Connections for sets and functions

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Abstract

Classically, connectivity is a topological notion for sets, often introduced by means of arcs. An algebraic definition, called *connection,* has been proposed by Serra to extend to complete sup-generated lattices. A connection turns out to be characterized by a family of openings parametrized by the sup-generators, which partition each element of the lattice into maximal components. Starting from a first connection several others may be designed, e.g. by applying dilations.

The present paper applies this theory to numerical functions. Every connection leads to *segmenting* the support of the function under study into regions. Inside each region, the function is φ-continuous, for a modulus of continuity φ given *a priori*, and characteristic of the connection. However, the segmentation is not unique, and may be particularized by other considerations (self duality ; large, or low, number of point components, etc.). These variants are introduced by means of examples for three different connections : flat zones, jumps, and smooth regions. They turn out to provide remarkable segmentations, depending only on a few parameters.

In the last section,some morphological filters based on flat zone connection, are described, namely opening by reconstruction, flattenings and levellings. **Keywords :** connectivity, complete lattice, mathematical morphology, connected operator, flattening, leveling, grain operator, segmentation.

1 The concepts of connectivity

1.1 Classical connectivity and image analysis

In mathematics, the concept of connectivity is formalized in the framework of topological spaces and is introduced in two different ways. First, a set is said to be connected when one cannot partition it into two non empty closed (or open) sets [CHO66]. This definition makes precise the intuitive idea that $[0,1] \cup [2,3]$ consists of two pieces, while $[0,1]$ consists of only one. But this first approach, because it is very general, does not derive any advantage from the possible regularity of some spaces, such as the Euclidean ones. In such cases, the notion of *arcwise connectivity* turns out to be more convenient. According to this notion, a set A is connected when, for every $a, b \in A$, there exists a continuous mapping ψ from $[0,1]$ into A such that $\psi(0) = a$ and $\psi(1) = b$. Arcwise connectivity is more restrictive than the general one ; however, in $\mathbb{R}^d$, any open set which is connected in the general sense is also arcwise connected.

A basic result governs the meaning of connectivity ; namely, the union of connected sets whose intersection is not empty is still connected :

$$\{A_i \text{ connected}\} \text{ and } \{\cap A_i \neq \emptyset\} \Rightarrow \{\cup A_i \text{ connected}\} \tag{1}$$

In image analysis, one often defines digital connectivities based on the Euclidean notion of arcwise connectivity, by introducing some elementary arcs between neighboring pixels. This results in the classical 4- and 8-square connectivities, as well as the hexagonal one, or the cuboctahedric one in 3-D spaces. During the seventies, these connectivities have been extensively used to design thinning and thickening operations [ROS70] [DIG78] [SER82].

Is such a topological approach to connectivity well adapted to image analysis? We can argue that

a/ each topology, hence each metric, imposes a *unique* connectivity, so that by changing the unit disc in the digital plane (square, hexagon or diamond) we obtain three different connectivities. However, they all suppose to represent a same and unique Euclidean concept ;

b/ The plane sectioning of 3-D sets, or the time sampling in image sequences, tend to disconnect objects, or trajectories. Topological connectivity, or its digital avatars, does not help very much to reconnect them on sections ;

c/ In applied sciences, a "good" definition should be operating, i.e. should introduce specific operations for image processing, such as segmentations and filterings. The criterion of "connectivity preservation", used in thinnings may be quoted here, but it is a passive one and cannot by itself characterize any connectivity ;

d/ The whole classical approach (i.e. discrete and more general cases) holds on *sets*. In Image Analysis, we also deal with numerical and vector functions, to which a convenient approach should also apply.

1.2 The notion of a connection

The previous three criticisms led G. Matheron and J. Serra to propose a new approach to connectivity, in 1988 [SER88] where they take not (1) as a consequence, but as a starting point.

Definition 1 *(G. Matheron and J. Serra) Let E be an arbitrary non empty space. We call connected class or* connection $\mathcal{C}$ *any family in $\mathcal{P}(E)$ such that*

(o) $\emptyset \in \mathcal{C}$

(i) for all $x \in E$, $\{x\} \in \mathcal{C}$

(ii) for each family $\{C_i\}$ in $\mathcal{C}$, $\cap C_i \neq$ implies $\cup C_i \in \mathcal{C}$.

Based on this definition, any set C of the connected class is said to be connected. In addition, the empty set as well as the singletons $\{x\}, x \in E$ are always connected.

As we can see, such a definition does not involve any topological background. The classical notions (e.g. connectivity based on digital or Euclidean arcs) are indeed particular cases, but the emphasis is put on another aspect, that indicates the following theorem ([SER88], Chap. 2) :

Theorem 2 *The datum of a connected class $\mathcal{C}$ on $\mathcal{P}(E)$ is equivalent to the family $\{\gamma_x, x \in E\}$ of openings such that*
(iii) for all $x \in E$, we have $\gamma_x(x) = \{x\}$
(iv) for all $A \subseteq E$, $x, y \in E$, $\gamma_x(A)$ and $\gamma_y(A)$ are equal or disjoint
(v) for all $A \subseteq E$, and all $x \in E$, we have $x \notin A \Rightarrow \gamma_x(A) = \emptyset$.

In addition, each γ_x , as an opening on $\mathcal{P}(E)$, is an increasing, anti-extensive and idempotent oprator. Fig.1a illustrates Theorem 2 : the set A under study is shown in gray, and its connected component $\gamma_x(A)$ at point x , also called "grain", appears in white. Many other connections could have been defined on the same space, such as

(a) Connections based on extensive dilations (Theorem 10), or on closings (Proposition 11). Both of them act by clustering the grains defined by a previous connection. An example of connection by dilation is given in fig.2a and b, and a connection by closing (but for numerical functions) is presented in fig.10 ;

(b) Given a partition of the space, all the subsets of each class of the partition generate a connected class [SER88]. This technique is used for obtaining the "smooth path connection" in section 5, and is illustrated by fig.8a and b ;

(c) In [RON98], Ch. Ronse starts from a first connection in $\mathcal{P}\left(\mathbb{R}^2\right)$ and proposes a new connected class generated by the points and the connected sets opened by a disc B. If $x \in X \circ B$, then $\gamma_x(X)$ is the initial connected component of $X \circ B$ containing x, and when point $x \in X \backslash X \circ B$, then $\gamma_x(X) = x$. A digital version of this so called "connection by opening" is shown in fig.12a below, in $\mathcal{P}\left(\mathbb{Z}^2\right)$, with B the square of side two.

The reader will find a number of other instructive connections in [RON98]and in [HEI97]. Historically speaking, the number of applications or of theoretical developments which was suggested (and permitted) by Theorem 2 during the nineties is considerable. The references [PAR92] [SER93] [VIN93] [MAR94] [MEY94] [CRE95] [BRE96], among others, show that it has opened the way to an object-oriented approach for segmentation, compression and understanding of still and moving images. They also show how fruitful are the exceptional properties of the connected filters.

But there is a paradox in that. On the one hand connected filtering has proved its efficiency in applications which hold on grey or color images, but on the other hand, the underlying axiomatics for connectivity (i.e. Definition 1 above), is strictly binary. An alternative (and equivalent) axiomatics has been proposed by Ch. Ronse [RON98] ; it contains, as a particular case, another one by R.M. Haralick and L.G. Shapiro [HAR92] ; however, both approaches are still set-oriented.

An extension from sets to the general framework of complete lattices has been proposed by J. Serra [SER98a]. The roundabout way which goes from sets to functions via complete lattices is understandable : one of the tasks was to generalize correctly the role of the points of the set case, and the sup generators, which replace the points, are not, in turn, more generalized when we go from functions to complete lattices.

The present paper aims at studying connections on numerical functions. Such a goal may approach from two points of view. We can enter the scope of the axiomatics developed for complete lattices and specify it for numerical functions. This will be proposed in section 4. Alternatively, we can consider that the above notion of a set connection is powerful enough, as it is, to yield convenient segmentations (section 5) and filters (sections 6 to 8) on numerical functions. But before exploring both ways, we remind briefly, in the second section, the lattice notions needed in the following and, in the third one, we summarize the major results concerning connections on complete lattices.

All proofs involved in sections 3 and 4 can be found in [SER98a], except that of proposition 11, which is new.

2 Reminder on lattices

A common feature to sets and numerical functions is that both satisfy the algebraic structure of a complete lattice. Recall that a complete lattice L is a partly ordered set in which every family $\{a_i : i \in I\}$ of elements admits a least upper bound $\bigvee a_i$ called *supremum* of the a_i's and a greatest lower bound $\bigwedge a_i$ called *infimum* of the a_i's. Both bounds belong to L. In particular, the supremum of all elements of L is called the *unit* and will be denoted by m ; the infimum of all elements of L is called the *zero* and will be denoted by 0.

In this paper, the term "*lattice*" always means "*complete lattice*" and is denoted by the generic symbol L. The elements of L are denoted by small letters, such as x, y, a, b... , whereas capital letters such as X, Y, A, B, ... , denote subsets of L, i.e. elements of $\mathcal{P}(L)$. Such a notation, very practical in the general case, is less suitable for $\mathcal{P}(E)$ lattices. In these cases, the two levels of E and $\mathcal{P}(E)$ are usually distinguished by using small letters for elements of E and capital ones for subsets of E. Therefore, in this case only, we shall introduce letter $\mathcal{C}$ to denote a connection in $\mathcal{P}(E)$. Here are now a few classical definitions.

- A non zero element a of lattice L is an *atom* if $x \leq a$ implies $x = 0$ or $x = a$. For example, when $L = P(E)$, the points of E are atoms in $P(E)$.

- An element $a \in L$, $a \neq 0$ is said to be co-prime when $a \leq x \bigvee y$ implies $a \leq x$ or $a \leq y$ in a non exclusive manner.

- A subset $X \subseteq L$ is called a sup-*generator* when every element $a \in L$ is the supremum of the elements of X that it majorates:

$$a = \bigvee \{x \in X, x \leq a\} \quad .$$

Lattice L is said to be *atomic (resp. co-prime)* when it is generated by a class of atoms (resp. co-prime).

- A subset L' of L is called a *sub-lattice* if it is closed under $\vee$ and $\wedge$ and contains 0 and m

- A lattice L is *complemented* when for every $a \in L$, there exists one b at least such that $a \bigvee b = m$ and $a \bigwedge b = 0$.

Distributivity

Several useful properties involve distributivity, or rather, distributivities. Recall that a lattice L is *distributive* if

$$x \bigwedge \left(y \bigvee z\right) = \left(x \bigwedge y\right) \bigvee \left(x \bigwedge z\right)$$
$$x \bigvee \left(y \bigwedge z\right) = \left(x \bigvee y\right) \bigwedge \left(x \bigvee z\right)$$

for all x, y, $z \in L$. The two equalities are equivalent. When the collection of elements between parentheses is allowed to extend to infinity, i.e. when

$$x \bigwedge (\bigvee y_i \ , i \in I) = \bigvee \{(x \bigwedge y_i) \, , i \in I\} \qquad \text{(infinite } \bigvee\text{-distributivity)}$$
$$x \bigvee (\bigwedge y_i \ , i \in I) = \bigwedge \{(x \bigvee y_i) \, , i \in I\} \qquad \text{(infinite } \bigwedge\text{-distributivity)}$$

for any subset $\{y_i : i \in I\} \subseteq L$ and for $x \in L$, then lattice L is *infinite distributive.*

Basic operators

An operator $\psi : L \to L$ is said to be a *morphological filter* on L when it is increasing :

$$a, b \in L, \quad a \leq b \Rightarrow \psi(A) \subseteq \psi(B)$$

and idempotent :

$$\psi(\psi(a)) = \psi(a).$$

In particular, a filter φ that is extensive, i.e.

$$a \subseteq \varphi(a), \qquad a \in L$$

is called a *closing.* Dually, a filter γ that is anti-extensive, i.e. :

$$\gamma(a) \subseteq a, \qquad a \subseteq L$$

is called an *opening.*

An operator $\delta : L \to L$ that commutes under supremum is called a *dilation*, i.e.

$$\delta(\vee a_i) = \vee \delta(a_i), \qquad i \in I$$

Examples

There are numerous lattices associated with image processing. A series of them will be introduced in section 4, in association with connections for numerical functions. Here, we just recall the three usual ones, which are of interest for the present study, namely sets, functions and partitions.

(1) Consider the lattice $\mathcal{P}(E)$ of the subsets of an arbitrary set E. The points of E are atoms, co-primes and sup-generators of $\mathcal{P}(E)$. The lattice $\mathcal{P}(E)$ is also complemented, and totally distributive: it accumulates all nice features.

(2) Lattice of all functions $f : E \to L$ from an arbitrary space E into a lattice L. The *pulse functions* or *pulses* :

$$\begin{cases} i_{x,t}(y) = t \quad if \quad y = x \\ i_{x,t}(y) = 0 \quad if \quad x \neq y \end{cases} \tag{2}$$

associated with each $x \in E$ and $t \in L$ are sup-generating co-primes but not atoms (except when $L = \{0, 1\}$); this lattice is infinite distributive, but not complemented.

(3) Given an arbitrary set E, a partition D of E is a mapping $D : E \to \mathcal{P}(E)$ such that, for all $x, y \in E$, we have

$x \in D(x)$

if $\quad D(x) \cap D(y) \neq \emptyset \quad$ then $\quad D(x) = D(y)$

The set $D(x)$ is called the *class of point x*. The family $\mathcal{D}$ of all partitions D form a complete lattice [SER88]. A partition D is smaller than partition D' when, for any point x we have $D(x) \subseteq D'(x)$. The infimum of a family $\{D_i\}$ is obtained by taking at each point x the intersection of the classes that contain x, i.e. $D(x) = \cap D_i(x)$. The supremum of the D_i's is the smallest partition D for which each class of each D_i is contained in a class of D (Fig.1b). In particular the coarsest partition has set E itself as a unique component, and the finer one is the pulverization of all points of E into different components.

Lattice $\mathcal{D}$ of the partitions is not distributive and does not admit co-primes.

3 Connections on lattices

We will now introduce the notion of a connection on lattice L. We previously observed that (1), which is a consequence of the topological definition of connectivity, does not involve topology by itself, but exclusively unions and intersections. Hence, by replacing unions and intersections by sup and inf respectively we can try and start from (1) to define a connectivity concept on lattices. In addition, another idea has to be axiomatized, namely that any lattice element can be decomposed into connected components (just as for the setwise case). This results in the following definition.

Definition 3 *Let L be a lattice. A class $C \subseteq L$ is said to be* connected*, or to define a* connection*, when*

(o) $0 \in C$

(i) C is sup-generating: $\forall\, a \in L, \quad a = \bigvee(c : c \leq a,\, c \in C)$

(ii) C is conditionally closed under supremum

$$X \subseteq C \quad , \quad \bigwedge X \neq 0 \quad \Rightarrow \quad \bigvee X \in C \tag{3}$$

Generally, class C is neither closed under $\bigvee$or $\bigwedge$. However, if C_x stands for the subclass of C that contains the majorants of a given $x \in C \setminus \{0\}$, i.e.

$$C_x \ = \{c : x \leq c,\ c \in C\}$$

then, the supremum of each non empty family of elements of C_x is again in C because of *(ii)*. In other words, the class $C_x\ \cup \{0\}$, closed under supremum, characterizes the invariant sets of a unique opening γ_x, which is called the *connected opening* of origin x (fig.1a). For all $x \in C \setminus \{0\}$, we have

$$\gamma_x(a) = \bigvee \{c : c \in C,\ x \leq c \leq a\} \qquad a \in L. \tag{4}$$

We then say that $\gamma_x(a)$ is *the connected component of* a *marked by* x, and that x (which is itself a connected component) is called a *marker*. Clearly for $x, y \in C \setminus \{0\}$ such that $x \leq y$, we have $C_x \supseteq C_y$, hence $\gamma_x \geq \gamma_y$.

Indeed, the family of openings $\{\gamma_x\ ;\ x \in C \setminus \{0\}\}$, induced by the connected class C *characterizes* it, as shown by the following key theorem.

Theorem 4 *Let C be a sup-generator in a lattice L. Class C is a connection if and only if it coincides with the invariant elements of a family $\{\gamma_x\,, x \in C \setminus \{0\}\}$ of openings such that*
(iii) for all $x \in C \setminus \{0\}$, we have $\gamma_x(x) = x$,
(iv) for all $a \in L$, $x, y \in C \setminus \{0\}$, $\gamma_x(a)$ and $\gamma_y(a)$ are equal or disjoint, i.e.
$\gamma_x(a) \bigwedge \gamma_y(a) \neq 0 \Rightarrow \gamma_x(a) = \gamma_y(a)$,
(v) for all $a \in L$ and for all $x \in C \setminus \{0\}$, we have $x \nleq a \Rightarrow \gamma_x(a) = 0$.

3.1 Partitioning

A number of attractive properties of the connected classes come from their ability to *partition*, or to *segment*, each element of L into its components.

We have previously defined partitions for sets. The approach extends to the notion of a partition D_a for an arbitrary element a of lattice L. Denote M^a the family of the minorants of element a

$$M^a = \{x : x \in L,\ x \leq a\}$$

Then a partition of element a is a mapping D_a from $C \cap M^a$ into M^a such that

(i) for all $x \in C \cap M^a : x \leq D_a(x) \leq a$

(ii) for all $x, y \in C \cap M^a : D_a(x) = D_a(y)$ or $D_a(x) \wedge D_a(y) = 0$

$D_a(x)$ is called the class of the partition of origin x. The two key following results highlight the partitioning effect of a connection C on all elements of lattice L.

Theorem 5 *Let C be a connection on L. For each $a \in L$ the family D_a of the maximal connected elements in $C \cap M^a$ a partition of a, whose classes are the connected components of a. This partition is increasing in that if $a \leq a'$, then any connected component of a is majorated by a connected component of a'.*

Proposition 6 *If lattice L is infinite $\bigvee$-distributive, and if C and C' are two connections on L with $C \subseteq C'$, then for all elements $y \in L$, every C'-component of y is the supremum of all C-components of y that it majorates.*

3.2 Canonic markers

The generalization we have just developed avoids to favour any sup-generating class in lattice L. Indeed, the lattice of the open sets in $\mathbb{R}^2$, for example, may be sup-generated by the open square, or, as well, by the open discs, although the intersection of these two classes is empty. In this sense, such a case differs from the set situation where the points belong to all sup-generating classes. Now the major application we have in mind here concerns function lattices (grey tone, color, equi-continuous, etc.), where one can assume that a basic sup-generator class belongs to all connections. It will be, for example, the class of the pulses for the numerical functions.

This circumstance suggests to slightly reduce the generality of the previous approach by choosing a sup-generating class S in L, with $0 \notin S$, and by replacing assumption *(i)* in Definition 3 by the following one:

$$\text{assumption } (vi)\text{: } C \textit{ contains the sup-generator } S. \tag{5}$$

The sup-generator S, which does not need to be itself a connection, will be called *canonic*. By so doing, we restrict the possible connections on L to those that contain S only. In return, the connected openings are better specified, and the family of all connections on L has more structure. First, the canonic markers S are sufficient to label all openings, for all connections on L. Indeed we can state the following result.

Proposition 7 *Given a canonic sup-generator S, every connection C on L consists in the invariant elements of the openings γ_s , $s \in S$, such that*

$$\gamma_s(a) = \bigvee \{x \in C,\ s \leq x \leq a\} \qquad s \in S \qquad a \in L$$

Second, one can define a new connection C by specifying all γ_x for $x \in S$, having the required properties. Third, the class of all connections on L turns out to form a complete lattice; more precisely, we have

Proposition 8 *The set of all connections on L is closed under intersection; it is thus a* complete lattice*; in this lattice the infimum of a family $\{C_i,\ i \in I\}$ of connections is given by the intersection $\cap C_i$ whereas the supremum is given by the least connection containing the union $\cup C_i$.*

3.3 Second generation connection

Operations such as dilations or closings can be used to remodel connected classes. Starting from a first class C, of connected openings $\{\gamma_x , x \in C\}$, we may use a clustering approach to define a new connection. Here, the first result we can quote is the following [SER98a].

Proposition 9 *Let C be a connection on lattice L, and $\delta : L \to L$ be an extensive dilation that preserves C (i.e. $\delta(C) \subseteq C$). Then the inverse image $C' = \delta^{-1}(C)$ of C under δ is a connection on L, which is larger than C (i.e. $C \subseteq C'$).*

The proposition is already instructive, but it does not give any information about the relations between the connected component of some $a \in L$ and its image $\delta(a)$. To achieve this, we need an assumption of distributivity over L, which then allows to state the following theorem.

Theorem 10 *Let C be a connection on an infinite $\bigvee$-distributive lattice L, and $\delta : L \to L$ be an extensive dilation that preserves C. Then the C-component of $\delta(a)$, $a \in L$, are exactly the images $\delta(y')$ of the C'-components y' of a, where C' is the connection $C' = \delta^{-1}(C)$.*

If γ_x stands for the connected opening associated with connection C and γ'_x for that associated with C', we have

$$\gamma'_x(a) = \gamma_x \delta(a) \bigwedge a \quad \text{when} \quad x \leq a \quad ; \quad \gamma'_x(a) = 0 \quad \text{otherwise.}$$

Examples

The connected openings γ'_x of theorem 10 not only give the theoretical access to their connection, but also provide the actual algorithm which extracts the components of a given a.

In practice, the openings γ'_x characterize the *clusters* of objects which lie a given distance d apart (fig.2a and 2b). Consequently, such an approach also provides a means for extracting the objects which are isolated. Imagine, for example, that we want to detect the particles (or components) in set X of (fig.2a) whose distances to their neighbours are $\geq 2d$. The two pieces of information we need for this problem are the dilate $X \oplus dB$ of set X, and the skeleton $skiz(X)$ by zones of influence, which is made of all points that have the same distance to at least two particles of X. If Y_1 denotes a particle of $X \oplus dB$ which intersects $skiz(X)$, then $Y_1 \cap X$ is a cluster of particles. If Y_0 is a particle of $X \oplus dB$ which does not intersect $skiz(X)$, then $X \cap Y_0$ is a particle whose all neighbours are at least distance $2d$ apart. Fig.2c shows the set of such isolated objects among those of fig.2a.

Other types of dilation may also be useful. To illustrate this point, we consider a sequence of sixty successive optical sections of an osteocyte, in confocal microscopy (kindly provided by Dr. V. Howard Liverpool). The whole volume contains three cells located at different depths, that one wishes to extract. Fig.3a is a 8-bit image of section n° 15 of the series. A high threshold (level 200, in a range of 256) easily extracts the osteocytes of the section, but also a number of artifacts. But we have another indirect piece of information : confocal microscopy generates a sort of halo around the thick objects. Indeed by selecting with a low threshold (60), we extract the halo and we take the union Z of all halos for the images of the series. Set Z is presented in Fig.3b. Finally, an osteocyte is recognized as a connected component γ'_x when the initial 3-D set is the high thresholded sequence, and when the dilation is restricted inside mask Z. A perspective view of the result is given by Fig.3c.

3.4 Clustering by closings

Dilations are not the only possible operators to provide second generation connections. We can use also closings, as the following result shows.

Proposition 11 *Assume that a given closing φ on a lattice L preserves the connection C, i.e.*

$$\varphi(0) = 0 \text{ and } \varphi(C) \subseteq C \tag{6}$$

Then the inverse image C' of C under φ is also a connection on L, which is larger than C.

Proof. By definition

$$a' \in C' \qquad \Leftrightarrow \qquad \varphi(a') \in C$$

Let $a'_i \in C', \quad i \in I$, with $\bigwedge a'_i \neq 0$. We have to prove that $\bigvee a'_i \in C'$. By extensivity of φ, we have

$$\varphi(a'_i) \geq a'_i \geq \bigwedge a'_i \neq 0 \quad \text{hence} \quad \bigvee \varphi(a'_i) \in C\,,$$

and by inclusion (6), we have $\varphi(\bigvee \varphi(a'_i)) \in C$. Now

$$\bigvee \varphi(a'_i) \geq a'_i \Rightarrow \varphi\left(\bigvee \varphi(a'_i)\right) \geq \varphi\left(\bigvee a'_i\right) \quad \text{(increasingness)}$$

and

$$\varphi\left(\bigvee a'_i\right) \geq \bigvee \varphi(a'_i) \Rightarrow \varphi\left(\bigvee a'_i\right) = \varphi\varphi\left(\bigvee a'_i\right) \geq \varphi\left(\bigvee \varphi(a'_i)\right) \quad \text{(idempotence)}$$

$$\text{hence} \quad \varphi\left(\bigvee a'_i\right) = \varphi\left(\bigvee \varphi(a'_i)\right) \in C, \quad \text{i.e.} \quad \bigvee a'_i \in C'$$

The two other axioms for a connection are clearly satisfied, hence C' is a connection. ∎

This result is related to Proposition 9 for dilations, but unlike that dilation case, we have here no equivalent of Theorem 10 here. Note also that for any dilation δ on L we have $\delta(0) = 0$, so that the relation (6) of connection preservation is the same as that introduced in Proposition 9. An application of Proposition 11 is provided in section 6 below, and illustrated by fig.10. The proposition is applied to the lattice of the weighted partitions by flat zones.

4 Connections on lattices of numerical functions

In this section, we look for interesting connections in the important case of numerical functions from a space E into a totally ordered set L. We assume the latter either finite, or isomorphic to $\overline{\mathbb{R}}$ or to $\overline{\mathbb{Z}}$ (e.g. $[0,1]$, $[0,+\infty]$, etc.). Concerning the domain space E, we assume that it is metric, with distance function d, and that $\mathcal{P}(E)$ is equipped with connection $\mathcal{C}$. These definition and properties concerning L and E are assumed to be valid throughout this entire section.

4.1 Numerical lattices of numerical functions

The set L^E of all mappings f from E into L, provided with the product order :

$$f \leq g \qquad \text{iff} \qquad f(x) \leq g(x) \qquad \text{for all } x \in E,$$

becomes in turn a complete lattice, where the sup and the inf are defined by the relations

$$\left(\bigvee f_i\right)(x) = \bigvee f_i\,(x) \qquad \left(\bigwedge f_i\right)(x) = \bigwedge f_i\,(x) \tag{7}$$

(we keep the same symbols for ordering, supremum and infinum in both lattices L and L^E). The *support* of a function is the set of points x where $f(x) > 0$. The support of function which is identically zero is the empty set; that of a pulse is reduced to one point.

We are often more interested in sublattices of L^E, rather than in L^E itself. A sublattice $L' \subseteq L^E$ is a set of functions which is closed under $\bigvee$ and $\bigwedge$ of L^E and which contains the zero and the unit. For example, the Lipschitz functions L_k of module k, which are defined by

$$f \in L_k \qquad \Leftrightarrow \qquad |f\,(x) - f\,(y)| \leq kd\,(x,y) \qquad \forall\ x,y \in E \quad .$$

turn out to generate a complete sublattice of L^E. More generally, if we replace $kd(x,y)$ by $\rho(d(x,y))$ in the above inequality, where function $\rho : \mathbb{R}_+ \to \overline{\mathbb{R}_+}$ is continuous at the origin, we delineate the class of the so-called "ρ-continuous functions" [CHO66]. For each ρ, the class of the ρ-continuous functions constitutes a complete sub-lattice of L^E, i.e. a lattice where $\bigvee$ and $\bigwedge$ are nothing but the *numerical* supremum and infimum at each point. It has been proved (under rather general conditions), that usual operators, such as dilations, openings, morphological filters, etc. map every ρ-continuous lattice into another one [SER92b], and that $\bigvee$ and $\bigwedge$ are continuous operators [MAT96]. Moreover, every function lattice where $\bigvee$ and $\bigwedge$ are generated by the numerical ones is infinite distributive. Henceforth, we refer to such lattices of numerical functions as *numerical lattices.*

4.2 Two straightforward connections

To define a connection on L^E, the first straightforward approach consists in extending to L^E the connectivity associated with E, by taking as connected functions (in L^E) those with connected support (in E). This class comprises function $f = 0$ and the pulses, and satisfies axiom *(ii)*. But it is rather coarse: for example, if $E = \mathbb{R}^2$ and if function f stands for the values of a grey tone photograph of a landscape, it is likely to attribute function f a single component most of the time. In addition, the connected components $\{\gamma_x (X_t (f)), t \in L\}$, of the horizontal cross-sections $X_t (f)$ of f at level t generate, as t varies, a function which completely differs from the connected component of f at point x.

The second straightforward approach takes as a connected class all pulses along with the function $f = 0$. The three axioms of a connection are obviously satisfied in this case. Then, for any pulse located at point x and of intensity $t \leq f(x)$, the associated connected component of f is the pulse at x with intensity $f(x)$. Although, mathematicallty speaking, both cases correspond to connections, they lack the property of being a shape and feature descriptor.

4.3 Numerical functions as weighted partitions

Such a drawback comes from the underlying lattice L^E, which is too poor, and does not allow to express structural partitions of the grey tone functions. On the other hand, partitions divide the space, but, by themselves, they do not say anything about possible parameters, or numerical entities, which may be associated with their classes. To overcome this shortcoming, we will introduce now the notions of a *weighted set* and of a *weighted partition*, in the framework of an arbitrary, but fixed, numerical sublattice of L^E.

Given a modulus ρ, associate, with every pair $(D, g) \in \mathcal{P}(E) \times G_\rho$, the restriction g_D of a ρ-function g to set D, given by

$$\begin{array}{lclll} g_D\ (u) & = & g\,(u) & \text{when} & u \in D \\ g_D\ (u) & = & 0 & \text{othewise} & \end{array}$$

Thus, the usual indicator function of a set D is replaced by the non constant weight g. The numerical function g_D may be considered as a *weighted set.* For example, if we take for G_ρ the class of the constant function, which corresponds to $\rho \equiv 0$, then each weighted set g_D is nothing but the cylinder of base D and of height the weight of g_D . If we take for class G_ρ the k-Lipschitz functions, each weighted set g_D turns out to be the restriction of a k-Lipschitz function to set D. As the doublet (D, g) spans $\mathcal{P}(E) \times G_\rho$, the set generated by the g_D's is denoted by $\mathcal{P}_\rho(E)$.

Proposition 12 *The set $\mathcal{P}_\rho(E)$ forms a complete lattice for the numerical ordering $\leq$; in this lattice, the supremum $\sqcup (g_D)_i$ of family $\{(g_D)_i\}, i \in I$, is the smallest upper bound of $\bigvee (g_D)_i$ in G_ρ and the infimum is given by $\sqcap (g_D)_i = (\bigwedge g_i)_{\cap D_i}$.*

Fig.4 illustrates the proposition. Two cylinders (fig.4a), and two Lipschitz functions (fig.4b) are represented in dotted lines. Their infima in $\mathcal{P}_\rho$ are given by their numerical infima over the intersection of the supports (dash zones), whereas their suprema in $\mathcal{P}_\rho$ are the smallest cylindric, or Lipschitz, extensions of the functions, restricted to the unions of the supports.

The notion of a weighted partition derives from that of a weighted set just as a usual partition derives from that of a set.

Definition 13 *A weighted partition with connected classes* $x \to (g_D)_x$ *is a mapping* Δ *from* E *into* $\mathcal{P}_\rho(E)$ *such that*

(i) $\forall x \in E,\ \ x \in D\,(x)$

(ii) $\forall x, y \in E,$ *either* $(g_D)_x$ *or* $(g_D)_x \bigwedge (g_D)_y = 0$

(iii) $\forall x \in E$ *class* $D(x)$ *is connected set of* E

The sub-mapping $x \to D(x)$ is a partition D in the usual sense, and $f = \bigvee \{ (g_D)_x , x \in E\}$ is a numerical function. Therefore, one can represent the weighted partition Δ as a pair (D, f) which determines Δ completely. However, any function $f : E \to L$ admits several representations as $\bigvee\ (g_D)_x$. For example, for $\rho = 0$, any subset of the flat zone of f that contains a given point x may serve as class $D(x)$ jointly with the constant function $g = f(x)$.

By considering the numerical functions under study as weighted partitions, we will be able to provide them with connections which are descriptive. But first of all, we must be sure that the weighted partitions fit into enter the convenient framework for connections, *i.e.* that they satisfy a complete lattice structure. Indeed, we have the following result[SER98a]

Proposition 14 *Let* G_ρ *be the lattice of* ρ*-continuous numerical functions* $E \to L$ *and* $\mathcal{L}$ *the class of the weighted partitions with connected classes of* E*, of base* G_ρ*. Then* $\mathcal{L}$ *is a complete lattice for the product ordering*

$$(f, D) \sqsubseteq (f', D') \Leftrightarrow \begin{array}{lll} f \leq f' & \textit{in} & L^E \\ D \leq D' & \textit{in} & \mathcal{D} \end{array}$$

In this lattice, the infimum $\sqcap(f_i, D_i)$ *of a family* $\{f_i, D_i, i \in I\}$ *is given by the doublet* $\left(\bigwedge f_i \left(\textit{in } L^E\right), \bigwedge D_i \left(\textit{in } \mathcal{D}\right)\right)$ *. The supremum* $\sqcup(f_i, D_i)$ *admits* $\bigvee D_i$ *for partition, and in each class* $(\bigvee D_i)\,(x)$*, is equal to the smallest function of lattice* G_ρ *which is above the* f_i *involved in* $(\bigvee D_i)\,(x)$*.*

(Easy proof). Here, the infimum $\wedge D_i$ of a family of partitions of E with connected classes is the partition whose class at point $x \in E$ is the connected component of $\cap D_i(x)$ that contains point x.

In order to build connections on the weighted partitions lattice $\mathcal{L}$, a convenient class is provided by the so-called G-cylinders.

Definition 15 *Let* G *be a numerical lattice of* L^E*, and* $\mathcal{C}$ *a connected class on* $\mathcal{P}(E)$*. For any* $Y \in \mathcal{C}$ *and* $g \in G$*, the weighted partition of function*

$$\begin{array}{lllll} h_{y,g}(y) & = & g(y) & \textit{when} & y \in Y \\ h_{y,g}(y) & = & 0 & \textit{when} & \textit{not} \end{array}$$

and of partition classes Y *and* $\{z\}$ *(for* $z \in Y^c$*) defines the* G*-cylinder of connected base* Y *and of values* g*.*

In particular, the pulses, equipped with the minimum partition d, are G-cylinders, for all lattices G.

Consider now the supremum in $\mathcal{L}$ of a family $\{h_i = h_{y_i,g_i}, i \in I\}$ of G-cylinders, whose infimum is not zero. Since $\cap Y_i$ is not empty $\cup Y_i$ is connected. Now the partition whose classes are $\cup Y_i$ and all the points of $(\cup Y_i)^c$, is nothing but $\bigvee D_{h_i}$. By adjoining the function equal to the smaller G-bound of the h_i on $\cup Y_i$, and equal to 0 elsewhere, we obtain the supremum in $\mathcal{L}$ of the h_i. This weighted partition is itself a G-cylinder and since the G-cylinders form a sup-generating family (they contain the pulse cylinders), hence we may state.

Theorem 16 *Let G be a numerical sublattice of L^E. The class C_G of the G-cylinders, together with function 0, constitutes a connection on lattice $\mathcal{L}$.*

Fig.4 shows two examples of suprema and of infima of connected G-cylinders, based on constant functions and on Lipschitz functions respectively. Note that the smallest element of $\mathcal{L}$, namely the weighted partition whose all classes are reduced to points, and function constant minus infinity, is itself a G-cylinder whose base is a point and having minus infinity on it. Note also that the theorem does not extend to weighted partitions with possibly not connected classes. Such classes could not be generated by the G-cylinders of Definition 15.

4.4 Two examples of connections

As particular cases of the approach developed here we obtain the two straightforward connections presented at the beginning of the section. However, this approach opens the way to a number of other connections, which turn out to be extremely pertinent descriptors for images. Two of them are presented below; the associated moduli ρ are drawn in fig.5.

Flat zone connections

Take for ρ the function $\rho \equiv 0$. Class G_φ is then that of the *constant functions*, for which the following result holds.

Proposition 17 *When the numerical lattice G is the class of the constant functions from E into L, then the associated connection on $\mathcal{L}$ is formed by the zero function and the cylinders (stricto sensu) of connected bases, i.e. by functions*

$$\begin{array}{lll} h_{c,t}(t) = t & \textit{when} & y \in C \\ h_{c,t}(t) = 0 & \textit{when} & \textit{not} \end{array}$$

$t \in L$, $C \in \mathcal{C}$.

The operations $\sqcup$ and $\sqcap$ on the lattice $\mathcal{L}$ associated with the constant functions are illustrated in fig. 4a , which shows the $\sqcup$ supremum of two cylinders.

We saw that given module ρ, there is generally an infinity of variants for modelling a numerical function f as a weighted partition. However, in the flat zone case (i.e. $\rho \equiv 0$), the class of these variants is closed under the supremum in $\mathcal{L}$. The corresponding non weighted partition D_f is the largest partition of E where f is constant in each class.

Fig.10 describes the action of a closing $\varphi : \mathcal{L} \to \mathcal{L}$, when $\mathcal{L}$ is the lattice of weighted partitions by flat zones. The evolution of one individual connected component is indicated by a dark contour (comments are given in section 6.1). In practice, the flat zones connection yields significant results when the images under study exhibit large zones with uniform grey tones. It is typically not the case for faces, for example, since the variation of the skin orientation creates a continuous gradation of the grey tones. Indeed, in the case of fig.6a, there are 114,519 flat zones for a total amount of 167,424 pixels in the photograph.

Jump connection

Such a counter example shows that the requirement of a strict flatness is too demanding, and should be relaxed. We shall weaken it by taking for class G of the numerical ρ-continuous functions whose modulus reaches range h at a finite distance d_0, as shown in Fig.5b. Each function in G varies between two bounds which are h apart. Therefore, by applying base G to a numerical function f, we obtain a segmentation of f into connected zones where the grey fluctuation is $\leq h$. Fig.6 illustrates such a "jump connection" in the digital framework by taking $d_0 = 1$ and $h = 10$ (between two neighbour points, the function cannot jump by more than 10 levels). As a matter of fact, this example deals with a color image. In the 3D Euclidean space of red, green and blue axes, the "gray" at point [red(x), green(x), blue(x)] has been defined by the distance from this point to the origin. After segmentation of the gray image according to the jump connection, the three colors are averaged in each class, so that the final result is still a color image.

Unlike the case of flat zone connection, there are no longer largest segmentations. This is not necessarily a disadvantage, since we can play with different partitions. For example, we can extract the largest connected zone $Z(x_0)$, around each minimum $f(x_0)$, where $|f(x) - f(x_0)| \leq h, \quad x \in Z(x_0)$, and iterate this procedure. By taking the union of the $Z(x_0)$'s when x_0 spans all minima of the function, we extract a binary version of the grains. The following example illustrates the use of such a transformation. Fig. 7a represents alumine grains, Fig.7b the partition of the space under jump connection, and Fig.7c the superposition of its skeleton by zones of influence and the initial image.

5 Induced binary connections

Smooth path connection

In this section, we investigate the access to segmentation of numerical functions $f : E \to L$ when a connection $\mathcal{C}$ is defined on $\mathcal{P}(E)$ *only*. This approach is more limited than the previous one, but it has the twofold advantage of being simpler, and of leading to rather powerful results.

Indeed, if our current goal is to segment a unique function $f : E \to L$, it becomes cumbersome to consider it as a variable element of the lattice L^E. The question should reformulated as follows : "is there a largest partition of space E into connected classes such that function f satisfies a given criterion inside each class ?

For example, the above segmentation of f by flat zones may alternatively be presented in this new framework. Given point $x \in E$, denote by Z_x the subclass of those elements of $\mathcal{C}$ that contains point $\{x\}$, and over every element of which function f is constant, i.e.

$$Z_x = \{x \in Z \subseteq \mathcal{C}, y \in Z \Rightarrow f(x) = f(y)\}.$$

Class Z_x is closed under union, so that its supremum

$$Z_x = \cup \{Z \in Z_x\}$$

is the largest connected component containing point x and on which function f is constant. Consequently, the mapping $x \to Z(x)$ is the largest partition of space E into flat zones of f, and further, family $\{Z_x \cup \emptyset\}$ defines the invariant sets of a connected opening at point x. This means that any set $Y \subseteq E$ is partitioned into the classes $Z(x) \cap Y$, as x spans E. By so doing, we have generated a new connection on E, by combining the initial one with some features of function f.

Unfortunately, the technique we used here does not necessarily extend to other types of ρ-continuous functions. For example, if we replace the modulus $\rho \equiv 0$ (flat zones connection) by that of a bounded variation, shown in fig.5b (jump connection), we are no longer able to define a largest connected component containing point x and on which function f is ρ-continuous : class Z_x is no longer closed under union.

However, there exist other criteria, which fulfill closure under union, and of great interest. Provide, for example, $\mathcal{P}(\mathbb{R}^n)$ with the arcwise connection. Consider an arbitrary, but fixed, function $f \in L^{\mathcal{P}(\mathbb{R}^n)}$ and the class $\mathcal{C}$ made of

- *i)* all singletons plus the empty set ;
- *ii)* and all open connected sets Y of $\mathcal{P}(\mathbb{R}^n)$ such that function f is k-Lipschitz, for the induced arcwise metric, along all paths included in set Y (in practice, this metric is often called "geodesic metric").

It is easy to see that class $\mathcal{C}$ defines a *second connection* on $\mathcal{P}(\mathbb{R}^n)$, where the variation of f over each set of class *ii)* is "*smooth*" in the k-Lipschitz sense.

In $\mathbb{Z}^2$, the implementation of this "*smooth path*" criterion is particularly easy. If H_x stands for the unit disc at point x (five, seven, or nine points), then the partition has for non point classes the arcwise connected components of all sets X such that

$$X = \cup \left\{x \in \mathbb{Z}^2, \vee \{|f(x) - f(y)|, y \in H_x\} \leq k\right\}.$$

An example of smooth path connection is given in fig8. Fig.8a represents an electron micrograph of concrete made of three phases : a white one and two grey ones. The histograms of the two grey phases are almost identical, but one is more continuous than the other. By segmenting fig.8a according to the smooth path connection, with a slope $k = 6$, we obtain a correct pre-extraction, which has to be amended by some filtering. For the same image, the best jump connection is obtained by taking a range $h = 15$, and yields the rather poor result depicted in fig.8c.

6 A few connected filters

The development of connected filtering, for the past ten years, has been directly related to indexing and coding. References [PAR94], [MAR96], [SAL96], among others, show that connected filters have opened the way to an object-oriented approach for compression and understanding of still and moving images.

Our purpose here is not to provide an exhaustive list of relevant applications, nor to reproduce existing theories. We just would like to relate these operations to the previously defined connections, and to complete them with some additional properties, in the case of leveling. We will restrict this overview to the *flat* numerical filters. A profound theorem due to H. Heijmans says that there exists a unique *binary* operator that allows us to construct the grey tone one, via the cross sections (see below, sect. 5.2.3). Moreover flat filters are by far the most commonly used, and the set formalism allows easy geometrical interpretations.

Throughout section 6 to 8, E is an arbitrary set, and $\mathcal{P}(E)$ is assumed to be equipped with connection $\mathcal{C}$. For every set $A \in \mathcal{P}(E)$, the two families of the connected components of A (the "grains") and of A^c (the "pores") yield a partition of the underlying space E. An operation $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ is said to be *connected* when the partition associated with $\psi(A)$ is coarser that that of A [SER93]. Clearly, an operator that takes the complement of a set, or removes some grains, or fills pores, is a connected operator. The major class of mappings we have in mind are the morphological filters, as defined in section 2, when in addition they are connected, and particularly the strong ones and the granulometries. Let us briefly recall these last two concepts.

A granulometry is a family $\{\gamma_d\, ,\, d > 0\}$ of decreasing openings (i.e. $d \geq d' \Rightarrow \gamma_d \subseteq \gamma_{d'}$) and an anti-granulometry is a family $\{\varphi_d\, ,\, d > 0\}$ of increasing closings, both depending on a positive parameter. Every granulometry and every anti-granulometry satisfy a semi-goup structure of law

$$\psi_n \psi_p = \psi_p \psi_n = \psi_p \qquad \text{if} \quad p \geq n \tag{8}$$

A morphological filter ψ is said to be *strong* if it satisfies the following robustness condition

$$\psi(A) \cap A \subseteq \psi(B) \subseteq \psi(A) \cup A \quad \Rightarrow \quad \psi(B) = \psi(A)$$

for $A, B \subseteq E$. In particular, openings and closings are strong filters.

Set opening by reconstruction and some derivatives

A comprehensive class of connected filters derives from the classical *opening by reconstruction*. Its definition appears in [SER88], ch.7.8. Significant studies which use this notion may be found in the literature, such as [SER93] (connected operators), [VIN93] (area criteria), [MEY94], (spanning trees), [CRE95], (connected filters)[HEI97], (grain oprators).

An opening by reconstruction is obtained by starting from an increasing binary criterion τ (e.g. "the area of A is ≥ 10"), to which one associates the trivial opening

$$\begin{array}{ll} \gamma^{\tau}(A) = A & \text{when } A \text{ satisfies the criterion} \\ \gamma^{\tau}(A) = \emptyset & \qquad\qquad \text{otherwise} \end{array}$$

The corresponding opening by reconstruction γ is then generated by applying the criterion to all grains (or connected components) $\gamma_x(A)$ of A, independently of one another, and by taking the union of the results :

$$\gamma(A) = \cup \{\gamma^{\tau}\gamma_x(A), \quad x \in E\}$$

The *closing by reconstruction* φ (for the same criterion) is the dual of γ for the complement, *i.e.* if $\complement$ stands for the complement operator, then

$$\varphi = \complement\gamma\complement\,.$$

For example, in $\mathcal{P}(\mathbb{R}^2)$, if we let τ to be the area criterion mentioned above, that is $\tau(A) = 1$ if the area of A is larger than a given threshold S and 0 otherwise, then $\gamma(A)$ equals the union of grains of A whose area is $\geq S$, and $\varphi(A)$ is the union of A and all its pores whose area is $\geq S$. Similarly, if criterion τ is expressed by "intersecting a given marker set M", then $\gamma(A)$ is the union of the grains that hit M, whereas $\varphi(A)$ is composed of A and of all pores that miss M .

Consider now a family $\{\gamma_i, i \in I\}$ of openings by reconstruction associated with criteria $\{\tau_i\}$. Clearly, their infimum $\gamma = \cap\gamma_i$ is still an opening, where a grain of A must fullfill all criteria τ_i to be retained. Since, on the other hand, the supremum of any collection of openings is also an opening, we may state the following result:

Proposition 18 *In the lattice of the increasing and anti-extensive operators from $\mathcal{P}(E)$ into itself, the openings and the closings by reconstruction constitute a complete sub-lattice. By duality, the same result holds for the closings by reconstruction.*

Assume now that we have a decreasing sequence of criteria $\{\tau_i\,;\, i \geq 1\}$, that is

$$\tau_{i+1} \leq \tau_i\,, \qquad i \geq 1$$

this can be interpreted by saying that the criteria are increasingly restrictive. It is obvious that the corresponding sequence of openings γ_i is decreasing as well :

$$\gamma_{i+1} \leq \gamma_i\,, \qquad i \geq 1$$

In other words, the family $\{\gamma_i\}$ is a granulometry, called *granulometry by reconstruction.* Two theorems govern the specific properties we obtain in that case. The first one is due to J. Crespo et al. [CRE95].

Theorem 19 *Let* $\{\gamma_i,\ i \geq 1\}$ *and* $\{\varphi_i,\ i \geq 1\}$*, be respectively a granulometry and an anti-granulometry by reconstruction, and define*

$$\Psi_n = \wedge \{\varphi_i \gamma_i, \quad 1 \leq i \leq n\} \text{ and } \Psi_n^* = \vee \{\gamma_i \varphi_i, \quad 1 \leq i \leq n\}$$

The operators Ψ_n *and* Ψ_n^* *are strong filters, and satisfy*

$$\Psi_n \Psi_p = \Psi_p \text{ and } \Psi_n^* \Psi_p^* = \Psi_p^* \text{ for } p \geq n.$$

Although the families $\{\Psi_n\}$ and $\{\Psi_n^*\}$ are both ordered, they do not satisfy the semi-group relationship 8 (namely the stronger filter imposes its law). To achieve this, we must introduce the so called *alternating sequential filters.*

$$\rho_n = \varphi_n \gamma_n \cdots \varphi_i \gamma_i \cdots \varphi_1 \gamma_1 \quad \text{and} \quad \sigma_n = \gamma_n \varphi_n \cdots \gamma_i \varphi_i \cdots \gamma_1 \varphi_1.$$

The following result holds

Theorem 20 *Let* $\{\gamma_i,\ i \geq 1\}$ *and* $\{\varphi_i,\ i \geq 1\}$*, be a granulometry and an anti-granulometry by reconstruction. Then the alternating sequential filters* ρ_n *and* σ_n *of primitives* $\{\gamma_i\}$ *and* $\{\varphi_i\}$*, generate, as* n *varies, a semi-group of law the relationship (8) .*

When this theorem appeared, in [SER93], it was also stated, in addition that the ρ_n's and σ_n's were strong filters. It has been shown in [HEI97] that this is true under the assumption of finite paths connecting each pair of points in any connected component. Note that this condition is fulfilled by all usual connections, and by those they induce by dilation or by closing.

Anyway, Theorem 21 turns out to be a central pillar in compression schemes based on morphological operators. It allows to build pyramids of filters where the additional information to get finer levels is concentrated in the flat zones [SAL92] and where the non-zero gradients may only be reduced to zero or kept unchanged. An example of such a behaviour is presented in Fig.9. Although the algorithm does not work level by level, each cross section of the grey tone image results in a set which is processed by an alternating sequential filter by reconstruction, according to rel.(19) below. The underlying binary criterion is the size of the disc inscribable in each grain.

Similarly, any set closing by reconstruction φ can be extended to numerical functions. Moreover, in such a case φ turns out to close not only the numerical functions, but also the partitions induced by the flat zones of the functions. In other words, φ is a closing on the lattice $\mathcal{L}$ of the weighted partitions by flat zones. Then, according to Proposition 11, closing φ generates a new connection of clustering type. This approach is illustrated in fig.10 which treats a vortex of clouds over the Atlantic Ocean. For a more efficient effect, only the four higher bits of the initial image have been processed, and the closing used here is the dual version of the reconstruction from the erosion by a disc of size 12. Under closing, the sky is segmented into more regular, hence more representative, clouds than those of the initial image.

7 Set flattening and leveling

Flattenings and leveling have been introduced by F. Meyer as grey tone connected operators on digital spaces[MEY98a], [MEY98b]see also [MEY99] and [MAR99]. Meyer chooses the connection defined by the classical arc-connectivity on digital grids. In [MAT97], G. Matheron proposes a generalization to the functions $f : E \to T$, when E is an arbitrary space (hence, without a priori connection). In his approach, the connection arrives as a final result, and is generated by an extensive dilation.

In both approaches, leveling, and flattenings (when the latter are increasing), turn out to be *flat* operators, *i.e.* that treat each grey level independently of the others. This circumstance suggests to try and generalize F. Meyer's approach by focusing on *set* flattenings and leveling, but re-interpreted in the framework of an arbitrary set connection $\mathcal{C}$. We will enter this way of thinking by taking one of the characteristic properties of the flattenings (Theorem 7 in [MEY98a]) as their definition.

Independently of Meyer and Matheron approach, H.Heijmans has introduced and studied the class of the "grain operators" in[HEI97]. Flattenings and leveling, in the sense of definitions 22 and 30 below, are particular grain operators. However, for reasons which will be discused in section 7.3, we prefer to restrict ourselves to openings and closings which are based on markers (for example, we will not accept or reject a particle according to its area).

Definition 21 (21) *Let E be an arbitrary set, and $\mathcal{C}$ be a connection on $\mathcal{P}(E)$. Let γ be an opening and φ a closing, both by marker reconstruction, from $\mathcal{P}(E)$ into itself. The operator $\theta : \mathcal{P}(E) \to \mathcal{P}(E)$, given by*

$$\theta = \gamma \cup (\complement \cap \varphi) = \varphi \cap (\complement \cup \gamma) \tag{9}$$

is called the flattening of primitives γ and φ. *Here $\complement$ stands for the complement operation on $\mathcal{P}(E)$.*

The set $\theta(A)$ satisfies the relations

$$\begin{aligned} A \cap \theta(A) &= A \cap \gamma(A) \\ A^c \cap \theta(A) &= A^c \cap \varphi(A) \quad (\Longleftrightarrow A \cup \theta(A) = A \cup \varphi(A)) \end{aligned} \tag{10}$$

i.e. $\theta(A)$ acts as an opening, and inside A^c as a closing. System (10) also relates to the *activity lattice*, where a mapping ψ on $\mathcal{P}(E)$ is said to be *less active* than ψ', when $\psi'(A)$ changes more points of A than $\psi(A)$, $\forall A \in \mathcal{P}(E)$, (see chapter 8 in [SER88]). If Id stands for the identity operator, the activity ordering is as follows

$$\begin{array}{ccc} Id \cap \psi & \supseteq & Id \cap \psi' \\ Id \cup \psi & \subseteq & Id \cup \psi' \end{array}$$

and one notes $\psi \preceq \psi'$. The activity ordering gives rise to a complete lattice structure, where the supremum and the infimum of a family $\{\psi_i \, , \, i \in I\}$ are given by

$$\begin{aligned} \curlyvee\psi_i &= \left[\complement \cap (\cup\psi_i)\right] \cup [\cap\psi_i] \\ \curlywedge\psi_i &= [Id \cap (\cup\psi_i)] \cup [\cap\psi_i] \ . \end{aligned}$$

When applying this system to the family $\{\gamma, \varphi\}$ of the two flattening primitives, we draw from (9) that

$$\gamma \curlyvee \varphi = \theta \qquad \gamma \curlywedge \varphi = Id \, .$$

An interesting situation occurs when γ and φ are dual under complement $i.e.$when $\varphi = \complement\gamma\complement$. Then we have

$$\complement\theta\complement = \complement\left[\gamma\complement \cup \left(Id \cap \varphi\complement\right)\right] = \varphi \cap \complement\left(Id \cap \varphi\complement\right) = \theta,$$

which means that θ is self -dual with respect to the complement.

Proposition 22 *Any flattening $\theta : \mathcal{P}(E) \to \mathcal{P}(E)$ is the activity supremum of its two primitives γ and φ. When these primitives are dual from each other, then the corresponding flattening is self-dual.*

7.1 Markers based flattenings

When we established Proposition 22, we did not use the reconstruction structure of operations γ and φ. However, since it will be directly involved in what follows, we now give explicit expressions for γ and φ, based on markers M and N^c respectively, $i.e.$

$$\begin{aligned} \gamma(A) = \gamma_M(A) &= \cup\{\gamma_x(A)\, , \, x \in E\, , \, \gamma_x(A) \cap M \neq \emptyset\} \\ \varphi^c(A) = [\varphi_{N^c}(A)]^c &= \cup\{\gamma_x(A^c)\, , \, x \in E\, , \, \gamma_x(A^c) \cap N^c \neq \emptyset\} \end{aligned} \tag{11}$$

Therefore, opening $\gamma_M(A)$ is the union of those grains of A that hit marker M, and the complement of closing $\varphi_{N^c}(A)$ is the union of those pores of A that hit marker N^c, hence

$$A^c \cap \varphi_{N^c}(A) = \cup\{\gamma_x(A^c)\, , \, x \in E\, , \, \gamma_x(A^c) \subseteq N\} \tag{12}$$

so that

$$\theta(A) = \gamma_M(A) \cup [A^c \cap \varphi_{N^c}(A)]$$

Therefore $\theta(A)$ is the union of all grains of A that intersect marker M and of all pores that are included in marker N.

Consider now a family $\{\theta_i \ , \, i \in I\}$ of flattenings with markers $\{M_i \ , N_i^c \ \}$. We have

$$\begin{aligned} (\cup\gamma_i)(A) &= \{ \text{ all grains of } A \text{ that hit } \cup M_i\} \\ (\cap\gamma_i)(A) &= \{\text{all grains of } A \text{ that hit each of the } M_i\}. \end{aligned}$$

Similarly for the closings, we have that

$(\cap\varphi_i)(A) = \{A$ and all pores of A included in $\cap N_i\}$
$(\cup\varphi_i)(A) = \{A$ and all pores of A included in one of the $N_i\}$.

Now, we conclude from Proposition 18 that the first two operators are openings and that the last two are closings. Hence, we may state

Proposition 23 *The class of flattenings θ on $\mathcal{P}(E)$ provided with the activity ordering constitutes a complete lattice. Given two families of primitives $\{\gamma_i\}$ and $\{\varphi_i\}$, $i \in I$, the infimum and the supremum of the associated flattenings admit the following expressions:*

$$\curlywedge\theta_i = (\cup\gamma_i) \cup [\complement \cap (\cap\varphi_i)]$$
$$\curlyvee\theta_i = (\cap\gamma_i) \cup [\complement \cap (\cup\varphi_i)].$$

Note that the opening $\cap\gamma_i$ (resp. the closing $\cup\varphi_i$) involved here is no longer based on a single marker : each grain (resp. each pore) is still judged individually, but with respect to several markers. As A, M and N vary, $\theta(A, M, N)$ is a mapping from $[\mathcal{P}(E)]^3$ into $\mathcal{P}(E)$. We will now see under which condition this mapping is increasing. Part of Theorem 24 has already been established in [HEI97], namely when M and N are fixed. But for extending increasingness from sets to numerical functions, we need to vary M and N.

Theorem 24 *The flattening $\theta : [\mathcal{P}(E)]^3$ into $\mathcal{P}(E)$ is increasing if and only if the two operands M and N are ordered by $N \subseteq M$.*

Proof. We begin with the proof of the "if" part. Firstly, we observe that, given A, when $M \subseteq M'$ and $N \subseteq N'$ more grains of A are hit and more pores of A are covered, hence

$$\theta(A, M, N) \subseteq \theta(A, M', N') \tag{13}$$

Consider now the variation of θ as a function of A only, and assume that $N \subseteq M$. The first term in (9)an opening, hence increasing. As for the second one, it suffices to prove that, when $A \subseteq A'$, the inclusion $\gamma_x(A^c) \subseteq N$ implies that $\gamma_x(A^c) \subseteq \theta(A', M, N)$. Now, for every point $z \in \gamma_x(A^c) \subseteq N$, only two cases may occur

i/ $\gamma_z(A'^c) \neq \emptyset$, then z belongs to a pore of A'^c. But since $A'^c \subseteq A^c$, we have $\gamma_z(A'^c) \subseteq \gamma_z(A^c) = \gamma_x(A^c) \subseteq N$, hence $z \in \theta\,(A', M, N)$;

ii/ $\gamma_z(A'^c) = \emptyset$, then z belongs to the grain $\gamma_z(A')$ of A'. Now $z \in N \subseteq M$, hence $\gamma_z(A') \cap M \neq \emptyset$ so that $z \in \theta(A', M, N)$.

Finally, we have

$$A \subseteq A' \quad ; \quad N \subseteq M \quad \Rightarrow \quad \theta(A, M, N) \subseteq \theta(A', M, N) \tag{14}$$

The "only if" part may be proved by means of a counterexample. Indeed, if $N \nsubseteq M$, take for set A a grain that misses M and that contains a pore included in N, and take for A' the same grain, but without its pore (fig.11a). We see that

$$A \subseteq A' \quad \nRightarrow \quad \theta(A, M, N) \subseteq \theta(A', M, N).$$

The proof is then achieved by combining rel.(13) and (14). ∎

An interesting feature of flattening concerns self duality. Indeed, flattenings have been introduced by F. Meyer with the goal of finding self dual connected operators. Firstly, we may consider the behaviour of the triple mapping $(A, M, N) \rightarrow \theta(A, M, N)$ when taking the complements. We have

$$[\theta(A^c, M^c, N^c)]^c = [\gamma_{M^c}(A^c)]^c \cap [A \cap [\gamma_N(A)]^c]^c = \varphi_{M^c}(A) \cap [A^c \cup \gamma_N(A)]\,,$$

hence

$$[\theta(A^c, M^c, N^c)]^c = \gamma_N(A) \cup [A^c \cap \varphi_{M^c}(A)] = \theta(A, N, M)$$

Therefore self duality of $\theta(A, M, N)$ is reached if and only if the two markers N and M are identical (a result that also follows from proposition 8.3 in [HEI97]). Since, in addition, condition $M \equiv N$ implies the inreasingness of θ, we arrive to the following result

Proposition 25 *The flattening $(A, M, M) \rightarrow \theta(A, M, M)$ is an increasing self dual mapping from $\mathcal{P}(E) \times \mathcal{P}(E)$ into $\mathcal{P}(E)$.*

In this approach, we implicitely supposed that the data of A end of M are independent pieces of information. In practice, it often occurs that marker M derives from a previous tranformation of A itself, $M = \mu(A)$, say. Then the proposition shows that the flattening $\theta : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$, with $\theta = \theta\,(A, \mu(A), \mu(A))$ is self-dual if and only if the operator μ is already self dual.

We now consider the question of idempotence for flattenings.

Proposition 26 *Given two sets M and N in $\mathcal{P}(E)$, with $N \subseteq M$, the flattening $A \rightarrow \theta(A, M, N) = \theta_{M,N}(A)$, from$\mathcal{P}(E)$ into itself, is idempotent.*

Proof. The set $\theta(A, M, N)$ comprises three types of connected components, namely some $\gamma_x(A)$, some $\gamma_z(A^c)$ and unions of some $\gamma_x(A)$ and $\gamma_z(A^c)$. By construction, components of type one and three intersect M, and since $N \subseteq M$, components of type two (which are included in N), do intersect M. Therefore none of them is removed *i.e.*

$$\theta(A, M, N) \subseteq \theta[(\theta(A, M, N), M, N].$$

In particular, we also have $\theta(A^c, N^c, M^c) \subseteq \theta[\theta(A^c, N^c, M^c), N^c, M^c]$, hence by duality

$$\theta(A, M, N) \supseteq \theta[\theta(A^c, N^c, M^c)^c, M, N] = [(\theta(A, M, N), M, N]$$

which concludes the proof. ∎

This result is to be related with Theorem R-2 in[MAT97]. There, Matheron establishes the idempotence of some asymptotic operator which coincides with the flattening θ when the connection $\mathcal{C}$ is obtained as the limit under iteration with a given extensive dilation.

In the same way, as for Proposition 26, one also proves easily that, given A, mapping $M \to \theta(A, M, M) = \theta_A(M)$ is idempotent. Therefore, by combining these results with Theorem 24, and by putting $\theta_{M,N}(A) = \theta(A, M, N)$ we arrive to the following result

Theorem 27 *When $N \subseteq M$, the two flattenings $A \to \theta_{M,N}(A)$ and $M \to \theta_A(M)$ are connected morphological filters from $\mathcal{P}(E)$ into itself.*

Since flattenings turn out to be morphological filters, the following question arises. The flattenings have been introduced using openings γ_M and closings φ_{N^c}. Now, both products $\gamma_M\varphi_{N^c}$ and $\varphi_{N^c}\gamma_M$ are themselves morphological filters (as products of openings and closings) which in addition satisfy the inclusion $\gamma_M\varphi_{N^c} \supseteq \varphi_{N^c}\gamma_M$ (as filters by reconstruction). Does there exist any relationship between these two product filters and the flattening θ of same primitives ? Unfortunately the answer is negative. To make it clear that, take, for example, A to be a single grain with an internal pore B, and take $M = N = B$ (fig.11b). Clearly, we have

$$\begin{array}{lll} & \varphi_{M^c}(A) = A \cup M & \Longrightarrow \quad \gamma_M\varphi_{M^c}(A) = A \cup M, \\ \text{but} & \gamma_M(A) = \varnothing & \Longrightarrow \quad \varphi_{M^c}\gamma_M(A) = \varnothing \end{array}$$

whereas $\theta(A) = M$ is neither $\gamma_M\varphi_{M^c}(A)$ nor $\varphi_{M^c}\gamma_M(A)$. Moreover, the example shows that $\theta(A \cap \theta(A)) = \varnothing$ and that $\theta(A \cup \theta(A)) = A \cup M$. This implies that θ cannot be decomposed into the product of an opening by a closing or *vice versa* (Theorem 6-11, corollary 2 in[MAT88c]). Finally, note that in the example of fig.11b the boundary between the grain and its internal pore is preserved, but not the sense of variation (i.e. grain G becomes a pore whereas a pore adjacent to G becomes a grain). As a matter of fact, such a "flip-flop" effect, as well as other drawbacks, are due to pathological cases when M contains a pore of A, but misses the surrounding grain(s).

Therefore, to improve the situation, we have to slightly modify the marking criteria involved in γ_M and in φ_{N^c}.

7.2 Set leveling

F. Meyer [MEY98b] defines a leveling as a flattening on grey tone functions which preserve the sense of variation on every two neighboring pixels of a digital grid. When transposed in terms of set mappings acting in the framework of an arbitrary connection $\mathcal{C}$, this condition concerns uniquely the pairs (X, Y) of sets that are *adjacent*.

Definition 28 *Let $\mathcal{C}$ be a connection on $\mathcal{P}(E)$, and let $X, Y \in \mathcal{P}(E)$. Sets X and Y are said to be* adjacent *when $X \cup Y$ is connected, whereas X and Y are disjoint.*

Definition 29 *Given a connected component $A \in \mathcal{C}$ and a set $M \in \mathcal{P}(E)$, one says that A* touches M, *and one writes $A \parallel M$ when either A intersects M or is adjacent to a subset of M. By duality, one says that A* lies in M *when A does not touch M^c; one writes $A \subseteqq M$ or $A \nparallel M^c$.*

The duality under complement provides the two following equivalences

$$A \parallel M \Longleftrightarrow A \nsubseteqq M^c \text{ and } A \nparallel M \Longleftrightarrow A \subseteqq M^c$$

Note that relation $A \parallel M$ (A touches M) is less demanding than $A \cap M \neq \varnothing$ (A intersects M), since it accepts in addition that A and M be adjacent. Similarly, $A \subseteqq M$ (A lies in M), is more severe than $A \subseteq M$, since none of the grains of A and of M must be adjacent to each other.

When $\gamma_x(A) \neq \gamma_y(A)$ for an arbitrary $A \in \mathcal{P}(E)$, one cannot have $\gamma_x(A) \parallel \gamma_y(A)$ since $\gamma_x(A)$ is the largest element of $\mathcal{C}$ included in A. But $\gamma_x(A)$ may not touch some pores Y_i of A and, nevertheless, touch their union $\cup Y_i$. For example, for the connection by opening introduced in section 1.2, none of the two point pores of the lateral gulf, in fig.12a, is adjacent to set A, whereas their union touches it. The most powerful connections are those which prevent this perverse effect, i.e. which fullfill the following condition

Condition 30 *A connection $\mathcal{C}$ on $\mathcal{P}(E)$ is* adjacency preventing *when, for any element $M \in \mathcal{P}(E)$ and any family $\{B_i\ , i \in I\}$ in $\mathcal{C}$, to say that M is adjacent to none of the B_i is equivalent to saying that M is not adjacent to $\cup B_i$.*

Let us return to the flattenings θ. Replace opening γ_M, as defined by relation (11) by

$$\overline{\gamma}_M(A) = \cup \{\gamma_x(A)\, , \, x \in E\, , \, \gamma_x(A) \Uparrow M\}$$

Operator $\overline{\gamma}_M$ is still a marker-based opening by reconstruction. For sake of duality, we have to provide φ_{N^c} with a similar structure, hence to replace it by

$$\left[\underline{\varphi}_{N^c}(A)\right]^c = \cup \{\gamma_x(A^c)\, , \, x \in E\, , \, \gamma_x(A^c) \Uparrow N\}\, .$$

This results in the following definition of a leveling

Definition 31 *A flattening of primitives $\overline{\gamma}_M$ and $\underline{\varphi}_{N^c}, M, N \subseteq E$, is called leveling λ, i.e.*

$$\lambda = \overline{\gamma}_M \cup (\complement \cap \underline{\varphi}_{N^c}) \quad . \tag{15}$$

This definition recovers Meyer's one, in the digital case. Notice that in the situation of fig.11b it yields $\lambda(A) = A$ (neither the grain nor its internal pore are modified). Mapping λ is thus less active than θ. One can easily verify that, *mutatis mutandis,* Propositions and Theorems 21 to 27 are still valid for leveling. Furthermore, Theorem 27 is strengthened by Theorem 33 below, according to which the leveling λ may be written as the product of $\overline{\gamma}_M$ by $\underline{\varphi}_{N^c}$ composed in an arbitrary order, hence is a strong filter.

Lemma 32 *Let* $\mathcal{C}$ *be an adjacency preventing connection on* $\mathcal{P}(E)$*, let* $A, N \in \mathcal{P}(E)$*, and let* Y *be a pore of* A*. If* Y *lies in* N*, then all those grains of* A *which are adjacent to* Y *intersect* N*. By duality, if a grain* X *of* A *does not touch set* N*, then none of the pores of* A *adjacent to* X *is included in* N *.*

Proof. Consider a pore Y of A, with $Y \subsetneqq N$, and a grain X of A which is adjacent to Y. The assumption of adjacency prevention implies that there exists at least one point $x \in X$ which is adjacent to Y ; moreover, since N^c is not adjacent to Y , (hence none of its points is adjacent to$Y$), x belongs necessarily to N , i.e. $X \cap N \neq \emptyset$.

Now take a grain X of A that does not touch N, i.e. such that $X \subsetneqq N^c$. We draw from the first part of the proof that every pore Y of A that is adjacent to X meets N^c, hence is not included in N. ∎

Theorem 33 *Let* $\mathcal{C}$ *be a connection on* $\mathcal{P}(E)$*. Given* $M, N \subseteq E$ *with* $M \subseteq N$*, the leveling* $\lambda_{M,N} : \mathcal{P}(E) \to \mathcal{P}(E)$ *of primitives* $\overline{\gamma}_M$ *and* $\underline{\varphi}_{N^c}$ *is a* strong *connected filter, and admits the double decomposition*

$$\lambda = \overline{\gamma}_M \ \underline{\varphi}_{N^c} = \underline{\varphi}_{N^c}\overline{\gamma}_M \tag{16}$$

if and only if connection $\mathcal{C}$ *is adjacency preventing.*

Proof. For the "if" proof, we have to show that the three following operations are identical:

i/ to take the union of the pores Y of A lying in N and of the grains X of A touching M;

ii/ to take the union $A\prime$ of the grains of A touching M, and to add this union to the pores of $A\prime$ that lie in N ;

iii/ to add to A all its pores lying in N, and to extract from the result the union of all grains touching M.

We begin by showing that operations *i/* and *iii/* are identical. Clearly, every grain of *i/* is also a grain of *iii/*. But a grain of *iii/* is either a grain X of A that touches M (hence it belongs to union *i/*) , or the union of a pore Y of A lying in N and of all its adjacent grains X_i in set A. Then according to the lemma, each of these grains X_i intersects set N, and a fortiori set M, since $N \subseteq M$, and consequently belongs to the result of operation *i/* . Therefore the two processings *i/* and *iii/* are identical. The proof is achieved by observing that *i/* is a self-dual procedure, and that *ii/* and *iii/* are dual of each other.

The "only if" part may be proved by means of a counterexample. Equip $\mathbb{Z}^2$ with the connection by square openings of size 2×2, and consider the set A of $\mathbb{Z}^2$ represented in fig.12a. This set is composed of five grains, namely two 2×2 squares, plus three points. The marker $N = M$ is composed of the two points located between the two squares of A (they are surrounded by a rectangle in fig.12a) . We have $\underline{\varphi}_{N^c}\overline{\gamma}_M = \overline{\gamma}_M \neq \overline{\gamma}_M \ \underline{\varphi}_{N^c} = \ \underline{\varphi}_{N^c}$, as shown in fig.12b and fig.12c respectively. Hence a non adjacency preventing connection yields a result that does not satisfy relation (16). ∎

Relation (16) can alternatively be obtained by introducing the notion of a stable operator [CRE93] and by applying proposition 10.3 of Heijmans study [HEI97]. Our above approach focuses on a characteristic property of the connection itself, and not of some particular types of operators. To conclude this section, we examine some possible laws of variation for leveling and flattenings. First, we draw from the double decomposition of a leveling that

$$\begin{matrix} M_1 \supseteq M_2 \\ \\ N_1 \subseteq N_2 \end{matrix} \quad \Rightarrow \quad \lambda_{M_1,N_1}.\lambda_{M_2,N_2} = \lambda_{M_2,N_2}\lambda_{M_1,N_1} = \lambda_{M_2,N_1}.$$

But if we also require the self duality for λ, *i.e.* $M = N$, then the above semi group property reduces to one element, which limits its interest. However, we can take the problem differently, and more realistically, by making M vary, *given* A. For sake of simplicity, we take $M = N$, although self-duality is not required here. The relevant formalism to develop the point is that of the activity ordering for sets (and no longer for set mappings)[MAT97]. As a matter of fact, any fixed set A generates an ordering on $\mathcal{P}(E)$ denoted by $\preceq_A$:

$$M_1 \preceq_A M_2 \qquad \text{if} \qquad M_1 \cap A \supseteq M_2 \cap A \qquad \text{and} \qquad M_1 \cap A^c \subseteq M_2 \cap A^c$$

From this ordering derives the so-called A-*activity lattice*, where the supremum and the infimum of a family $\{M_i, i \in I\}$ of sets are given by

$$\begin{aligned} \curlyvee_A M_i &= [A^c \cap (\cup M_i)] \cup [\cap M_i] \\ \curlywedge_A M_i &= [A \cap (\cup M_i)] \cup [\cap M_i] \end{aligned}$$

A itself is the least element, and A^c as the greatest element (Note that these concepts are very similar to those presented earlier for the operator activity lattice).

Given A, consider the operators $M \to \lambda_A(M) = \lambda(A, M)$ and $M \to \theta_A(M) = \theta(A, M)$ from $\mathcal{P}(E)$ into itself. These operators turn out to be openings on the A-activity lattice. Indeed, according to relations (10), $\theta_A(M)$ coincides with $\gamma_M(A)$ (or $\lambda_A(M)$ with $\overline{\gamma}_M(A)$) on A. Hence, it suffices to reduce $A \cap M$ for reducing $A \cap \lambda_A(M)$, and similarly to increase $A^c \cap M$ for enlarging $A^c \cap \lambda_A(M)$. In other words, we have

$$M_1 \preceq_A M_2 \Rightarrow \lambda_A(M_1) \preceq_A \lambda_A(M_2) \quad \text{and} \quad \theta_A(M_1) \preceq_A \theta_A(M_2)$$

i.e. θ_A and λ_A are increasing on the A-activity lattice. They are also anti-extensive, since we can write

$$A \cap M \subseteq A \cap \gamma_M(A) = A \cap \theta_A(M) \subseteq A \cap \overline{\gamma}_M(A) = A \cap \lambda_A(M)$$

and, similarly, $A^c \cap M \;\supseteq\; A^c \cap \theta_A(M) \supseteq A^c \cap \lambda_A(M)$.

Equivalently, we can write

$$\theta_A(M) \preceq_A M \quad \text{and} \quad \lambda_A(M) \preceq_A M.$$

The third and last axiom, i.e., idempotence, derives from Theorem 27.

A pyramid, as defined in [SER93], is a family $\{\psi_\lambda, \lambda > 0\}$ of operators from a space E into a space F, which depend on a positive parameter λ such that for each $\lambda \geq \mu > 0$, there exists a $\nu \geq 0$ with $\psi_\nu \psi_\mu = \psi_\lambda$ (for any $\lambda \geq \mu$, the operator ψ_λ can be obtained either directly, or by starting from ψ_μ and composing it by ψ_ν). Is it possible to generate pyramids by means of such openings ? The answer is yes. Consider two markers M_1 and M_2 with $M_1 \preceq_A M_2$, and perform the opening γ_{M2} on the result $\theta_A(M_1)$ of the smaller flattening. As $M_1 \cap A \supseteq M_2 \cap A$, the grains of A that are intersected by M_2 are exactly the grains of $\theta_A(M_1)$ intersected by M_2, so that

$$A \cap \theta_{\theta_A(M_1)}(M_2) = A \cap \gamma_{M_2}[\theta_A(M_1)] = A \cap \gamma_{M_2}(A) = A \cap \theta_A(M_2).$$

By duality, we can write also

$$A^c \cap \theta_{\theta_A(M_1)}(M_2) = A^c \cap \theta_A(M_2)$$

i.e., finally

$$\theta_{\theta_A(M_1)}(M_2) = \theta_A(M_2).$$

If we start with the stronger flattening, we find similarly

$$\theta_{\theta_A(M_2)}(M_1) = \theta_A(M_2).$$

Since the approach is also valid for leveling, we can combine all these results in the following theorem

Theorem 34 *Given set A, the flattening $M \to \theta_A(M)$ and the leveling $M \to \lambda_A(M)$ from the A-activity lattice of $\mathcal{P}(E)$ into itself are openings. Moreover, if $M_1 \preceq_A M_2$ we have*

$$\theta_{\theta_A(M_1)}(M_2) = \theta_{\theta_A(M_2)}(M_1) = \theta_A(M_2)$$
$$\lambda_{\lambda_A(M_1)}(M_2) = \lambda_{\lambda_A(M_2)}(M_1) = \lambda_A(M_2).$$

This last granulometric type pyramid is specially useful in practice, for it allows us to grade the activity effects of markers: it means that we can directly implement a highly active marker, or, equivalently, reach it by intermediary steps. An example is given in fig.14a and b. Notice that the first part of the theorem has already been established by Matheron (Theorems R-2 and R-4 in [MAT97]) for flattenings, among various other algebraic properties.

7.3 Discussion

We are now in a better position to compare the Meyer-Matheron approach, based on flattenings and leveling, with the more general grain operators of Heijmans, where the binary criteria for changing the status of a grain or a pore are not *a priori* submitted to any condition such as increasingness, or existence of markers.

What happens if we accept, in a flattening $\theta = \gamma \cup (\complement \cap \varphi)$, openings and closings by reconstruction in the broad sense of section 6 ? Then θ depends on two binary increasing criteria, m for the opening and n for the closing, say. The constraint for increasingness becomes the condition that the supremum of m and n is identically one (proposition 8.5 in [HEI97]). This means, in case of markers M and N^c that every set of $\mathcal{P}(E)$ must hit M or N^c (which is true iff $N \subseteq M$). But such a condition is hardly compatible with the area criteria. For example, take $m(A) = [area(A) \geq a]$ and $n(A) = [area(A) \geq b]$, then the supremum $m(A) \vee m(A)$ is not identically one in general, and if we change the sense of one of these two inequalities, we no longer deal with openings (or closings). Finally, markers based operations seem to provide the most convenient type of primitives for increasing flattenings. Among others, they directly yield the self-dual ones.

However, there exist other ways than flattenings to obtain self-dual connected filters. When γ and φ are by reconstruction (in the broad sense), then $\gamma\varphi$ and $\varphi\gamma$ are strong filters and $\gamma\varphi \supseteq \varphi\gamma$, hence the center $\alpha = (Id \cap \gamma\varphi) \cup \varphi\gamma$ is in turn a strong filter which becomes self-dual as γ and φ are dual of one another. However, experience shows that this operator, which was the first to be developed (chapter 7.8 in[SER88]), does not allow a wide range of activities [SER93].

8 Function flattenings and levelings

8.1 From sets to functions

We now return to the function lattice L^E. An increasing operator Ψ on L^E is said to be *flat* if there exists an increasing set operator ψ such that

$$X(\Psi(f), t) = \bigcap_{s<t} \psi\left[X(f, s)\right] \tag{17}$$

where $X(f, t)$ stands for the thresholding of function f at level t, i.e. :

$$X(f, t) = \{x : x \in E, \quad f(x) \geq t\} \tag{18}$$

H. Heijmans [HEI91] has shown that every increasing flat function operator Ψ admits a unique set generator ψ. Moreover, when the set generator ψ is $\downarrow$-continuous, then (17) takes the simple form [SER82]

$$X\left[\Psi(f), t\right] = \psi\left[X(f), t\right] \tag{19}$$

In particular, in the finite cases of digital imagery, relation (18) is sufficient to characterize the function operator Ψ associated with an increasing set operator ψ. From these two relations, and Theorem 25, we define the numerical versions Θ and Λ of the increasing flattenings and levelings using their cross sections.We start with the discrete case.

Definition 35 *Let f, g, h, be three functions from Z^n into L, and let $g \leq h$. The relations*

$$X\left[\Theta(f), t\right] = \theta\left[X(f,t)\right], X(g,t), X(h,t)]$$
$$X\left[\Lambda(f), t\right] = \lambda\left[X(f,t)\right], X(g,t), X(h,t)]$$

define one and only one flattening $\Theta(f)$ as well as one and only one leveling $\Lambda(f)$ on L^{Z^n}.

If connection $\mathcal{C}$ is obtained using iterations of an elementary dilation δ, with adjoint erosion ε, then a digital algorithm for $\Theta(f)$ and $\Lambda(f)$ as functions of f, g and h has been given by Matheron in the self dual case, *i.e.* for $g = h$ [MAT97]. According to this algorithm, the flattening $\Theta(f)$ and the leveling $\Lambda(f)$ turn out to be the limits of the recursive sequence

$$g_n = (f \vee \varepsilon g_{n-1}) \wedge \delta g_{n-1} = (f \wedge \delta g_{n-1}) \vee \varepsilon g_{n-1}$$

$$\text{with} \quad \begin{array}{lll} g_1 = & (f \wedge \delta(g \wedge f) \vee \varepsilon(g \vee f) & \text{for flattenings} \\ g_1 = & (f \wedge \delta g) \vee \varepsilon g & \text{for levelings} \end{array}$$

For the same type of connections, but for any pair (g, h) with $g \leq h$, one can obtain the leveling by applying the decomposition Theorem 31, and computing the opening by reconstruction $g_\infty(f)$ and successively $\Lambda(f) = h_\infty[g_\infty(f)]$. The first operation is thus given by the limit of the sequence

$$g_n = (f \wedge \delta g_{n-1}) \qquad \text{with} \qquad g_1 = (f \wedge \delta g)$$

and the second one by

$$h_n = [g_\infty(f) \vee \varepsilon h_{n-1}] \qquad \text{with} \qquad h_1 = [g_\infty(f) \vee \varepsilon h]$$

This second technique is usually better than the previous one, since the two primitives are implemented by fast recursive algorithms in most of the morphological packages. One can check numerically that the two factors h_∞ and g_∞ do commute in algorithm $\Lambda(f) = h_\infty[g_\infty(f)]$ (Theorem 31).

Let us now consider the case of non discrete grey tone values. We will suppose for the sake of simplicity that E is a separable topological space, and that $\mathcal{C}$ is a connection such that for each $x \in E$, the connected γ_x opening is $\downarrow$-continuous on the closed sets of E, and $\uparrow$-continuous on the open sets of E. If we consider

- an upper semi continuous function $f : E \to L$,
- a marker g whose cross sections $X(g,t)$ are compact,
- and a second marker h smaller or equal to g,

then the definition of the discrete case extends directly (without these assumptions, we should have to consider the whole stack of sections).

8.1.1 Self-duality

All properties concerning self-duality, found in the binary case, such as Proposition 26, extend directly to numerical cases. If 0 and m stand for the two extreme bounds of the grey axis L, we have for example

$$m - \Lambda(m - f, m - g, m - g) = \Lambda(f, g, g)$$

which means that the leveling $f, g \to \Lambda(f, g)$ is always a self-dual mapping (here the complement operation is replaced by its function analogue $f \to m - f$). In addition, if the marker is computed with a self-dual operator, i.e. $g = g(f)$, where the operator g satisfies

$$m - g(m - f) = g(f)$$

for every f, then the leveling $f \to \Lambda(f, g(f), g(f))$ is self-dual and we have

$$m - \Lambda[m - f, g(m - f), g(m - f)] = \Lambda[f, g(f), g(f))]$$

The same comments apply to flattenings as well. Operators g, such as convolution, or median operators do provide self-dual markers. The reader will find another provider of self-duality for markers in the example of fig.14a and 14b below. Notice that the above relation of self-duality for g is distinct from that of *invariance* under complementation, namely

$$g(m - f) = g(f)$$

This latter relation is satisfied by every symmetrical function of the minima and of the maxima of f, or by any function of the module of its gradient, for example. But it does not imply the general self-duality of $\Lambda(f, g, g)$.

8.1.2 Examples

In practice, the role of the marker is crucial. In fig.13 to 15, three markers g are compared when the planar digital connection is that of the hexagonal grid. For the three markers we take $h = g$. The first marker g is an alternating sequential filter of size two of f, starting with the opening (observe that it is not self-dual). The corresponding leveling is presented in fig.13b.

The second pair of markers is obtained by replacing f by zero on the extended maxima and minima of f, and by leaving f unchanged elsewhere.

To compute the extended maxima of f , we take the following steps. Perform the opening by reconstruction $\gamma_{\mathrm{rec}}(f)$ of f from $f - k$, where k is a positive constant. Then the maxima of $\gamma_{\mathrm{rec}}(f)$ define the so called *extended maxima* of f, and those points x where $f(x) - \gamma_{\mathrm{rec}}(f)(x) = k$ define the (non extended) maxima of f of dynamics $\geqslant k$; the extended minima are obtained by duality.

The corresponding levelings are shown in fig.14a and 14b, for markers g_{30} and g_{60}, of dynamics 30 and 60 respectively (over 256 grey levels).

These two markers are self-dual by construction, and satisfy the condition of activity increasingness of Theorem 32. Their progressive leveling action appears clearly when confronting fig. 14a and 14b. Notice the relatively correct preservation of some fine details such as buttons, eyes, eyebrows, fingers, etc.. These details are preserved because of their high dynamics. We also point out that, in the current example, there is no visual difference between these levelings and the corresponding flattenings.

Alternatively, we could have replaced g_{30} and g_{60} by the suprema of those maxima and minima of f which have dynamics greater than 30 and 60, respectively. The contrasts would have been better protected, but the self-duality would have been lost in this case.

One can compare the above three processings by computing the total area occupied by the flat zones which have non empty interiors. In the initial image, which is composed of 58,240 pixels, the area of such flat zones is 24,568. Under the leveling action, they increase up to 41,670 ; 41,621 and 43,173 pixels for images 13b, 14a and 14b, respectively. Hence, for a same compression rate in terms of flat zones, the asf marker is perceptually worse, since it loses more fine structures.

In the third example, the initial image has been corrupted by Poisson noise (fig.15a). A gaussian convolution reduces the noise, but also smoothes all transitions (fig.15b). However, if we use the convolved image 15b as a (self-dual) marker acting by leveling on the corrupted image, this second operation recovers the initial sharpness of the edges, while keeping down the noise (fig.15c).

A last word. The approach we have followed in this paper emphazises the increasing operators. Alternatively, we could have dropped this condition from the flattening step. Then, starting from any (binary) grain operator, one can always extend it to the numerical case by using equation (16). A number of the above propositions would disappear, but the algorithms may present some practical interest.

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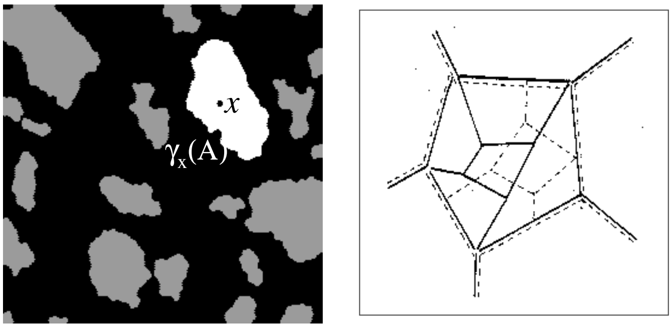
Figures



a) set A and $\gamma_x(A)$ *b) sup of two partitions*

Figure 1: *a) lattice $\mathcal{P}(E)$ of the sets of E, and connected component $\gamma_x(A)$ of A at point x; b) an example of partition lattice: two partitions and their supremum.*

a) *b)* *c)*

figure 2: *a) initial set, made of 28 grains;*

b) for the connection according to a disc dilation (here of radius 8) there are 5 grains only;

c) the isolated grains are those which are the same in a) and b), i.e. whose dilate misses the skeleton by zones of influence of set a).

a) Osteocytes *b) mask* *c) 3D cells*

Figure 3: *a) section n*°15 *in a stack of* 60 *microscopical confocal sections. Three osteocytes appear;*

b) mask obtained by the union of a low thresholding over the 60 sections ;

c) reconstruction of a high threshold of the cells inside marker b) (perpective view).

a) constant weights *b) Lipschitz function weights*

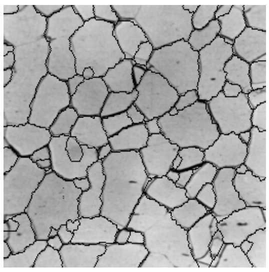
Figure 4: *suprema (continuous lines) and infima (dashed zones), in lattice* $\mathcal{P}_\rho$*,of two weighted sets (dotted lines). In a), the weights are constant, in b), they are Lipschitz functions. These weighet sets generate connected G-cylinders, in the sense of Definition 15, when they are combined with the partitions they induce on* $\mathbb{R}^1$

a) flat zones *b) bounded variation*

figure 5: *two basic moduli of continuity, which associated with the two connections by flat zones and by bounded variation (also called "jump connection of range k ") respectively.*

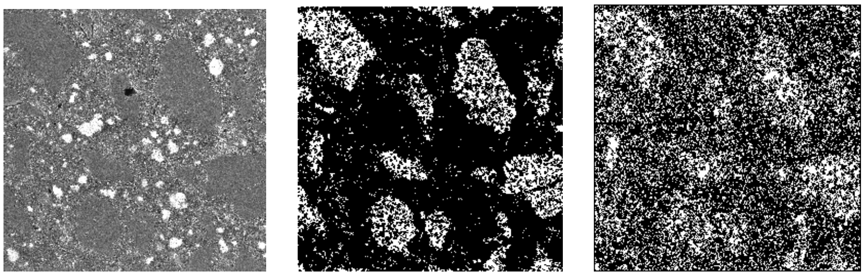
a) Initial image *b) Jump connection*

Figure 6: *A jump connection of range 14 (over 256 grey tones) on image a) yields image b) which comprises 94 regions. Each of them is represented by its average grey.*



a) Initial image *b) Jump connection* *c) Derived SKIZ*

Figure 7*: a) optical micrograph of a polished section of alumina; b) jump connection of range 12 (in dark, the point connected components; in white, each particle represents the base of a cylinder in the jump connection); c) skeleton by zones of influence of the set provided by image b).*

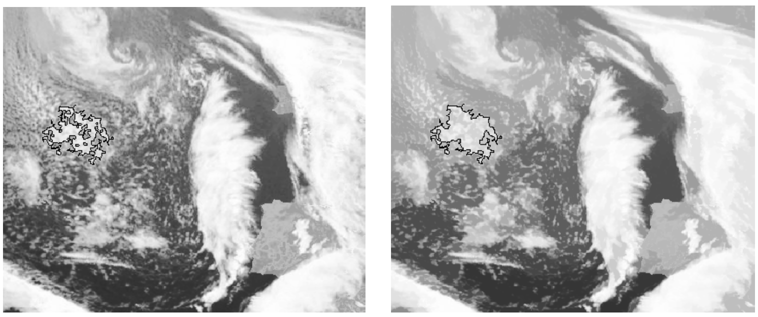


a) Initial image *b) Smooth connection* *c) Jump connection*

Figure 8 *: a) Electron micograph of a petrographic polished section; the two phases have the same histogram, but one is more uniform than the other; b) smooth path connection of slope 6; c) jump connection of range 15. In b) and c) the blacks are the union of all connected components that are reduced to points.*

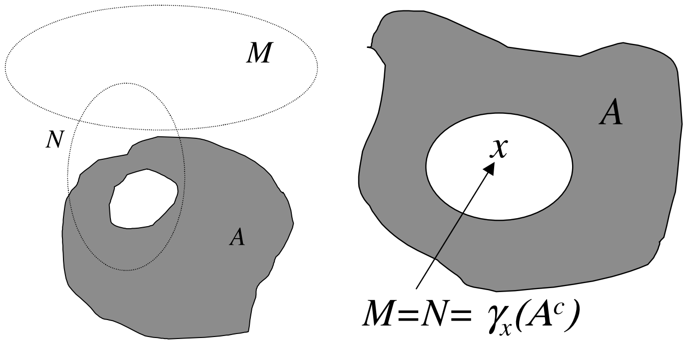
a) initial image *b) ASF of size 4* *c) ASF of size 8*

Figure 9 : *An example of a pyramid of connected alternated sequential filters. Each contour is preserved or suppressed, but never deformed : the initial partition increases under the successive filters, which are strong and form a semi-group.*



a) *b)*

Figure 10 : *a) Satellite image of the Acores Islands area; b) connected closing by reconstruction of a) . According to proposition 11, the connected closing generates a new connection on the lattice of the weighted partitions by flat zones. In order to show this effect, one connected component has been followed through the process (the cloud surrounded by a dark line).*



a) *b)*

Figure 11: *a) Non increasingness of θ when $N \nsubseteq M$.Take for A' grin A plus its pore; then A$\subseteq$ A' whereas $\theta(A') \subseteq \theta(A)$. b) Take the internal pore of grain A as M and N, then $\theta(A)$ equals the pore without the grain (flip-flop effect).*

a) *b)* *c)*

Figure 12 : *a) In black: initial set A ; inside the small rectangle: marker M (made of two points). For the opening connection by a 2×2 square, set A is composed of five grains, namely the two 2×2 squares and three point connected components ;*

b) Set $\underline{\varphi}_{N^c}\overline{\gamma}_M(A)$;

c) Set $\overline{\gamma}_M\ \underline{\varphi}_{N^c}(A)$; the difference between b) and c) comes from that the opening connection is not adjacency preventing.

a) b)

Figure 13 : *a) Manet' Joueur de fifre (detail);*
b) leveling of a) by taking an ASF filter as marker.

a) b)

Figure 14: *Levelings of picture 11a, by taking a marker based on extended extrema; the dynamics is equal to 30 in a) and 60 in b), over 256 gray levels.*

a) b) c)

Figure 15: *a) initial image plus Poisson noise;*
b) convolution of a) by a disc of radius 5;
c) leveling of a) by marker b) (the noise is removed, but the contours are those of the initial image).