\{\gamma_s \ , s \in S\}$ of openings, for a reduced class S of labels which is common to all connections. Note also that proposition 4 becomes more significant, since when applied to the canonic sup-generator S it holds on the lattice of *all* connections that exist on the lattice L under study.

Examples

There is a number of interesting lattices. The brief list which follows is

(a) In a topological space, both connectivities based on disjoint closed sets, and on arcs.

(b) arcwise connectivities on digital spaces.

(c) connections based on extensive dilations (theorem 4 below).

(d) set segmentation according to a fixed partition of the space [9, p. 54].

(e) lattice of the Minkowski dilates, as presented above in section 2.1. In this lattice, two discs which intersect (in the set sense) nevertheless form two *disjoint* particles! (see fig. 2a and 2b). We find just the opposite effect of the clustering by an extensive dilation of fig. 1, where a group of seemingly disjoint objects was seen as a unique particle.

(f) In [19], Ch. Ronse proposes, among several other ones, the instructive following example. In $\mathcal{P}(\mathbb{R}^2)$, the class generated by the points and the connected sets opened by a disc B forms a new connection. If $x \in X \circ B$, then $\gamma_x(X)$ is the topologically connected component of $X \circ B$ containing x, and when point $x \in X \backslash X \circ B$, then $\gamma_x(X) = x$. This connection, illustrated in fig. 5 below, is useful, not only for possible practical uses, but also for building instructive counter examples (see prop. 17 below).

2.4 Sup-generating markers and partitioning

A number of attractive properties of the connected classes come from their ability to *partition* each element of L into its components. In order to describe them, we will first define the notion of a partition D (D as "division") for a lattice element.

Definition 4 *Partition: Let L be a complete lattice, and C be a connection on L. A partition of $a \in L$ is a mapping D_a from $C \cap M^a$ into M^a such that*
(i) for all $x \in C \cap M^a : x \leq D_a(x) \leq a$
(ii) for all $x, y \in C \cap M^a : D_a(x) = D_a(y)$ or $D_a(x) \bigwedge D_a(y) = 0$

$D_a(x)$ is called the class of the partition of origin x. As connection C is sup-generating, the supremum of all classes $D_a(x)$ restitutes a:

$$a = \bigvee \{D_a(x), x \in C \cap M^a\} \tag{6}$$

The kinship between the disjunction axioms for partitions (*(ii)* in definition 4) and for connected classes (axiom *iv*) is highlighted by the following result:

Proposition 7 *Let C be a connection on L. For each $a \in L$ the family D_a of the maximal connected elements in $C \cap M^a$ is a partition, whose classes are the connected components of a.*

Proof. For any $a \in L$, the class $C \cap M^a$ is never empty from axioms *(o)* and *(i)*. By definition of $\gamma_x(a)$, for $x \in (C \setminus \{0\}) \cap M^a$, the maximal connected components D_a coincide with the $\gamma_x(a)$, which are disjoint, according to theorem 3. Q.E.D.

Corollary 2 *Let D_a be as in Prop. 7. If $b \neq 0$ is connected and $b \leq a$, then $y \in D_a$ implies either $b \leq y$ or $b \bigwedge y = 0$.*

Clearly, we meet again the connected openings, since given $x \in C$ and $a \in L$, $\gamma_x(a)$ is nothing but the element $y \in D_a$ larger than x. But proposition 7 investigates the connection C from the point of view of the lattice elements, whereas theorem 3 starts from the markers.

Corollary 3 *Opening γ_x partitions any $a \in L$ into the smallest possible number of components belonging to the class C, and this partition is increasing in that if $a \leq a'$, then any connected component of a is majorated by a connected component of a'.*

[Proof identical to that of corollary 1, p. 53, in [9].

3 Connectivity and increasing mappings

3.1 Dilations and connectivity

The dilations that we will consider here map lattice L into itself. We remind that a dilation is defined as an operator δ on L which commute with the supremum [33][31][34]:

$$\delta(\bigvee a_i) = \bigvee \delta(a_i) \qquad \{a_i\} \in \mathcal{L}.$$

Similarly, an erosion ϵ on L is an operator which commutes with the infinum:

$$\epsilon(\bigwedge a_i) = \bigwedge \epsilon(a_i) \qquad \{a_i\} \in \mathcal{L}.$$

With every dilation δ is associated a unique *adjoined erosion* ϵ, by the following equivalence

$$\epsilon(a) \geq b \qquad \Leftrightarrow \qquad a \geq \delta(b) \qquad a, b \in L.$$

The composition product $\gamma = \delta \circ \epsilon$ is an opening, it is said to be derived from dilation δ. By many respects, dilations and connected classes interact. In this section and in the next one (i.e. 3.2), we propose to analyze some of these effects.

Proposition 8 *Let L be a complete lattice, and S be a sup-generating family of co-primes in L. Suppose that L is equipped with a connected class C that contains S. If an extensive dilation preserves the connectivity over class S, i.e. if $\delta(x)$, $x \in S$, is connected, or, equivalently $\delta : S \to C$, then δ preserves connectivity over the whole connected class C, i.e. $\delta : C \to C$.*

Proof. For $z \in C$ and $x \in S \cap M^z$, we have $x \leq \delta(x) \bigwedge z$ and as both $\delta(x), z \in C$, we deduce that $\delta(x) \bigvee z \in C$. Now $z = \bigvee (S \cap M^z)$, so that $\delta(z) = \bigvee \{\delta(x), x \in S \cap M^z\}$ and as δ is extensive, $z \leq \delta(z)$, so that

$$\begin{aligned} \delta(z) &= \delta(z) \bigvee z = (\bigvee \{\delta(x), x \in S \cap M^z\}) \bigvee z \\ &= \bigvee \{\delta(x) \bigvee z, x \in S \cap M^z\} \end{aligned} \tag{7}$$

As $z \leq \bigwedge \{\delta(x) \bigvee z, x \in S \cap M^z\}$ and each $\delta(x) \bigvee z \in C$, we deduce that the right member of eq. 7 is connected, that is $\delta(z) \in C$. Q.E.D.

Remark 1. The extensivity of δ is not always necessary. For example, in the Minkowski addition

$$\delta(X) = X \oplus A, \; = \cup \{x + a\, ;\, x \in X,\, a \in A\} \qquad X, A \in \mathcal{P}(\mathbb{R}^d),$$

translation invariance compensates for extensivity. Indeed, choose $a \in A$; then A_{-a} contains the origin and dilation by A_{-a} is extensive. If X and A are connected, then so are X_a and A_{-a}, and by Proposition 8, $X \oplus A = X_a \oplus A_{-a}$ is connected. We can state

Proposition 9 *Let E be $\mathbb{R}^d$ or $\mathbb{Z}^d$, equipped with an arbitrary connected class $\mathcal{C}$. When A and X belong to $\mathcal{C}$, then $X \oplus A$ is $\mathcal{C}$-connected too.*

Remark 2. We will illustrate proposition 8 by taking the case of binary *standard dilations*. They are used in all morphological software packages as substitutes for Minkowski additions, whereas they are *not* translation invariant. In two dimensions for example, Z stands for a rectangle in the 2-D space, provided with the usual connectivity. Then the standard dilation of X by A is defined by the relation

$$\delta\,(X) = [(X \cap Z) \oplus A] \cap Z = \cup\,[A_x \cap Z,\, x \in X \cap Z]\ , \quad X \in \mathcal{P}\,(Z)\ .$$

It is a dilation in the lattice-theoretical sense, i.e. it distributes the supremum operation $\cup$.

Proposition 10 *In $\mathbb{R}^d$ or $\mathbb{Z}^d$ the extensive standard dilations by convex sets preserve connectivity on $\mathcal{P}\,(Z)$.*

Proof. Let A be a convex set, take $a \in A$ and write $B = A_{-a}$; for every $X \in \mathcal{P}\,(Z)\,, \delta\,(X) = (B \oplus X_a) \cap Z$ is the union of all $B_y \cap Z$ for $y \in X_a$. As B_y and Z are convex, their intersection $B_y \cap Z$ is connected. Now $X_a \in \mathcal{P}\,(Z_a)$, and applying prop. 8 to $\mathcal{P}\,(Z_a)$, it follows that δ preserves connectivity on subsets of Z. Q.E.D.

Among other things, the proposition teaches us that the implicit assimilation of Minkowski addition to standard dilation, which is often found in literature, turns out to be somewhat improper as soon as the structuring elements are not convex.

We will conclude this section by extending a classical result of Euclidean morphology, but which remains valid for any C-connectivity.

Proposition 11 *Let L be a complete lattice, equipped with a connection C. Then, for any dilation δ on L, that preserves connectivity and for all $a \in L$, the adjoined erosion ϵ and opening γ treat the connected components of a independently of one another.*

Proof. Let $a \in L$, having connected components c_i, $i \in I$. By increasingness, we have $\bigvee \epsilon(c_i) \leq \epsilon(a)$. To show the reverse inequality, consider $x \in C$, with $x \leq \epsilon(a)$. By adjunction, we have $\delta(x) \leq a$. But $\delta(x)$ is connected, i.e. from corollary 3, $\delta(x) \leq c_i$ for some i. Hence by adjunction $x \leq \epsilon(c_i) \leq \bigvee \epsilon(c_i)$, and, since class C is sup-generating, $\epsilon(a) \leq \bigvee \epsilon(c_i)$. This achieves the first part of the proof. From $\gamma = \delta\epsilon$, and since δ distributes the supremum, we L

$$\gamma(a) = \delta\left(\bigvee \epsilon(c_i)\right) = \bigvee \delta\epsilon(c_i) = \bigvee \gamma(c_i)$$

Q.E.D.

3.2 Second generation connectivity

Dilations can be used to remodel connected classes. Starting from a first class C, of connected openings $\{\gamma_x, x \in C\}$, we may try and cluster some disjoint connected components into new ones. The approach which follows generalizes, and improves, Serra's proposition 2.9, in [9]. We will begin with a general, and rather immediate, property.

Proposition 12 *Let C be a connection on lattice L, and $\delta : L \to L$ be an extensive dilation that preserves C (i.e. $\delta(C) \subseteq C$). Then the inverse image $C' = \delta^{-1}(C)$ of C under δ is a connection on L, which is richer than C.*

Proof. By definition, class C' is the family of elements x' such that $\delta(x') \in C$. Since δ preserves C, we have $C' \supseteq C$ hence $0 \in C'$ and also C' is sup-generating. Let $x'_i \in C'$, with $\bigwedge x'_i \neq 0$. By extensivity of δ, $\bigwedge \delta(x'_i) \geq \bigwedge x'_i$ is *a fortiori* $\neq 0$. Since $\delta(x'_i) \in C$, we have $\delta(\bigvee x'_i) = \bigvee \delta(x'_i) \in C$, therefore $\bigvee(x'_i) \in C'$. Q.E.D.

This first result may be made more precise by introducing the adjoint erosion ϵ and opening $\theta = \delta\epsilon$, whose family of invariant elements is B_θ. Remark that B_θ is nothing but $\delta(L)$, i.e. the et of the $\delta(a)$ for $a \in L$. Since δ distributes the supremum, for every sup-generating family S of L, $\delta(S)$ is a sup-generating family of $B_\theta = \delta(L)$.

Proposition 13 *When C' is the connection defined by proposition 12, then $\delta(C')$ equals $B_\theta \cap C$ and is a connection over B_θ.*

Proof. $x' \in C' \Rightarrow \delta(x') \in C$, moreover, as a dilate, $\delta(x')$ is also an invariant set of θ, hence $\delta(C') \subseteq C \cap B_\theta$. Conversely, if $z \in B_\theta \cap C$, then $z = \delta(a) \in C$ for an $a \in L$, hence $a \in C'$. Hence $\delta(C') = B_\theta \cap C$.

To prove that $\delta(C')$ is a connection over B_θ, note first that since $C' \supseteq C$ is sup-generating, by the above remark $\delta(C')$ is also sup-generating (and $0 = \delta(0) \in B_\theta \cap C$). Let $b_i \in \delta(C')$ with $\perp b_i \neq 0$, where $\perp$ stands for the inf symbol in B_θ. Since $\perp b_i = \theta(\bigwedge b_i)$, we have *a fortiori* $\bigwedge b_i \neq 0$, hence $\bigvee b_i \in B_\theta \cap C = \delta(C')$, which achieves the proof. Q.E.D.

The last two propositions are already instructive, but they do not inform us about the relations between the connected component of some $a \in L$ and its image $\delta(a)$. To go further, we need an assumption of distributivity over L, as it appears in the following lemma.

Lemma. *If lattice L is infinite $\bigvee$-distributive, and if C and C' are two connections on L with $C \subseteq C'$, then for all elements $y \in L$, every connected component x' of y (according to C') is the supremum of the connected components x_i (according to C) that it majorates.*

Proof. Let x'_i, $i \in I$ and x_j, $j \in J$ the C' and C connected components of y. Since $C \subseteq C'$, each $x_j \in C'$, hence is majorated by one and only one C'-connected component, say $x'_{i(j)}$, and $x_j \bigwedge x'_i = 0$ for $i \neq i(j)$ (corollary of theorem 3). Conversely, each x'_i majorates the x_j, $j \in J(i)$ and remains separated from the others. Thus we may write :

$$x'_i = x'_i \bigwedge y = x'_i \bigwedge \left(\bigvee x_j\right)$$

and , since L is infinite $\bigvee$-distributive:

$$x'_i = \bigvee(x'_i \bigwedge x_j \ , \ j \in J) = \bigvee(x'_i \bigwedge x_j \ , \ j \in J(i)) = \bigvee(x_j \ , \ j \in J(i))$$

which achieves the proof. Q.E.D.

We now come back to the dilation of propositions 12 and 13, and we try to interpret the connected components of $\delta(a)$ in terms of *images* of the connected components of a. We will proceed in two steps :

Proposition 14 *Suppose that L is infinite $\bigvee$-distributive. Let $a \in L$ and $y = \delta(a)$. Then the $\delta(C')$-components of y (in lattice B_θ) coincide with the C-components of y in lattice L.*

Proof. Let z_i^*, $i \in I$ be the $\delta(C')$-components of y in B_θ and $z_j, j \in J$ be the C-components of y in L. Since the $z_i^* \in C$ for each $j \in J$, we have either $z_i^* \leq z_j$ or $z_i^* \bigwedge z_j = 0$. Put $I(j) = \{i, i \in I,\ z_i^* \leq z_j\}$. Given j, for $i \in I \backslash I(j)$ we have $z_j \bigwedge z_i^* = 0$ so that

$$z_j = z_j \bigwedge (\bigvee z_i^*, i \in I\ ) = \bigvee (z_j \bigwedge z_i^*, i \in I\ ) = \bigvee (z_j \bigwedge z_i^*, i \in I(j))\ .$$

Now $z_i^* \in B_\theta$ which is closed under $\bigvee$, hence $z_j \in B_\theta$ and also $z_j \in B_\theta \cap C = \delta(C')$. But if $z_j \in\ \delta(C')$, we have (in lattice B_θ) $z_j \leq z_i^*$ for one of the $\delta(C')$-components of y. Therefore, class $I(j)$ comprises one index i only, and we have $z_j = z_i^*$ for that index i. Q.E.D.

Theorem 4 *Let C be a connection on an infinite $\bigvee$-distributive lattice L, and $\delta : L \to L$ be an extensive dilation with $\delta(C) \subseteq C$. Then the C-component of $\delta(a)$, $a \in L$, are exactly the images $\delta(y_i')$ of the C'-components of a, where C' is the connection $C' = \delta^{-1}(C)$. Thus, δ induces a bijection between the connected components of $C \cap B_0$ and those of C'.*

Proof. Let us compare z_i^*, as defined in the previous proposition, with the C'-components y_j' of a. Since $\delta(y_j') \in \delta(C')$, we observe, as previously, that

$$z_i^* = \bigvee \left\{\delta(y_j') : \delta(y_j') \leq z_i^*\right\} = \delta \bigvee \left\{y_j' : j \in J_i'\right\}$$

By putting $y_i^{*\prime} = \bigvee \left\{y_j' : \delta(y_j') \leq z_i^*\right\}$, we obtain $\delta(y_i^{*\prime}) = z_i^* \in B_\theta \cap C$. Therefore $y_i^{*\prime} \in \delta^{-1}(B_\theta \cap C) = C'$. Since $y_i^{*\prime} \leq a$, this implies that $y_i^{*\prime}$ is one of the y_j', $j \in J_i'$, which achieves the proof. Q.E.D.

Corollary 4 *If γ_x stands for the connected opening associated with connection C and ν_x for that associated with C', we have*

$$\nu_x(a) = \gamma_x \delta(a) \bigwedge a \quad \textit{when} \quad x \leq a \quad ; \quad \nu_x(a) = 0 \quad \textit{when not}$$

Proof. Let $x \in C\setminus\{0\}$ such that $x \leq a$. By theorem 4, $\delta\nu_x(a)$ is a C-connected component of $\delta(a)$, and as δ is extensive, $x \leq \nu_x(a) \leq \delta\nu_x(a)$. So $\delta\nu_x(a) = \gamma_x\delta(a)$. As $\nu_x(a) \leq \delta\nu_x(a)$ and $\nu_x(a) \leq a$, we get $\nu_x(a) \leq \delta\nu_x(a)\bigwedge a$. Let $y \in C\setminus\{0\}$, with $y \in \delta\nu_x(a)\bigwedge a$. Then $\delta\nu_y(a) = \gamma_y\delta(a)$ as before.Thus, $0 < y \leq \gamma_x\delta(a\bigwedge\gamma_y\delta(a))$, and so $\gamma_x\delta(a) = \gamma_y\delta(a)$, that is $\delta\nu_x(a) = \delta\nu_y(a)$. But δ is a bijection (theorem 4), hence $\nu_x(a) = \nu_y(a)$, and $y \leq \nu_x(a)$. Since C is sup-generating, we get $\delta\nu_x(a)\bigwedge a \leq \nu_x(a)$, and the equality follows. If $x \nleq a$, then $\nu_x(a) = 0$ by axiom *(v)*, which achieves the proof Q.E.D.

Comments

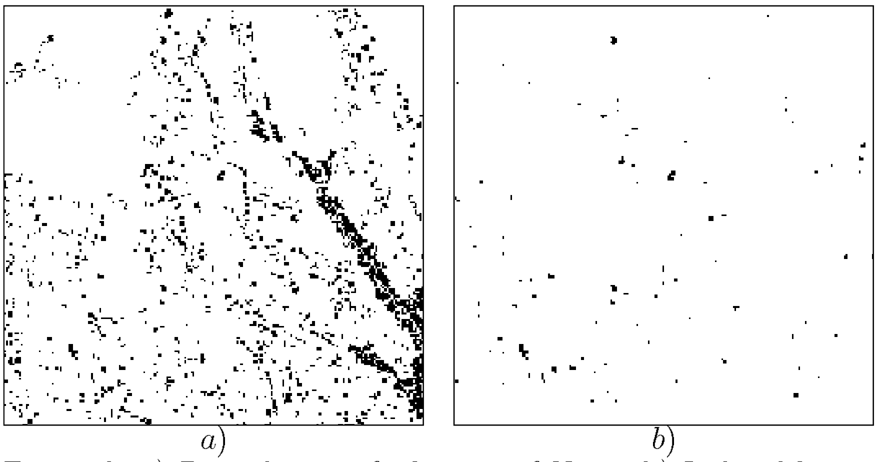


Figure 3 a) Partial view of the city of Nice b) Isolated houses

1/ Although the lemma does not really require infinite $\bigvee$-distributivity, the next two steps demand it. As a counter example, consider the lattice of the u.s.c. functions $\mathbb{R} \rightarrow [01]$ and take its impulses for connection C. Let δ be the dilation defined by

$$\begin{aligned} \delta(i_{x,t}) &= i_{x,1} && \text{when } x \text{ is rational} \\ \delta(i_{x,t}) &= i_{x,t} && \text{when not} \end{aligned}$$

(Impulsions have been introduced in eq. 2, above). Dilation δ satisfies the conditions of theorem 4, and the connection $C' = \delta^{-1}(C)$ it induces is identical to C. Under δ, the function $f = 0,5$ everywhere become $f = 1$ everywhere, however for $x \in \mathbb{R}$ irrational, we have

$$\delta(i_{x,0,5}) = i_{x,0,5}$$

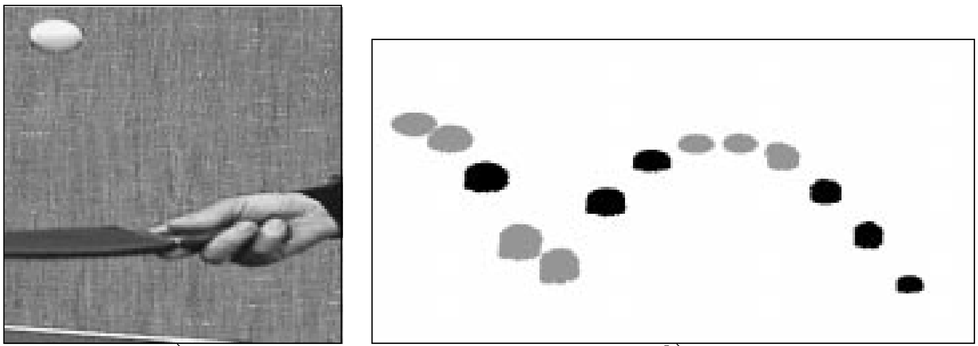
i.e. *the image of the pulse is not the pulse of the image.* Theorem 4 does not apply because of the lack of infinite $\bigvee$-distributivity for the lattices of u.s.c. functions.

2/ The connected openings ν_x of corollary 4 not only give the theoretical access to their connected class, but also provide the actual algorithm which extracts the components of a given X.

In practice, the openings ν_x characterize the *clusters* of objects from a given distance d apart. Consequently, such an approach also provides a means to extract the objects which are isolated. Imagine, for example, that we want to detect in the city map of Nice X (fig. 3a) the houses whose distances to their neighbours are $\geq 2d$ (the houses with large gardens, say). The two necessary pieces of information are the dilate $X \oplus dB$ of map X, and the skeleton by zones of influence $skiz(X)$, which is made of all segments at midway from neighbouring houses. If Y_1 denotes a particle of $X \oplus dB$ which is marked by $skiz(X)$, then $Y_1 \cap X$ is a cluster of houses, and if Y_0 is a particle of $X \oplus dB$ which misses $skiz(X)$, then $X \cap Y_0$ is a house whose all neighbours are at least from $2d$ apart. Fig. 3b shows the set of such houses among those of fig. 3a.

We now illustrate theorem 4 by a second example on motion analysis. Fig. 4a comes from a time sequence of thirteen images. The camera is fixed and the hand moves slightly up and down to make the ping-pong ball rebund. In each image, the connected component "ping-pong ball" is extracted and followed in the space time projection of fig. 4b.

In the space $\times$ time product, this ball is obviously connected. It will turn out to be also the case in fig. 4b if we use a dilation based connection (in the sense of theorem 4) by a time segment of size 15. But if we take a smaller size, 10 say, we generate three clusters (in grey in fig. 4b). They correspond to the positions when the speed is low. The remainder is made of six isolated particles, corresponding to a higher speed of the ball motion.



a) *b)*

Figure 4 a) First image of a sequence;
b) Space× Time repesentation of the "ping-pongball" connected components.
In a dilation based connection,
the three clusters in grey are considered as particles. They correspond to the slow motions

3.3 Filtering and connections : semi groups

We noticed, in introduction, the considerable use of connectivity for filtering purposes, in the recent past years. This is due to two series of properties which are satisfied in that case, namely semi-groups and strong filters. In this section, we approach the first ones in the general framework of the lattices. The second ones will be developed in the next section, in the set oriented case.

First of all, we will introduce the two major notions that we need for the following, namely reconstruction openings (and closings) and clean operators.

Definition 5 *Let C be a connection on lattice L, and γ_T be a trivial opening on L (i.e. such that either $\gamma_T(a) = a$ or $\gamma_T(a) = 0$). Then the supremum of the $\gamma_T\gamma_x$ over class C, i.e.*

$$\gamma = \bigvee \{\gamma_T\gamma_x \, , \, x \in C\} \tag{8}$$

is an opening called reconstruction opening with respect to criterion T. (This follows from the fact that $\gamma_T\gamma_x$ is clearly an opening, and that the supremum of openings is still an opening.) The reconstruction closings are introduced similarly, by duality.

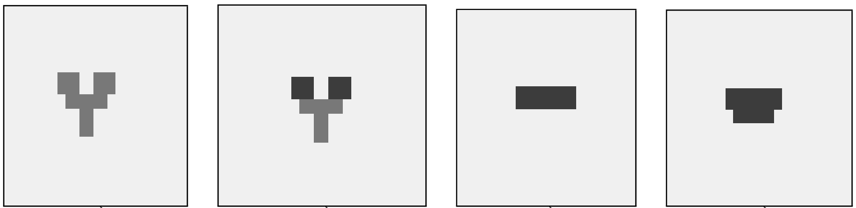
In set oriented cases, the various operators that can only keep or remove particles according to markers, or according to their sizes, their inscribable discs, or again some Ferret diameters, belong to the family of the reconstruction openings. Similarly, when numerical functions are treated via their cross sections, the area openings in L.Vincent's sense [16], or the volume ones, in C. Vachier's sense [23] are also examples of reconstruction openings. A number of properties of the γ_x extend to these openings, as we will see.

We focus now on the increasing operators that modify or possibly suppress connected components, but never create new ones.

Definition 6 *Given a connection C on lattice L, an operator $\psi : L \to L$ is* clean *when for all $a \in L$ and all $x \in C \setminus \{0\}$, the following implication is true*

$$x \leq \psi(a) \quad \Rightarrow \quad a \bigwedge \gamma_x \psi(a) \neq 0 \ . \tag{9}$$

Typically, in $\mathcal{P}(E)$, the complement mapping $X \to X^c$, $X \subseteq E$ is not clean. But there exist also *increasing* operations



a) *b)* *c)* *d)*

Figure5 Set X is the tomahawk a) and set B a square. For connection C,whose elements are the sets open by B, plus the points, set X is not connected, but composed of the two upper squares in dark grey in b), plus an infinity of isolated points, in light grey in b). Let γ (resp.φ) stand for the opening (resp.closing) by reconstruction that removes the point grains (resp. pores) in connection C. The filtered version $\varphi\gamma(X)$ of set X is shown in c); it differs from $\varphi\gamma(X \vee \varphi\gamma(X))$, which is shown in d) Thus, operator $\varphi\gamma$is not idempotent.

that are not clean. Take for example the above connection (f), of section 2-3, and apply it to the binary tomahawakof fig. 5a by taking for B the unit square. If we close the set in dark gray, in fig. 5b by a square large enough, the central channel is filled up, see fig. 5c. However, for the connection we have chosen, the two initial particles are not clustered, and each point of central channel constitutes a distinct connected particle, i.e. which is not adjacent to the other points of the channel. Implication (9) is not satisfied for any x belonging to the channel.

When dealing with closings φ, for example, it is always possible to build clean operators. It suffices to take the restriction of $\varphi(a)$ to those of its components which are larger than a component of a. This restriction, clean by construction, is still a closing. One will notice also that a *same* operation on L may be clean or not according to the connection to which it refers. Coming back to the closings, we may state the following proposition.

Proposition 15 *Let C be a connection on an infinite $\bigvee$-distributive lattice L, φ be a closing and γ be a reconstruction opening, both on L. If φ is clean, then*

$$\gamma\varphi\gamma = \varphi\gamma \quad \text{or equivalently} \quad \gamma\varphi \geq \varphi\gamma \quad \text{or again} \quad \varphi\gamma\varphi = \varphi\gamma \tag{10}$$

Proof. Since γ is extensive, relation (9) of cleanness of φ may be rewritten as follows

$$\gamma_x\varphi(a) \neq 0 \qquad \Rightarrow \qquad \exists y : y \leq \gamma_y(a) \leq \gamma_x\varphi(a) = \gamma_y\varphi(a)$$

Take $\gamma(b)$, $b \in L$, for element a, and compose both sides of the last inequality by γ_T, we obtain

$$\gamma_T\gamma_y\gamma(b) \leq \gamma_T\gamma_x\varphi\gamma(b) \tag{11}$$

Now by idempotence of γ, we have $\gamma(b) = \gamma^2(b) = \bigvee_z \gamma_T\gamma_z\gamma(b)$; by infinite $\bigvee$-distributivity $\gamma_y\gamma(b) = \gamma_y\gamma(b) \bigwedge \gamma(b)$ gives

$$\gamma_y\gamma(b) = \gamma_y\gamma(b) \bigwedge \left[\bigvee_z \gamma_T\gamma_z\gamma(b)\right] = \bigvee_z \left(\gamma_y\gamma(b) \bigwedge \gamma_T\gamma_z\gamma(b)\right)$$

and by axioms of connected openings, this gives

$$\gamma_y\gamma(b) = \gamma_y\gamma(b) \bigwedge \gamma_T\gamma_y\gamma(b)$$

that is $\gamma_y \gamma\,(b) = \gamma_T \gamma_y \gamma_b \neq 0$. Therefore the right member of (11) is non zero, i.e.

$$\gamma_T \gamma_x \varphi\gamma\,(b) = \gamma_x \varphi\gamma\,(b)$$

hence $\gamma\varphi\gamma\,(b) = \bigvee \{\gamma_T \gamma_x \varphi\gamma\,(b)\,,\, x \in \varphi\gamma(b)\} = \bigvee \gamma_x \varphi\gamma\,(b) = \varphi\gamma(b)$. Finally, according to criterion 6.6 in [35], $\gamma\varphi\gamma = \varphi\gamma$ is equivalent to $\varphi\gamma\varphi = \gamma\varphi$, and to $\gamma\varphi \geq \varphi\gamma$. Q.E.D.

This proposition governs the semi-group structure of the alternating sequential filters M_i and N_i based on reconstruction granulometries $\{\gamma_i\}$and clean anti-granulometries $\{\varphi_i\}$. One knows [9, ch. 10] that given a granulometry $\{\gamma_i\}$ and an anti-granulometry $\{\varphi_i\}$, both operators

$$\mu_i = \gamma_i \varphi_i \quad \text{and} \quad \nu_i = \varphi_i \gamma_i$$

are filters, and satisfy the inequalities [33, p. 205]

$$j \geq i \Rightarrow \quad \mu_j\, \mu_i \leq \mu_j \quad \mu_i\, \mu_j \geq \mu_j \quad ; \quad \nu_j\, \nu_i \geq \nu_j \quad \nu_i\, \nu_j \leq \nu_j$$

Are also morphological filters the composition products :

$$M_i = \mu_i\, \mu_{i-1} .... \mu_1 \quad ; \quad N_i = \nu_i \nu_{i-1} .... \nu_1 \ .$$

The latter are called alternating sequential filters. They satisfy the absorption law [33, p. 208][24]:

$$\begin{array}{rrcllll} j \geq i & \Rightarrow \quad M_j M_i & = & M_j & \quad \text{and} \quad & M_i M_j \geq M_j \\ & N_j N_i & = & N_j & \quad \text{and} \quad & N_i N_j \leq N_j \end{array} \tag{12}$$

In the present situation these properties are reinforced by the following ones

Proposition 16 *Let C be a connection on an infinite $\bigvee$-distributive lattice L, $\{\gamma_i\}$ be a granulometry by reconstruction on L, and $\{\varphi_i\}$ be an anti-granulometry of clean closings. Then we have $\mu_i \geq \nu_i$ and*

$$j \geq i \quad \textit{implies} \quad \mu_i \mu_j = \mu_j \quad \textit{and} \quad \nu_i \nu_j = \nu_j$$

and the alternating sequential filters M_i and N_i satisfy a semi-group structure of law

$$M_i M_j = M_j M_i = M_{\max(i,j)} \qquad (\textit{id. for } N_i) \tag{13}$$

Proof. We showed in Proposition 15 that for every i, j we have $\gamma_j \varphi_i \geq \varphi_i \gamma_j$; in particular for $j = i$ this gives $\mu_i \geq \nu_i$. We have then for $j \geq i$

$$\gamma_j \varphi_j = \gamma_i \gamma_j \varphi_i \varphi_j \geq \gamma_i \varphi_i \gamma_j \varphi_j \geq \gamma_j \varphi_j$$

that is $\gamma_j \varphi_j = \gamma_i \varphi_i \gamma_j \varphi_j$, in other words $\mu_j = \mu_i \mu_j$. Similarly, $\nu_j = \nu_i \nu_j$. Combining this with (12), (13) follows. Q.E.D.

Proposition 16 has been stated for discrete increments i. When the φ_i's and γ_i's are $\downarrow$continuous, it extends to real positive parameters by using the technique developed in [9, Chap. 10]. Physically speaking, the semi-group gives a meaning to the notion of slice $[i, j]$, or of *power spectrum*, in the sense that if $i \leq k \leq j$, then filtering M_k is not affected by smaller filterings $(M_i M_k = M_k)$ and does not modify larger ones $(M_j M_k = M_j)$.

3.4 Connected filters on $\mathcal{P}(E)$

In this section, we restrict ourselves to lattices $L = \mathcal{P}(E)$, where E is an arbitrary space. The properties we aim at finding, such as the strength for filters, extend from the sets of E to the functions $f : E \to T$ where T is an arbitrary lattice. Indeed, as soon as they are satisfied on the cross sections of such functions, they are also true for the corresponding flat operators.

When $\mathcal{P}(E)$ is equipped with a connection $\mathcal{C}$, every set $A \subseteq E$ forms a partition of E into its own connected components, called the grains, and those of its complement A^c, called the pores. This partition is not completely free, for two grains, as well as two pores, cannot be too close from each other. For example, if two connected components are described as *adjacent* when they are disjoint but admit a connected union, then neither two grains nor two pores can be adjacent (each of them is a maximum element) [26]. Below we will use adjacency in the case of *connected filters*, which are introduced as follows:

Definition 7 *An operator $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ is said to be connected for connection $\mathcal{C}$ when for each $A \in \mathcal{P}(E)$, the partition associated to $\psi(A)$ is coarser than that of A. In other words, we have for any $x \in E$:*

$$\gamma_x \psi(A) = [\cup \gamma_i(A),\ i \in I] \cup [\gamma_j(A^c),\ j \in J] \tag{14}$$

for some families of points $I \subseteq A$ and $J \subseteq A^c$. In particular when ψ is increasing and idempotent (i.e. a morphological filter), one says that ψ is a connected filter.

The connected filters turn out to be the corner stone in a number of seg mentation problems. The term, and the above definition, have been introduced by Serra and Salembier [10], but some of these operators go back to the beginning of the 80's (e.g. the geodesic operations in [6] or the reconstruction openings in [4]). The more recent developments on *grain operators*, due to H. Heijmans [41] belong to this class of notions.

Proposition 17 *Consider a closing by reconstruction φ_1 on $\mathcal{P}(E)$ which is clean for a connection $\mathcal{C}$ and similarly, an opening by reconstruction γ_1 on $\mathcal{P}(E)$ which is the dual of a clean closing . Then, the alternating filters $\gamma_1\varphi_1$ and $\varphi_1\gamma_1$ and the derived sequential alternating filters are also clean for connection $\mathcal{C}$.*

Proof. Saying that point $x \in \gamma_1\varphi_1(A)$ is equivalent to saying:

1/ $x \in \varphi_1(A)$, which implies $A \cap \gamma_x\varphi_1(A) \neq \emptyset$ (φ_1 clean)

2/ the grain of $\varphi_1(A)$ marked by x is not eliminated by γ_1, i.e.:

$$\gamma_x\varphi_1(A) = \gamma_1\gamma_x\varphi_1(A) = \gamma_x\gamma_1\varphi_1(A).$$

By combining the two statements, we have

$$x \in \gamma_1\varphi_1(A) \Rightarrow A \cap \gamma_x\gamma_1\varphi_1(A) \neq \emptyset$$

(Similar proof, by duality, for $\varphi_1\gamma_1$).

Consider now two pairs $\{\varphi_1, \gamma_1\}$ and $\{\varphi_2, \gamma_2\}$ of such filters, with $\varphi_2 \supseteq \varphi_1$ and $\gamma_2 \subseteq \gamma_1$. A point x belongs to $\gamma_2\varphi_2\gamma_1\varphi_1(A)$ iff x belongs to a grain of $\varphi_2\gamma_1\varphi_1(A)$ which is not suppressed by γ_2. Now, $\gamma_x\varphi_2\gamma_1\varphi_1(A)$ contains at least one grain of $\varphi_1(A)$ since φ_2 is clean. Let it be $\gamma_z\varphi_1(A)$. Since φ_1 is clean, we have $\gamma_z\varphi_1(A) \cap A \neq \emptyset$, hence $\gamma_x\varphi_2\gamma_1\varphi_1(A) \cap A \neq \emptyset$; and since $\gamma_x\varphi_2\gamma_1\varphi_1(A) = \gamma_x\gamma_2\varphi_2\gamma_1\varphi_1(A)$, the proof is achieved. (Similar induction when more than two indices). Q.E.D.

Remind that a morphological filter is defined as strong [35] when

$$\psi = \psi\left(I \bigvee \psi\right) = \psi\left(I \bigwedge \psi\right)$$

or, equivalently, when for any $A, B \subseteq E$, such that

$$A \cap \psi(A) \subseteq B \subseteq A \cup \psi(A)$$

we have $\psi(A) = \psi(B)$. This second formulation reveals the exceptional robustness of the strong filters, which has no equivalent in the domain of linear filtering. The following theorem relates the cleanness of some filters by reconstruction with their strength.

Theorem 5 *Let $\mathcal{C}$ be a connection on a lattice $\mathcal{P}(E)$, $\{\gamma_i\}$ be a granulometry, and $\{\varphi_i\}$ be an anti-granulometry both clean and by reconstruction. Then the alternating sequential filters M_n and N_n, of primitives $\{\gamma_i\}$ and $\{\varphi_i\}$ are strong.*

Proof. According to proposition 17, filter M_n is a clean operator. Hence, every grain $\gamma_x M_n(A)$ contains a non empty collection $\{G_j\}$ of grains of A, plus possible pores of A, say $\{P_k\}$. If there are no pores, then clearly $M_n [A \cap \gamma_x M_n(A)] = M_n \gamma_x M_n(A) = \gamma_x M_n(A)$.

Suppose that family $\{P_k\}$ is non empty. Since each pore P_k is adjacent to a grain G_j, each time a P_k disappears under the action of some $\varphi_i (1 \leq i \leq n)$, it is absorbed by a G_j. In turn, this G_j may vanish, but it will be permanently reconstituted by a further φ_i, since the internal elements of $M_n(A)$ evolve independently of the outside ones, and precisely lead to the unique grain $\gamma_x M_n(A)$. Therefore, in any case we obtain, finally:

$$M_n (A \cap \gamma_x M_n(A)) = \gamma_x M_n(A)$$

By summing up in x over $M_n(A)$, one can write:

$$M_n(A \cap M_n(A)) = M_n [\cup (A \cap \gamma_x M_n(A))] \supseteq \cup M_n(A \cap \gamma_x M_n(A)) = M_n(A).$$

One proves similarly that $N_n(A \cap N_n(A)) \supseteq N_n(A)$; thus the proof is achieved. Q.E.D.

Theorem 5 gives a better understanding of the connected filters, and also of some undesirable connections. For example, it rejects the closing by reconstruction based on the above connection (f), of section 2-3:

$$\mathcal{C} = \left\{x,\ x \in \mathbb{R}^2\right\} \cup \{A \oplus B,\ A \subseteq E,\quad A \oplus B \text{ arcwise connected component}\}$$

where B is the closed unit disc. Such connections cannot serve for building strong alternating filters because they may generate point grains without adjacent pores (see fig. 5).

The general result of theorem 5 admits weaker forms. In particular, if the closure φ_i's are clean, but not necessarily connected, then M_n is a $\bigwedge$-filter and N_n a $\bigvee$-filter. Also, the theorem is still true when the φ_i's are clean and connected, but with respect to a connection $\mathcal{C}' \subseteq \mathcal{C}$, where $\mathcal{C}$ is the connection associated with the γ_i's.

Another result of interest, that we give here without proof is the following. Take a granulometry $\{\gamma_\lambda \ , \ \lambda > 0\}$ of clean openings by reconstruction (in connection $\mathcal{C}$) and an anti-granulometry $\{\varphi'_\lambda \ , \ \lambda > 0\}$ of clean closings by reconstruction (in connection $\mathcal{C}'$). If $\mathcal{C}' \subseteq \mathcal{C}$, then $\bigwedge \left\{\gamma_\mu \varphi'_\mu \ , \ 0 < \mu \leq \lambda\right\}$ and $\bigvee \left\{\varphi'_\mu \, \gamma_\mu \ , \ 0 < \mu \leq \lambda\right\}$ are strong filters. This result extends a beautiful theorem established by J. Crespo in [26], when $\mathcal{C} = \mathcal{C}'$.

This impressive collection of strong filters is valid, not only for the set oriented case, but also for the filters which are derived from them for the functions $f : E \rightarrow L$ (L arbitrary lattice), via the cross sections. And the extension does not require the need of any connectivity over the functions $f : E \rightarrow L$.

4 Application to lattices of numerical functions

4.1 Numerical lattices of numerical functions

This section is a search for nice connections and connected operators, in the important case of numerical functions. The arrival space is thus a totally ordered set. We shall suppose it either finite, or isomorphic to $\overline{\mathbb{R}}$ or to $\overline{\mathbb{Z}}$ (e.g. $[0, 1]$, $[0, +\infty]$, etc.). We will denote L this numerical lattice. Concerning the starting space E, we assume that it is metric, with distance d, and that $\mathcal{P}(E)$ is equipped with connection $\mathcal{C}$. These definition and properties, about L and E, are valid for the **whole section 4**.

The set L^E of all mappings f from E into L, when provided with the product order :

$$f \leq g \qquad \text{iff} \qquad f(x) \leq g(x) \qquad \text{for all } x \in E,$$

becomes in turn a complete lattice, where the sup and the inf are defined by the relations

$$(\bigvee f_i)(x) = \bigvee f_i(x) \qquad (\bigwedge f_i)(x) = \bigwedge f_i(x) \tag{15}$$

(we keep the same symbols for ordering, supremum and infinum in both lattices L and L^E). The *support* of a function is the set of points x where $f(x) > 0$ (strictly). The support of function $f = 0$ is the empty set; that of a pulse is reduced to one point.

We are often more interested in some sublattices of L^E, rather than in L^E itself. A sublattice $L' \subseteq L^E$ is a class of functions which is closed under $\bigvee$ and $\bigwedge$ of L^E and which admits the same extrema as L^E itself. For example, the Lipschitz functions L_k of module k are defined by

$$f \in L_k \qquad \Leftrightarrow \qquad |f(x) - f(y)| < kd(x,y) \qquad \forall\, x, y \in E \quad .$$

It can be proved [1][30] that class L_k is a complete sublattice of L^E. More generally, if we replace $kd(x,y)$ by $\varphi(d(x,y))$ in the above inequality, were $\varphi : \mathbb{R}_+ \to \overline{\mathbb{R}_+}$ is continuous at the origin, we delineate the class of the so-called "φ-continuous functions" [1]. For each φ, the associated φ-continuous functions form a complete sub-lattice of L^E, i.e. a lattice where $\bigvee$ and $\bigwedge$ are generated by the *numerical* supremum and infimum at each point. It has been proved that under broad conditions, usual operators, such as dilations, openings, morphological filters, etc. map every φ-continuous lattice into another one [40], and that $\bigvee$ and $\bigwedge$ are continuous operators [30]. Moreover, every function lattice where $\bigvee$ and $\bigwedge$ are generated by the numerical ones is totally distributive. As a consequence, theorem (4) on second generation connections, or prop. (16), on semi-groups of filters do apply to them. We shall call *numerical lattices* (of numerical functions) these types of lattices.

In contrast, in the same context, the class $\mathcal{F}$ of the upper semi-continuous functions forms a complete lattice, where ordering and infinum are the same as for L^E, but where the supremum of $\{f_i\}$is the closure of $\bigvee f_i$. Therefore, $\mathcal{F}$ is not a sublattice of L^E, and does not satify the conditions for theorem (4) and prop. (16).

4.2 Two straightforward connections

For providing a connection with L^E, the first straightforward approach consists in extending to L^E the connectivity associated with E, by taking as connected functions (in L^E) those with a connected support (in E). This class comprises function $f = 0$ and the pulses, and satisfies axiom *(ii)*. But it is rather coarse: applied to visual scenes, it is likely to attribute them a single component almost always (except perhaps for the stars in a sky scenery, at midnight). In addition, the connected components $\gamma_x (X_t (f))$, $t \in L$, of the horizontal cross-section of f generate a function which completely differs from the connected component of f at point x.

The second straightforward approach takes as a connected class all pulses plus function $f = 0$. The three axioms of a connection are obviously satisfied. Then, for any pulse located at point x and of intensity $t \leq f(x)$, the associated connected component of f is the pulse $(x, f(x))$.

Although they are formally correct, as connections, both miss the target of being a shape and features descriptor.

4.3 Weighted partitions

In definition 4 of a partition, as given before, if we take $\mathcal{P}(E)$ for lattice L, set E for element a, and the connection reduced to all points of E, we meet again the classical definition of a partition D [9, p. 15], namely D is a mapping from E into $\mathcal{P}(E)$ such that

$$\begin{array}{llll} \forall & x \in E, & x \in D(x) & \\ \forall & x, y \in E, & \text{if} \quad D(x) \cap D(y) \neq \emptyset & \text{then} \quad D(x) = D(y) \end{array}$$

Partitions divide the space, but, by themselves, they do not tell anything about possible parameters, or numerical entities, which may be associated with their classes. To overcome this aspect, we will introduce now the notions of a *weighted set* and of a *weighted partition*, in the framework of a given (but arbitrary) numerical sublattice G_φ of L^E.

Given a modulus φ, associate, with every pair $(D, g) \in \mathcal{P}(E) \times G_\varphi$, the restriction g_D of φ-function g to set D, i.e.

$$\begin{aligned} g_D(u) &= g(u) \quad \text{when} \quad u \in D \\ g_D(u) &= 0 \qquad \text{when} \quad \text{not} \end{aligned}$$

Thus, the usual indicator function of set D is replaced by the non constant weight g. The numerical function g_D may be considered as a weighted set. As the doublet (D, g) spans $\mathcal{P}(E) \times G_\varphi$, the set generated by the g_D 's is denoted by $\mathcal{P}_\varphi(E)$.

Proposition 18 *The set $\mathcal{P}_\varphi(E)$ forms a complete lattice for the numerical ordering $\leq$; in this lattice the supremum $\sqcup (g_D)_i$ of family $\{(g_D)_i\}, i \in I$, is the smallest upper bound of $\bigvee (g_D)_i$ in G_φ and the infimum is given by $\sqcap (g_D)_i = (\bigwedge g_i)_{\cap D_i}$.*

[easy proof]

The notion of a weighted partition derives from that of a weighted set just as a usual partition from that of a set

Definition 8 *A weighted partition $x \to (g_D)_x$ is a mapping Δ from E into $\mathcal{P}_\varphi(E)$ such that*
(i) $\forall x \in E, \quad x \in D(x)$
(ii) $\forall x, y \in E,$ *either* $(g_D)_x = (g_D)_y$ *or* $(g_D)_x \bigwedge (g_D)_y = 0$

The sub-mapping $x \to (g_D)_x$ is a partition D in the usual sense, and $f = \bigvee \{ (g_D)_x , x \in E\}$ is a numerical function. Therefore, one can denote the weighted partition Δ by $\Delta = (D, f)$. However, conversely, any function $f : E \to T$ admits several representations as $\bigvee (g_D)_x$. For example, for $\varphi = 0$, any subset of the flat zone of f that contains a given point x may serve as class $D(x)$ jointly with the constant function $g = f(x)$.

Theorem 6 *Let G_φ be a numerical lattice of functions $E \to L$ and $\mathcal{L}$ the class of the weighted partitions of E, of base G. Then $\mathcal{L}$ is a complete lattice for the product ordering*

$$(f, D) \sqsubseteq (f', D') \Leftrightarrow \begin{array}{ll} f \leq f' & \text{in } L^E \\ D \leq D' & \text{in } \mathcal{D} \end{array}$$

In this lattice, the infimum $\sqcap(f_i, D_i)$ of a family $\{f_i, D_i, i \in I\}$ is given by the doublet $\left(\bigwedge f_i \left(\text{in } L^E\right), \bigwedge D_i \left(\text{in } \mathcal{D}\right)\right)$. The supremum $\sqcup(f_i, D_i)$ admits $\bigvee D_i$ for partition, and in each class $\left(\bigvee D_i\right)(x)$, is equal to the smallest function of lattice G which is above the f_i involved in $\left(\bigvee D_i\right)(x)$.

Proof. Concerning the infimum $\sqcap$, we observe that partition $\bigwedge D_i$ is defined by the intersections of the classes of the D_i's. In each class of this intersection, all f_i are assimilable to elements of G_φ. Since the inf in lattice G_φ is the numerical one, the restriction of $\bigwedge f_i$ to each class of $\bigwedge D_i$ is assimilable to an element of G_φ. Concerning the supremum, we observe that its smallest possible partition is $\bigvee D_i$. Now, in every class $\left(\bigvee D_i\right)(x)$, there exists a smaller element of G_φ that is above $\bigvee f_i$, and since G_φ admits a pointwise supremum, this smaller element depends only on the f_i that intervene on class $(\bigvee D_i)(x)$. Q.E.D.

The role of the pointwise supremum (or infimum) will be better understood by means of a counterexample. Consider in $\mathbb{R}^1$, the family of functions $\{f_r\ , r > 0\}$, which are equal to 0 in $]-r, +r[$ and to 1 outside. Take for partition D_r all the points of $]-r, +r[$ plus the outside (as a unique class). At point $x = 0$, the supremum $\bigvee f_r$ has a value 0. If we interpret functions $f = 1$ and $f = 0$ as elements of the lattice G of the constant functions, then $\sqcup (f_r, D_r)$ equals 0 at point $x = 0$. However, if both functions $f = 1$ and $f = 0$ are interpreted as s.c.s. functions, then $\sqcup (f_r, D_r)$ equals 1 at point $x = 0$: theorem 6 is no longer applicable.

In order to build connections on the weighted partitions lattice $\mathcal{L}$, a convenient class is provided by the G-cylinders.

Proposition 19 ***Definition 9*** *Let G be a numerical lattice of L^E, and $\mathcal{C}$ a connected class on $\mathcal{P}(E)$. For any $Y \in \mathcal{C}$ and $g \in G$, the weighted partition of function*

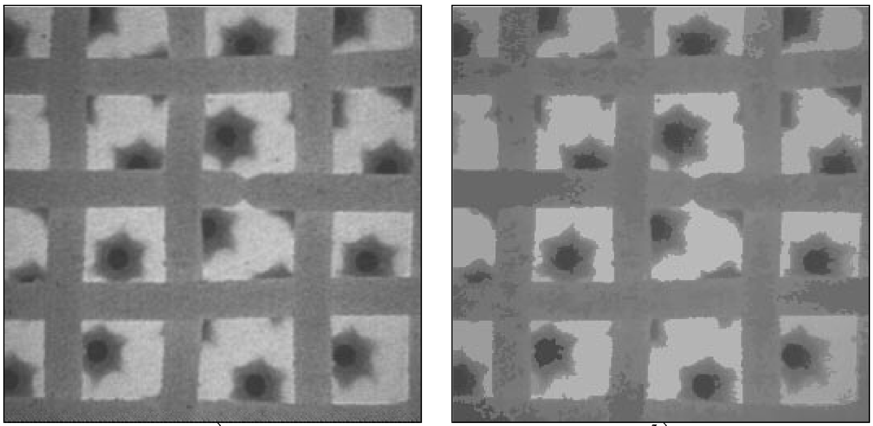
$$\begin{array}{rcll} h_{y,g}(y) & = & g(y) & \text{when } \ y \in Y \\ h_{y,g}(y) & = & 0 & \text{when } \ \text{not} \end{array}$$

and of partition classes Y and $\{z\}$ (for $z \in Y^c$) defines the G-cylinder of (connected) base Y and of values g.

In particular the pulses, equipped with the minimum partition d, are G-cylinders for all lattices G.

Consider now the supremum in $\mathcal{L}$ of a family $\{h_i = h_{y_i,g_i}, i \in I\}$ of G-cylinders, whose infimum is not zero. Since $\cap Y_i$ is not empty $\cup Y_i$ is connected. Now the partition whose classes are $\cup Y_i$ and all the points of $(\cup Y_i)^c$, is nothing but $\bigvee D_{h_i}$. By adjoining the function equal to the smaller G-bound of the h_i on $\cup Y_i$, and equal to 0 elsewhere, we obtain the supremum in $\mathcal{L}$ of the h_i. Since this weighted partition is itself a G-cylinder and since the G-cylinders form a sup-generating family (they contain the pulse cylinders) we may state

Theorem 7 *Let G be a numerical sublattice of L^E. The class C_g of the G-cylinders, plus function 0, constitute a connection on lattice $\mathcal{L}$.*



a) *b)*

Figure 6 a) Infrared image of a gaz burnern partly covered by a grid; b) Mosaic image of connected regions in which the grey fluctuation does not exceed 24 levels

The approach which has been developed here recovers as particular cases, the two straightforward connections presented at the beginning of the section. But it opens the way to a number of other connections, which turn out to be extremely pertinent descriptors of the images. For example, take for G the class G_1 of the numerical φ-continuous functions whose modulus reaches range h at a finite distance, d_0 say. Each function of class G varies between two bounds from h apart. Therefore, by applying base G to a numerical function f, we obtain a segmentation of f into connceted zones where the grey fluctuation is $\leq h$. Fig. 6 illustrates such a "jump connection" in the digital framework by taking $d_0 = 1$ and $h = 24$(between two neighbour points, the function cannot jump by more than 24 levels).

Another interesting use of theorem 7 is the search for continuous zones in a numerical function. Take for G the class G_2 the functions whose modulus is $\leq \alpha\rho^2$, until $\rho = \rho_0$, and is linear afterwards. Such a base allows to segment numerical functions into zones in which the variation is continuous, in the sense "smaller than $\alpha\rho^2$, for distances $\leq \rho_0$". To illustrate this continuous base G, we will take (fig. 7a) an image where one phase is relatively more continuous than the others. But the histograms of the grey tones, taken from the two phases are almost the same. Indeed, the segmentation of fig. 7a according tothe above jump connection with $h = 12$, leads to the mediocre result fig. 7b, whereas base G_2 allows a better, but not perfect, segmentation of the "flat spots" (fig. 7c).Now by taking the the inf of these two connections, which corresponds to class $G = G_1 \cap G_2$, we obtain the more satisfacory result of fig 7d.

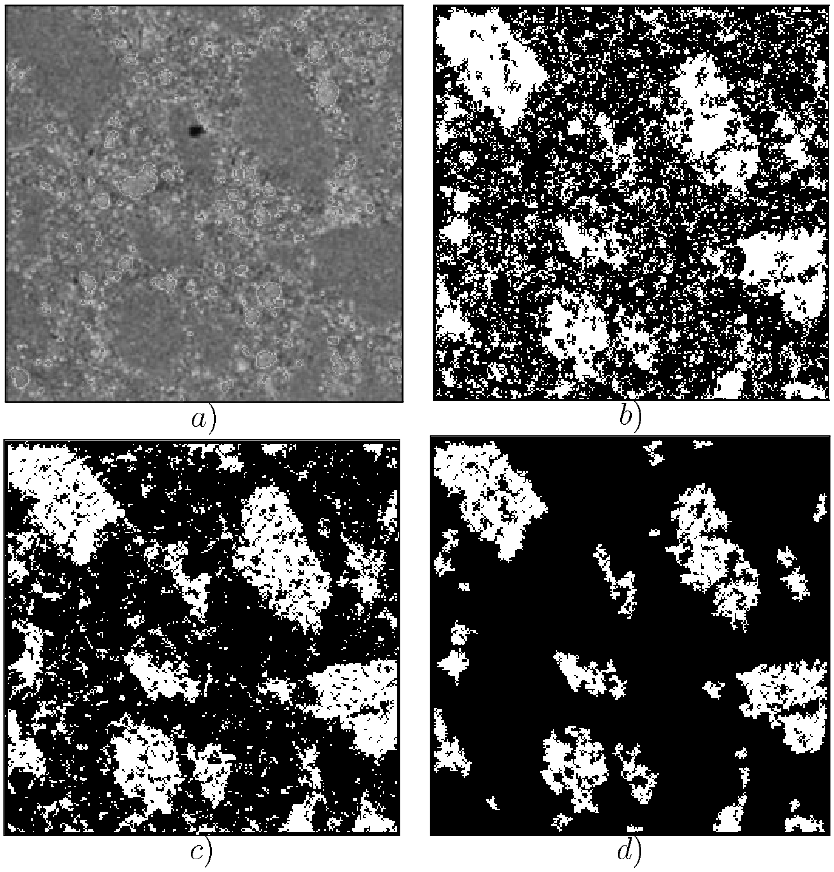


Figure 7 a) Electron micrograph of minerals; b) in white, the components where the fuctuation is ≤ 13 *(poor selectivity) ; c) in white, the components where the function is continuous d) in white, the components for the infimum of the two previous connections (which turns out to be a new connection)*

Other similar illustrations could easily be provided (think for example to Lipschitz functions for class G, or to colour image lattices). We will rather conclude this section by taking for G the class of the *constant functions.*

Proposition 20 *When the numerical lattice G is the class of the constant functions from E into L, then the associated connection on $\mathcal{L}$ is formed by* *the zero function and the cylinders (stricto sensu) of connected bases, i.e. by functions*

$$\begin{array}{ll} h_{c,t}(t) = t & \textit{when} \quad y \in C \\ h_{c,t}(t) = 0 & \textit{when} \quad \textit{not} \end{array}$$

$t \in L$, $C \in \mathcal{C}$.

The operations $\sqcup$ and $\sqcap$ on the lattice $\mathcal{L}$ associated with the constant functions are illustrated in fig. 8. Fig. 9 shows the cylindric supremum of two cylinders, and also that the notion of a connection does not extend by duality.

In the lattice $\mathcal{L}$ associated to constant functions, the openings by reconstruction suppress cylinders according to some criteria. The results look like sky scrapers on a flat plain. However, in the present situation, since we deal with flat zones, we can alternatively introduce openings and closings by reconstruction, based on *set* connections. These set operators extend to *flat* operators for the function of lattice $\mathcal{L}$, as well as their nice filtering properties, such as propositions 15 and 16 for example [10][26].

Such a favourable situation may occur only because the underlying connection is based on the constant functions G. One can note also that the passage from connected set operators to flat operators on numerical functions do not really require increasingness for the said operations. Salembier and Oliveras [27] on the one hand, and Breen and Jones, on the other hand [14] have developed this point.

5 Hyperconnections, hypoconnections

5.1 Hyperconnections

Given a metric space E, the watersheds of a numerical function $f : E \to \overline{\mathbb{R}}$ partition space E, and the restrictions of function f to each class of the partition, namely the catchment basins, have a flavor of connected objects. Can we transcribe this intuition by means of the above concepts? If we want that the connectivity concept extends to notions such as the watersheds, we need to relax axiom *(ii)* in the definition 3 of a connection. Instead of the condition that the $\bigwedge$ is not empty, we will replace $\bigwedge$ by another increasing mapping $\perp$ from $\mathcal{P}(L)$ into L, which is smaller than $\bigwedge$, i.e. we will introduce a more general axiom by putting *(ii) To constraint* $\perp$, class C *is conditionally closed under the supremum*

$$X \subseteq C \quad , \quad \perp X \neq 0 \quad \Rightarrow \quad \bigvee X \in C \tag{16}$$

Note that every connection may be considered as a hyperconnection, but the contrary is false. It is easy to check that axiom *(iv)*, in theorem 3, which characterizes the connection by means of openings, no longer applies. This major feature makes the difference between this broader approach and that of definition 3 : in the present case, all the properties involving partitions, (those which are grouped in sect. 2.4) do vanish.

We will now use the notion of a hyperconnection to try and define formally the watersheds of a relief. In order to be consistent with the role of $\bigvee$ in (eq. 16), we consider the watersheds of the maxima, and not of the *minima*, as is usual in practice. To be sure of the existence of such maxima, we model the reliefs as upper semi-continuous functions f from a compact zone K of the Euclidean space $\mathbb{R}^d$ into $[0, \infty]$. These functions are structured in a complete lattice $\mathcal{F}$, say, for the usual ordering [9]. Denote $X(t) = \{x : f(x) \geq t\}$ the cross sections of f, and consider the class W of those functions $f \in \mathcal{F}$ which admit a unique maximal connected component, called "maximum". The horizontal cross sections of functions $f \in W$ are either connected, for the usual arcwise connectivity, or empty. Therefore, given a family $\{w_i\, , i \in I\}$ in W, we have

$$\forall t \in [0, \infty]\, , \quad (i \cap X_i(t) \neq \emptyset \quad \Rightarrow \quad i \cup X_i(t) \text{ is connected}) \tag{17}$$

and

$$(\forall t \in [0, \infty]\, , \quad i \cup X_i(t) \text{ is connected}) \;\Rightarrow\; i \bigvee w_i \in W \tag{18}$$

We observe, in addition, that function $f = 0$ and all pulses belong to W, and that $\forall t \in [0, \infty]\, , \cap X_i(t) \neq \emptyset$ under the previous conditions is more demanding than $\bigvee w_i \neq 0$. Hence, if we take the two conditions (17) and (18) as constraint $\perp$, we can state.

Proposition 21 *In the lattice* $\mathcal{F}$ *of the u.s.c. function* $f : K \to [0, \infty]$*, those that admit a unique maximum generate a hyperconnection.*

Since W is a hyperconnection, the connected components w_i, $i \in I$ of a given function f may overlap. But we can restrict, conventionally, w_i to its

connected part wi
* which reaches the i-th maximum and which is higher than the other w_j of f , $j \neq i$.

Then the wi
* are nothing but the usual catchment basins of f, and their supports are the watersheds, whose complement of the union generates the divide *zones*. Note that these zones may be thick ; for $E = \mathbb{R}^2$ for example, they are not reduced to lines, as soon as a large flat zone separates two catchment basins. Also, the mapping $f \to w_i$ is an opening (unlike $f \to w_i^*$), and this may allow to construct watershed based filters.

One may remark the simplicity and the generality of the present approach. First of all it is formal, and not a concealed algorithm, as several definitions for the watershed. Second, it does not assume derivability conditions for functions f (as in Najman and Schmitt approach [38]), neither a metric on E (as in Meyer approach [39]), and it treats the question of the flat stairs, that all other formal approaches have avoided. Finally, it generalizes the notion introduced by Salembier and Oliveras [27] for the distance functions, under the name of "pseudo-connectivities".

5.2 Hypoconnections

Instead of relaxing axiom *(ii)* of a connection, we can also think of relaxing axiom *(i)*. This may be useful in particular when we deal with lattices of mappings, such as the lattice L' of the increasing transformations ψ from an initial lattice L into itself. In this kind of lattices, the condition for a connection to be sup-generating may be too demanding. We will see how to weaken it, and, as a counterpart, what do we lose by weakening it. Subdivide the two roles devoted to a connection, namely to serve as markers, and to be conditionally closed under sup, by dispatching them to the two different classes of *markers* and of a *hypoconnection*.

Definition 10 *Let L be a complete lattice. A class $M \subseteq L$ is said to be* marking *when $0 \notin M$ and when every non zero element $A \in L$ contains at least one x, $x \in M$. The elements of M are called* markers.

The marker class M, sufficient to point on every element of lattice L, opens the way to the notion of a hypoconnection.

Definition 11 *Let L be a complete lattice, and M be a marking set on L. A hypoconnection C over L is a class in L such that*

(o) $0 \in C$

(i) C contains the markers : $M \subseteq C$

(ii) for each family $\{C_i\}$ in C, if $\bigwedge C_i \neq 0$ then $\bigvee C_i \in C$.

Note that a connection C is a hypoconnection with C as a marking set. Just as in the case of connections, class C is associated with a characteristic opening family $\{\gamma_x\}$. More precisely, we have the following theorem

Theorem 8 *The datum of a hyperconnection C, of markers M, is equivalent to the family of openings γ_x, $x \in M$, such that (iii) for all $x \in M$ we have $\gamma_x(x) = x$*

(iv) for all $a \in L$, and all $x, y \in M$, the open elements $\gamma_x(a)$ and $\gamma_y(a)$ are equal or disjoint,

(v) for all $a \in L$ and for all $x \in M$, we have $x \not\leq a \Rightarrow \gamma_x(a) = 0$.

We find again theorem 2 of the connections. Similarly, corollary 1 and proposition 3 extend to hypoconnections. The trouble comes with the partitioning properties, that now fail. An element of L *cannot always be segmented* into its hypocomponents. This negative property also affects the second generation and the filtering (above sections 3.2 and 3.3 respectively).

It would remain to establish an inventory of the properties which are still valid, and of their adequation to lattices of operators. We shall not do it here. Nevertheless, the two attempts of hyperconnections and hypoconnections show that the notion of a connection realizes a good compromise between the amount of the assumptions needed and the pertinence of the properties obtained .

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5.3 (suppressed in the published paper)

This last result confronts us with a sort of reverse problem, since what is given, in practice, is only a numerical function f. Some partition(s) D acting on f has (have) to be found. But how ?

Consider a function $f \in L_E$, and join it to the coarse partition of E, in order to generate an element of $\mathcal{L}$. Given $x \in E$, denote by G_x the class of the G-cylinders whose weight $h_{y,g}$ is equal to f over their supports Y, i.e.

$$h_{y,g}(y) = f(y) \qquad y \in Y \, .$$

Class G_x is not empty, since it contains the pulse G-cylinder $((x, f(x)), d)$. Let $\{Y_i\} = \{Y_i(x)\}$ the family of the associated supports. All $Y_i(x)$ contain point x. According to Zorn theorem, each ordered family $\{Y_{i,j}\}$ in $\{Y_i\}$ admits a maximal element Y_j. Denote $D(x)$ the intersection $\cap Y_j$ of all maximal element Y_j associated with family $\{Y_i\}$. We have $x \in D(x)$; moreover if $z \in D(x)$, and if $D(z)$ stands for the homolog maximal intersection for z, then $D(x) \cup D(z)$ is connected, and function f is assimilable to a G function over the support $D(x) \cup D(z)$. Therefore all maximal elements $Y_j(x)$ contain $D(x) \cup D(z)$, as well as the maximal element $Y_j, (z)$, which results in $D(x) = D(z)$. Hence we may state

Theorem 9 *Let $f \in L^E$ be a numerical function, G be a numerical sublattice of L^E, and $\{Y_i(x)\}, x \in E$, be the family of the connected zones containing x, and where f is assimilable to a G-function. Then the intersection $D(x)$ of the maximal element of $\{Y_i(x)\}$ is the class at point x of a partition D which segments f into piecewise G-functions.*

It follows directly from the theorem that the weighted partition (f, D) admits as connected components the G-cylinders of bases D_x and weights: $f(y), y \in D_x$. So theorem 7 provides at once both a partition and the corresponding connected components. The obtained partition is neither optimal nor necessarily unique, but it carries an actual physical soundness.