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Geodesy and connectivity in lattices

Christian Ronse[*]
LSIIT UPRES-A 7005,
Université Louis Pasteur, Département d'Informatique
Boulevard Sébastien Brant, 67400 Illkirch (FRANCE)
ronse@dpt-info.u-strasbg.fr
URL: http://arthur.u-strasbg.fr/∼ronse/

Jean Serra
Centre de Morphologie Mathématique,
Ecole des Mines de Paris,
35, rue Saint-Honoré, 77305 Fontainebleau Cedex (FRANCE)
serra@cmm.ensmp.fr

Abstract. This paper generalizes the notion of symmetrical neighbourhoods, which have been used to define connectivity in the case of sets, to the wider framework of complete lattices having a sup-generating family. Two versions (weak and strong) of the notion of a symmetrical dilation are introduced, and they are applied to the generation of "connected components" from the so-called "geodesic dilations". It turns out that any "climbing" "weakly symmetrical" extensive dilation induces a "geodesic" connectivity. When the lattice is the one of subsets of a metric space, the connectivities which are obtained in this way may coincide with the usual ones under some conditions, which are clarified. The abstract theory can be applied to grey-level and colour images, without any assumption of translation-invariance of operators.

Keywords: neighbourhood, symmetry, connectivity, geodesy, complete lattices, mathematical morphology, metrics.

[*] Address for correspondence: LSIIT UPRES-A 7005, Université Louis Pasteur, Département d'Informatique, Boulevard Sébastien Brant, 67400 Illkirch, France

1. Introduction

In digital image processing, geometrical algorithms rely often on a choice of an adjacency relation between pixels, such as the well-known 4- and 8-adjacencies. Here the adjacency relation is equivalent to the notion of pixel neighbourhood: to every pixel x we associate its neighbourhood $N(x)$ made of all pixels y which are adjacent to x; since the adjacency is symmetrical (y is adjacent to x iff x is adjacent to y), the neighbourhood must be symmetrical in the sense that $y \in N(x)$ iff $x \in N(y)$. The adjacency (or neighbourhood) can be used to define such concepts as connected objects, or connected components. In mathematical morphology one encounters for example the so-called "reconstruction from a marker": we have an object M called the *mask,* which represents the data to be analysed, and a set X called the *marker*, corresponding to some properties, and we want to obtain all connected components of M which intersect X. The straightforward algorithm for this purpose is to initialize the reconstruction R with $X \cap M$, and to iteratively increment it by adding the neighbourhood in M of each of its pixels:

$$R := R \cup \Big(\bigcup_{p \in R} \big(N(p) \cap M \big) \Big) \ ;$$

this is repeated until no further point is added to R.

The operation of adding the neighbourhood of each pixel of the figure, is a dilation, and when this neighbourhood is restricted to the mask, it becomes a *geodesic dilation*. This leads immediately to the dual definition of a *geodesic erosion*, and other geodesic operators can then be devised (distance transform, zones of influences, etc.)

Historically speaking, this has been the starting point from which the axioms for morphological connectivity progressively arose. Trivially, changing the definition of adjacency (or of the neighbourhood) modifies the class of connected components; but in a subtler way, the operations of reconstruction from a marker, or simply of taking the connected component of a set containing a given point, have a set of algebraic properties, which can be taken to define abstractly the notion of a connected component, and hence of a connected set. This approach was used in [16] to characterize axioms for connectivity on sets; further examples were given in [10] and equivalent axioms have been proposed in [12]. But a number of questions come to mind: is such an approach limited to the case of sets, i.e., binary images? Otherwise, to which families of pictorial objects can it be applied? In the above reconstruction algorithm, does one need dilations to expand the markers, or can one take instead other operators which preserve connectivity? If the space of pictorial objects is not *a priori* equipped with a connectivity, to which extent does such a reconstruction algorithm lead to a connectivity? Finally, in the Euclidean spaces, what are the links between these operations and the classical topological connectivity? Surprisingly, the answers to these questions depend on the *symmetry* of the operators, a concept which will be clarified here, but also on their *extensivity* (the fact that they enlarge objects) and related properties (namely, what we call *climbing*).

The main conceptual problem is the generalization of the notions of a symmetrical neighbourhood and hence of a symmetrical dilation, which is straightforward for sets of points (binary images), to an abstract framework covering many families of pictorial objects, in particular grey-level or color images. There is another topic in mathematical morphology where steps towards a wider notion of symmetry were made: annular filters. The starting point comes from [16]: let A be a symmetrical structuring element ($a \in A \iff -a \in A$); then the set operator $X \mapsto X \cap (X \oplus A)$ is a translation-invariant algebraic opening which removes from a set X all *isolated* points, that is all $p \in X$ such that $A_p \cap X = \emptyset$; this operator is called the *annular opening by* A.

In [13], annular openings were generalized in two ways. First, in the case of sets, the annular opening can be defined without the assumption of translation-invariance: we replace the symmetrical structuring element A by a symmetrical neighbourhood $N(x)$ associated to every point x (the symmetry being that $y \in N(x) \iff x \in N(y)$), and the dilation by A becomes the dilation $\bigcup_{x \in X} N(x)$ by the neighbourhoods. Second, the notion of a symmetrical structuring element can be to extended grey-level functions, and more generally to an arbitrary complete lattice having a sup-generating family on which an abelian group of lattice automorphisms acts transitively; this led to the construction of annular openings on such a lattice. Finally, the approach of [13] was technically improved in [14]; in particular other types of annular filters were studied.

This work on annular filters in an abstract framework was our inspiration for the search of a general definition of symmetry that would lead to connectivity by geodesic reconstruction. As for many problems in mathematical morphology, the solution to a general question can often be found by looking at what happens with grey-level images, in other words numerical functions.

We will give below the definition of a symmetrical structuring function for grey-level images, which was presented in [13, 14], and compare it to more straightforward definitions of symmetry. The main lesson of this investigation of annular openings is that the notion of symmetry, which is clear and unambiguous for sets, is not uniquely defined for other objects. In particular for numerical functions, we will find 3 different definitions of symmetry, from the most exacting to the least demanding.

1.1. Symmetry for numerical functions

We consider grey-level images as functions $E \to T$, where E is the space of points, and T is the set of grey-levels, which can be $\overline{\mathbf{Z}}$ or $\overline{\mathbf{R}}$; T is a complete lattice for the ordering by $\leq$, and has $-\infty$ and $+\infty$ as least and greatest elements. As sets are generated by points, grey-level images are generated by joining together "grey-level points"; more precisely, given a point $p \in E$ and a grey-level $t \in T \setminus \{\pm\infty\}$, the "impulse" function $i_{(p,t)}$ defined by

$$\forall x \in E, \qquad i_{(p,t)}(x) = \begin{cases} t & \text{if } x = p, \\ -\infty & \text{if } x \neq p, \end{cases} \tag{1}$$

represents in some way the point p with grey-level t attached to it; now such impulse functions form a sup-generating family for grey-level functions, this means that every function $f : E \to T$ is the supremum of a family of impulses, namely those less than or equal to it:

$$f = \bigvee \{ i_{(p,t)} \mid p \in E,\ t \in T \setminus \{\pm\infty\},\ \text{and } t \leq f(p) \} \ . \tag{2}$$

It is customary to associate to f a subset of $E \times (T \setminus \{\pm\infty\})$ called its *umbra*, that is the set

$$U(f) = \{ (p,t) \mid p \in E,\ t \in T \setminus \{\pm\infty\},\ \text{and } t \leq f(p) \} \ . \tag{3}$$

The similarity between (2) and (3) shows the correspondence $(p,t) \leftrightarrow i_{(p,t)}$ between points in the umbra of a function and impulses generating that function.

Let us write $\mathrm{supp}(f)$ for the *support* of the function f, namely the set of all points $p \in E$ where $f(p) > -\infty$. Note that for $(p, t) \in U(f)$ we have $p \in \mathrm{supp}(f)$. For $p \in E$ and $t \in T \setminus \{\pm\infty\}$, the *translate* of the function f by (p, t) is the function $f_{(p,t)} : E \to T : x \mapsto f(x - p) + t$.

In order to define a "symmetrical" structuring function f, we need to extrapolate the condition "$a \in A \iff -a \in A$" characterizing symmetrical sets. A straightforward transposition would give

$$\forall p \in E,\ \forall t \in T \setminus \{\pm\infty\}, \qquad (p, t) \in U(f) \iff (-p, -t) \in U(f) \ , \tag{4}$$

which means that f verifies the following two conditions:

1. $\mathrm{supp}(f)$ *is a symmetrical set, and*

2. *for every* $p \in \mathrm{supp}(f)$, $f(p) = +\infty$.

In other words, f is a cylinder with symmetrical support, and having an infinite height all over it. This condition corresponds to what we will call *strong symmetry*.

In fact umbras are redundant descriptions of functions, in other words the family of impulses used in (2) to generate f is redundant. The basic idea in [13] was to require $(-p, -t) \in U(f)$ only for points $(p, t) \in U(f)$ in a portion of the umbra sufficient to reconstruct f; this was expressed as follows:

$$\forall (p, t) \in U(f),\ \exists s \geq t \text{ such that } (p, s) \in U(f) \text{ and } (-p, -s) \in U(f) \ .$$

Following [14], an equivalent formulation is:

$$\forall p \in E, \quad f(p) = \sup\{s \mid (p, s) \in U(f) \text{ and } (-p, -s) \in U(f)\} \ .$$

The above condition can then be expressed as follows:

1. $\mathrm{supp}(f)$ *is a symmetrical set, and*

3. *for every* $p \in \mathrm{supp}(f)$, $f(p) + f(-p) \geq 0$.

We call this condition *annular symmetry*. Then the operator $g \mapsto g \wedge (g \oplus f)$ is a translation-invariant algebraic opening, called again "annular opening" [13]. We will not use this annular symmetry in the remainder of this paper.

Finally, we can consider a weaker symmetry condition

$$\forall (p, t) \in U(f),\ \exists s \leq -t \text{ such that } (-p, s) \in U(f) \ . \tag{5}$$

This amounts to requiring only:

1. $\mathrm{supp}(f)$ *is a symmetrical set.*

This corresponds to what we will call *weak symmetry*. Clearly strong symmetry implies annular symmetry, and the latter implies weak symmetry. We illustrate strong and weak symmetry in Figure 1. These are the two notions that we will use in the following.

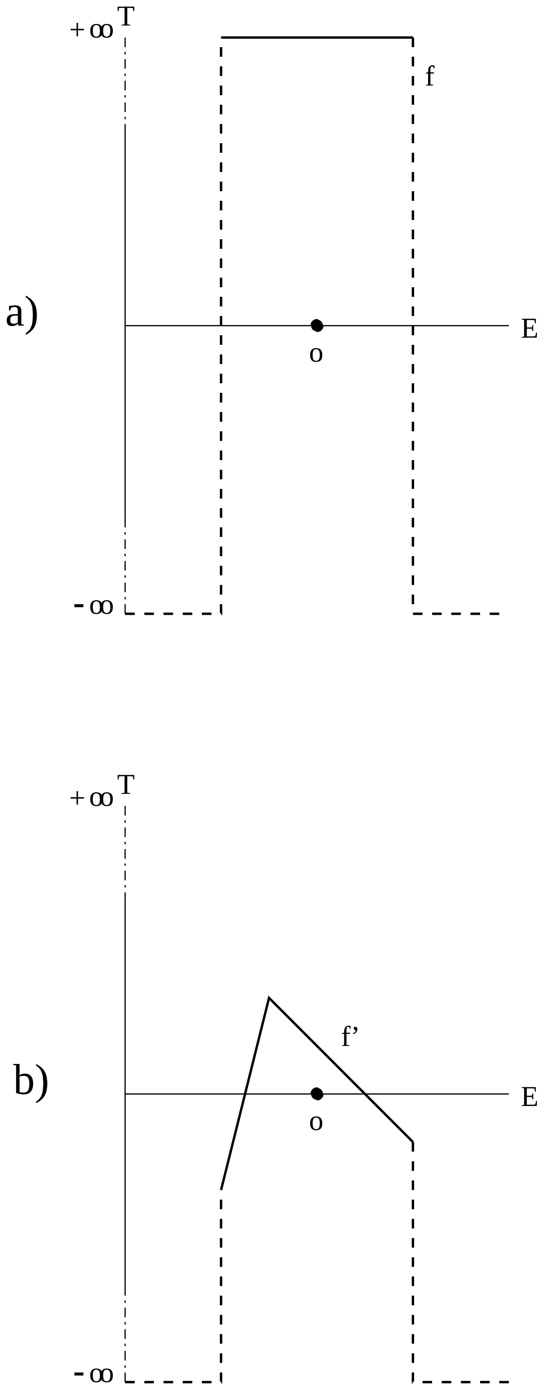

Figure 1. a) A strongly symmetrical function $f : E \mapsto T$. b) A weakly symmetrical function $f' : E \mapsto T$. (For both functions, the part of the graph corresponding to the support is shown with a plain line, and the rest with dashed lines.)

1.2. Symmetrical dilations, geodesic reconstruction, and connectivity

Both in annular openings and in geodesic reconstruction from markers, symmetrical structuring elements or neighbourhoods are used in a dilation. In the case of sets, the dilation by a neighbourhood function $N : x \mapsto N(x)$ is the map $\delta_N : \mathcal{P}(E) \to \mathcal{P}(E) : X \mapsto \bigcup_{x \in X} N(x)$. Given a "symmetrical" structuring function f, the dilation by f is the map $\delta_f : T^E \to T^E : g \mapsto g \oplus f$; here the role of points in the case of sets is taken by the impulse functions $i_{(p,t)}$ defined in (1), so the "neighbourhood" of $i_{(p,t)}$ ($p \in E$, $t \in T \setminus \{\pm\infty\}$) is given by $\delta_f(i_{(p,t)}) = i_{(p,t)} \oplus f = f_{(p,t)}$, the translate of f by (p,t). Then the strong symmetry condition on f can be expressed as follows:

$$\forall p, q \in E,\ \forall t, s \in T \setminus \{\pm\infty\}, \qquad i_{(p,t)} \leq f_{(q,s)} \iff i_{(q,s)} \leq f_{(p,t)} \ . \tag{6}$$

On the other hand, weak symmetry can be written as:

$$\forall p, q \in E,\ \forall s, t \in T \setminus \{\pm\infty\}, \qquad i_{(p,t)} \leq f_{(q,s)} \implies f_{(p,t)} \wedge i_{(q,s)} \neq \bot \ , \tag{7}$$

where $\bot$ designates the constant $-\infty$ function. These new interpretations of symmetry are illustrated in Figure 2.

This allows us to give an abstract expression for the strong and weak symmetry of a dilation δ. Write L for the set T^E of functions $E \to T$ and S for the set of impulses; L is a complete lattice for the order relation $\leq$ between functions. Here (2) becomes:

$$\forall f \in L, \qquad f = \bigvee \{i \in S \mid i \leq f\} \ ;$$

this means that S is a *sup-generating family* of L. Then the strong symmetry of dilation δ becomes

$$\forall i, j \in S, \qquad i \leq \delta(j) \iff j \leq \delta(i) \ , \tag{8}$$

while its weak symmetry is here

$$\forall i, j \in S, \qquad i \leq \delta(j) \implies \delta(i) \wedge j \neq \bot \ . \tag{9}$$

These definitions can then be extrapolated from the present case of the lattice of numerical functions with a sup-generating family made of impulses, to any complete lattice L with sup-generating family S.

We now introduce a few basic facts about geodesy; these will be studied in detail in Section 3. Given a *mask* $m \in L$, we consider the lattice $L[m]$ of all functions $f \in L$ such that $f \leq m$; here $L[m]$ has the sup-generating family $S[m]$ made of all impulses $i \in S$ such that $i \leq m$. We define the *geodesic restriction* of δ to m as the map $\delta_m : L[m] \to L[m] : f \mapsto \delta(f) \wedge m$. An interesting point is that both properties of strong and weak symmetry are preserved by geodesic restriction (in (8) and (9) we replace S by $S[m]$, and δ by δ_m). From δ_m we can build the geodesic reconstruction φ_m as the operator obtained by repeating **id** $\vee\, \delta_m$ until idempotence (where **id** is the identity on $L[m]$, in particular when δ_m is extensive, **id** $\vee\, \delta_m$ reduces to δ_m). This reconstruction φ_m is in fact the least algebraic closing on $L[m]$ which is above δ_m.

Since geodesic reconstruction, using repeated dilation by the 4-/8-neighbourhoods, was initially devised as a method for reconstructing 4-/8-connected components of sets touched by a marker, one can wonder whether it is possible to do the reverse, to define a new connectivity from the geodesic reconstruction φ_m obtained from a suitable "symmetrical" dilation δ_m. In the case of sets, everything is

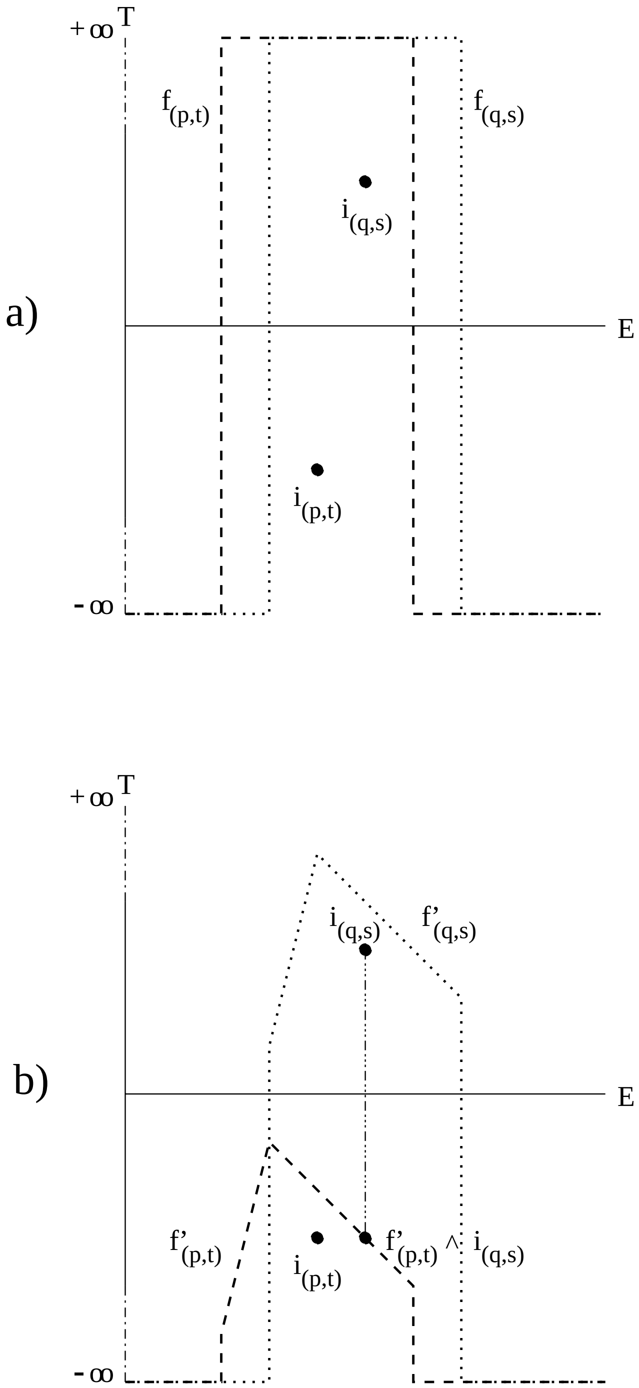

Figure 2. a) The strongly symmetrical function f of Figure 1.a satisfies (6). b) The weakly symmetrical function f' of Figure 1.b satisfies (7).

straightforward: the symmetrical dilation corresponds to symmetrical point neighbourhoods, and the latter are determined by a unique symmetrical adjacency relation between points; then the geodesic reconstruction produces connected components for the connectivity arising from this adjacency on points. In the more general framework of a complete lattice with a sup-generating family, geodesic reconstruction φ_m using an extensive strongly symmetrical dilation δ_m will indeed induce a connection in the sense of [17].

However in some lattices, such as the one of numerical functions, strong symmetry is an extremely severe requirement, so we can wonder whether this holds also for a weakly symmetrical dilation. There is a fundamental problem which did not arise with sets: the generators in S are not "atoms", we can have $i \leq i'$ for $i, i' \in S$; for example with (grey-level) numerical functions, taking $p \in E$ and $s, t \in T \setminus \{\pm\infty\}$ such that $s < t$, we have $i_{(p,s)} \leq i_{(p,t)}$; see Figure 3.a. Intuitively, for $i, i' \in S$ with $i \leq i'$, i must be in the same connected component (of a given function above i, i') as i', so that conversely i' must be in the same connected component (of that function) as i; thus i' should be obtained from i by geodesic reconstruction using dilation δ. In other words it should be possible to "climb" from i to i' with δ. We introduce then a new property: δ is *climbing* if for any $i, i' \in S$ with $i \leq i'$, there is a sequence $i_0, \ldots, i_n \in S$ such that $i_0 = i$, $i_n = i'$, and for each $k = 0, \ldots, n-1$, $i_k \leq i_{k+1} \leq \delta(i_k)$; see Figure 3.b. In particular, when δ is climbing, it is extensive. For example in the case of grey-level functions, the dilation by a structuring function f is climbing iff $f(o) > 0$, where o is the origin (or null vector) of E. In the case of sets, a dilation is climbing iff it is extensive. We show that, up to some technical assumptions on the lattice L (which are verified both for sets, grey-level and colour functions), when the geodesic dilation δ_m is climbing and weakly symmetrical, then the geodesic reconstruction φ_m is strongly symmetrical.

We obtain thus a progression from a weakly symmetrical and climbing dilation to a strongly symmetrical geodesic reconstruction, and from the latter to a definition of a connection on the lattice.

The paper is organized as follows. In Section 2 we introduce our notation and terminology for complete lattices, and then we define weak and strong symmetry and study their properties. In Section 3 we introduce geodesic operators, in particular reconstruction from markers, and give conditions for obtaining a connected opening, in other words a connection [17] on the lattice. Section 4 is devoted to examples of connectivities related to geodesic reconstruction, in particular for metric spaces and numerical functions; we also discuss there some cases where our theory does not apply directly, in particular image partitions. The last section is the conclusion, it links our results to other papers published on the topic of connectivity, and introduces several lines of investigation for the theory of symmetry and connectivity in lattices.

After we wrote the first version of this paper in 1999, Braga-Neto and Goutsias [5] obtained independently some results overlapping ours, in particular those of Subsection 3.2. We will indicate there the relations between their results and ours.

2. Lattices and symmetry

We use essentially the same theoretical framework and notation as in [17]. The *object space*, that is the family of images being considered, is written L; elements of L are called *objects* and written with lower-case letters $a, \ldots, z$; they represent individual images under consideration. Upper-case letters $A, \ldots, Z$ will denote subsets of L. However when L is the lattice of subsets of a set E, elements of L (subsets of E) will be written $A, \ldots, Z$, while $a, \ldots, z$ will designate points of E. We assume that L is ordered by

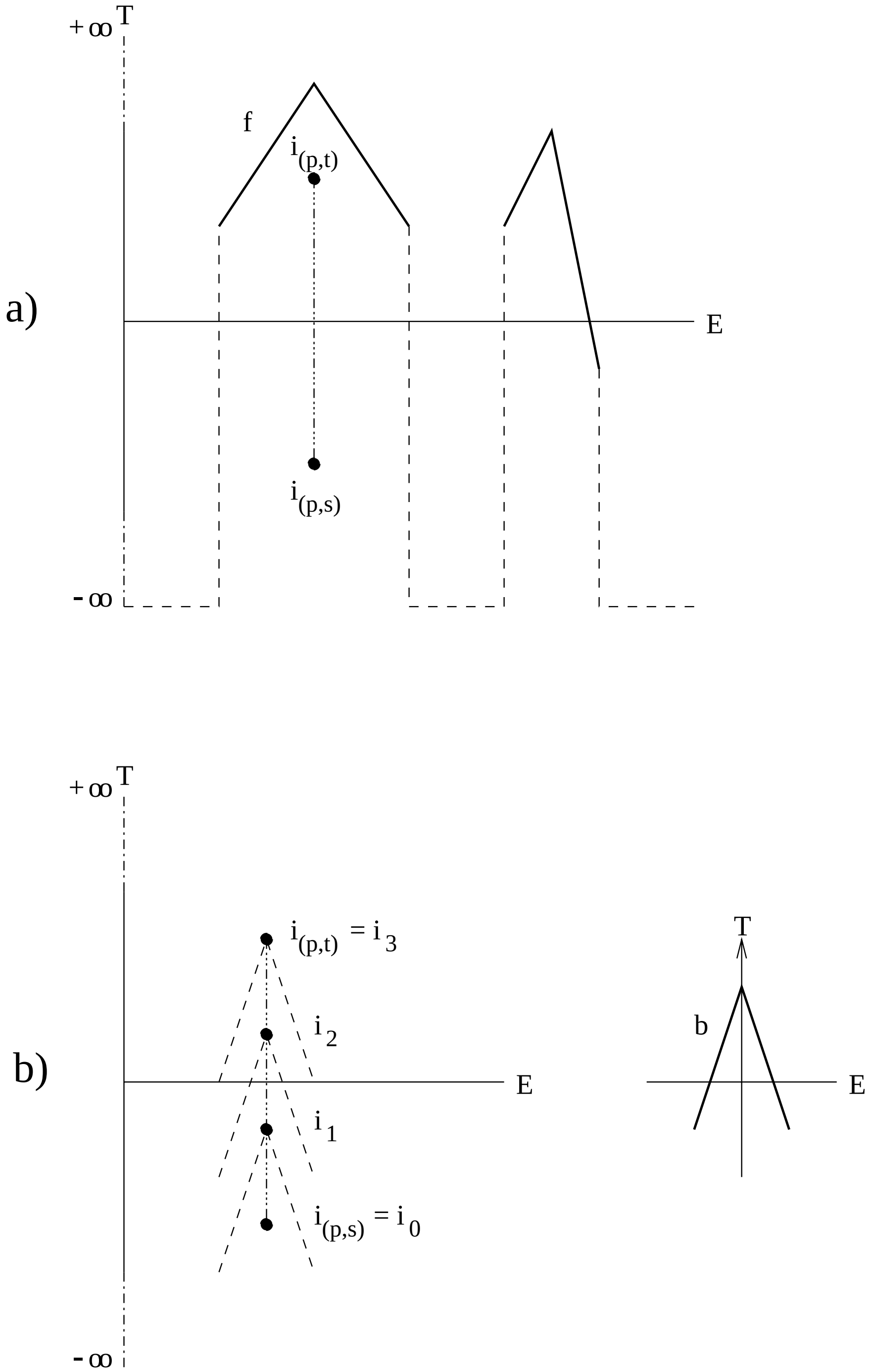

Figure 3. a) For $s, t \in T \setminus \{\pm\infty\}$ with $s < t$, $i_{(p,s)} \leq i_{(p,t)}$; here $i_{(p,s)}$ and $i_{(p,t)}$ belong to the same connected component of a function f above them (here f has two connected components). b) "Climbing" from $i_0 = i_{(p,s)}$ to $i_n = i_{(p,t)}$ with dilation δ by a structuring function b.

a partial order relation $\leq$ for which it is a *complete lattice* [3]; the supremum and infimum operations are written $\bigvee$ and $\bigwedge$. The least and greatest elements of L are written $\perp$ and $\top$ respectively; we have $\perp = \bigwedge L = \bigvee \emptyset$ and $\top = \bigvee L = \bigwedge \emptyset$. For any $a \in L$ and $X \subseteq L$, we write $X[a]$ (resp., $X^*[a]$) for the set of lower bounds (resp., upper bounds) of a in X:

$$\begin{aligned} X[a] &= \{x \in X \mid x \leq a\} \ ; \\ X^*[a] &= \{x \in X \mid x \geq a\} \ . \end{aligned}$$

Throughout this paper, we assume that L has a *sup-generating family* S; this means that for every $x \in L$ there is some $X \subseteq S$ for which $x = \bigvee X$; equivalently:

$$\forall x \in L, \qquad x = \bigvee S[x] \ .$$

Elements of S are called *generators*. We suppose also that $\perp \notin S$; indeed, $\perp$ is always redundant in a sup-generating family: $x = \bigvee X \implies x = \bigvee\bigl(X \setminus \{\perp\}\bigr)$.

For example, when L is the set $\mathcal{P}(E)$ of parts of a Euclidean or digital space E, S will be the set of all singletons in E; when L is the family Fun(E, T) of numerical functions $E \to T$ ($T = \overline{\mathbf{Z}}$ or $\overline{\mathbf{R}}$), S will consist of all "impulse" functions $i_{p,t}$ for $p \in E$ and $t \in T \setminus \{-\infty, +\infty\}$ (**Z** or **R**), defined by setting $i_{p,t}(p) = t$ and $i_{p,t}(x) = -\infty$ for $x \neq p$ (see (1)). Further examples can be found in [9].

A part X of a complete lattice L is said to be *sup-closed* if for any $Y \subseteq X$ we have $\bigvee Y \in X$. We define similarly an *inf-closed* part X by $Y \subseteq X \implies \bigwedge Y \in X$.

One calls an *atom* of L some $a \in L$ such that $a \neq \perp$ and for every $x \in L$, $\perp \leq x \leq a \implies x = a$ or $x = \perp$.

We say that the complete lattice L is *infinite supremum distributive* (in brief, *ISD*) if the binary meet operation $\wedge$ distributes the supremum operation $\bigvee$, in other words:

$$\forall x \in L,\ \forall y_i \in L\ (i \in I), \qquad x \wedge \Bigl(\bigvee_{i \in I} y_i\Bigr) = \bigvee_{i \in I} (x \wedge y_i) \ . \tag{10}$$

Note that both the complete lattice $\mathcal{P}(E)$ of parts of E and the one of numerical functions $E \to T$ are ISD.

Maps $L \to L$ will be called *operators* and will be designated by lower-case greek letters $\alpha, \ldots, \omega$; they represent image processing operations. The identity operator $L \to L : x \mapsto x$ will be written **id**. The set L^L of operators naturally inherits from L the partial order $\leq$ and the structure of a complete lattices with supremum and infimum operations $\bigvee$ and $\bigwedge$. The composition of operators ψ followed by λ is written $\lambda\psi$, where $\lambda\psi(x) = \lambda(\psi(x))$. Given an operator ψ and $n \in N$, we define ψ^n by $\psi^0 =$ **id**, $\psi^1 = \psi$, and $\psi^n = \psi\psi^{n-1}$ for $n > 1$. We write $\top_L$ and $\perp_L$ for the constant operators $x \mapsto \top$ and $x \mapsto \perp$ on L.

A part $\mathcal{F}$ of L^L is said to be *power-closed* if for every $\psi \in \mathcal{F}$ and any integer $n > 0$ we have $\psi^n \in \mathcal{F}$.

We recall from [7, 9, 13, 15, 16] some fundamental concepts. An operator ψ is *increasing* if $x \leq y$ implies $\psi(x) \leq \psi(y)$; it is *extensive* if $\psi \geq$ **id**, i.e., we always have $x \leq \psi(x)$; it is *anti-extensive* if $\psi \leq$ **id**, i.e., we always have $\psi(x) \leq x$; it is *idempotent* if $\psi\psi = \psi$, i.e., we always have $\psi(\psi(x)) = \psi(x)$. An *opening* is an increasing, idempotent, and anti-extensive operator; a *closing* is an increasing, idempotent, and extensive operator. A *dilation* is an operator δ which distributes the supremum operation, while an *erosion* is an operator ε which distributes the infimum operation:

$$\delta\Big(\bigvee_{i \in I} x_i\Big) = \bigvee_{i \in I} \delta(x_i) \ , \qquad \varepsilon\Big(\bigwedge_{i \in I} x_i\Big) = \bigwedge_{i \in I} \varepsilon(x_i) \ ;$$

in particular, for $I = \emptyset$, we get $\delta(\bot) = \bot$ and $\varepsilon(\top) = \top$. Note that every dilation or erosion is increasing.

We will now introduce our notions of strong and weak symmetry. Following (8) and (9), we make the following definition:

Definition 2.1. Let η and ζ be two operators on L. We say that η *strongly mirrors* ζ if

$$\forall s, t \in S, \qquad s \leq \eta(t) \implies t \leq \zeta(s) \ , \tag{11}$$

and that η *weakly mirrors* ζ if

$$\forall s, t \in S, \qquad s \leq \eta(t) \implies \zeta(s) \wedge t \neq \bot \ . \tag{12}$$

We say that η is *strongly* (*resp., weakly*) *symmetrical* if η strongly (*resp.,* weakly) mirrors itself.

Note that (11) implies (12), so strong mirroring (or symmetry) always implies weak one. Also for $\eta' \leq \eta$ and $\zeta' \geq \zeta$, if η strongly (*resp.,* weakly) mirrors ζ, then η' strongly (*resp.,* weakly) mirrors ζ'.

Let us consider some examples. For $L = \mathcal{P}(E)$ with S the set of singletons, strong and weak mirroring are equivalent, they both amount to:

$$\forall p, q \in E, \qquad p \in \eta(\{q\}) \implies q \in \zeta(\{p\}) \ .$$

In the translation-invariant case, if each $x \in E$ gives $\eta(\{x\}) = A_x$ and $\zeta(\{x\}) = B_x$ for two structuring elements $A, B \subseteq E$, the above mirroring property means that $A \subseteq \check{B} = \{-b \mid b \in B\}$.

When L is the lattice of numerical functions $E \to T$ (where $E = \mathbf{R}^n$ or $\mathbf{Z}^n$ and $T = \overline{\mathbf{R}}$ or $\overline{\mathbf{Z}}$), taking the translation-invariant dilations $\delta_f, \delta_{f'}$ by two structuring functions f, f', δ_f strongly mirrors $\delta_{f'}$ iff for every $p \in \mathrm{supp}(f)$ we have $f'(-p) = +\infty$, while δ_f weakly mirrors $\delta_{f'}$ iff for every $p \in \mathrm{supp}(f)$ we have $-p \in \mathrm{supp}(f')$. In particular δ_f is strongly (*resp.,* weakly) symmetrical according to the present definition, iff f is strongly (*resp.,* weakly) symmetrical according to (4,5).

We will now study the properties of strong and weak mirroring w.r.t. algebraic operations on operators, namely supremum, infimum, and composition:

Proposition 2.1. $\top_L$ and $\bot_L$ are strongly symmetrical, **id** is weakly symmetrical. Given a family of operators η_i, ζ_i ($i \in I$) on L:

1. If η_i strongly mirrors ζ_i for each $i \in I$, then $\bigwedge_{i \in I} \eta_i$ strongly mirrors $\bigwedge_{i \in I} \zeta_i$.

2. Let L be ISD, and suppose that each ζ_i ($i \in I$) is increasing. If η_i strongly (*resp.,* weakly) mirrors ζ_i for each $i \in I$, then $\bigvee_{i \in I} \eta_i$ strongly (*resp.,* weakly) mirrors $\bigvee_{i \in I} \zeta_i$.

Given operators $\eta, \eta', \zeta, \zeta'$ on L:

3. Let L be ISD, and suppose that ζ and ζ' are increasing, while η is a dilation. If η weakly mirrors ζ and η' strongly (*resp.,* weakly) mirrors ζ', then $\eta\eta'$ strongly (*resp.,* weakly) mirrors $\zeta'\zeta$.

Proof:
For every $s, t \in S$, we have *always* $s \leq \top_L(t)$ and $t \leq \top_L(s)$, while we have *never* $s \leq \bot_L(t)$ nor $t \leq \bot_L(s)$. Thus $\top_L$ and $\bot_L$ are strongly symmetrical. Now $s \leq \mathbf{id}(t)$ means $s \leq t$, so that $\mathbf{id}(s) \wedge t = s \wedge t = s \neq \bot$, therefore **id** is weakly symmetrical.

1) For $I = \emptyset$, we have $\bigwedge_{i \in I} \eta_i = \bigwedge_{i \in I} \zeta_i = \top_L$, which is strongly symmetrical. We have thus only to consider the case where $I \neq \emptyset$. Let $s, t \in S$ such that $s \leq \bigwedge_{i \in I} \eta_i(t)$. Then for each $i \in I$ we have $s \leq \eta_i(t)$, and as η_i strongly mirrors ζ_i, we get $t \leq \zeta_i(s)$. Hence $t \leq \bigwedge_{i \in I} \zeta_i(s)$. This means that $\bigwedge_{i \in I} \eta_i$ strongly mirrors $\bigwedge_{i \in I} \zeta_i$.

2) For $I = \emptyset$, we have $\bigvee_{i \in I} \eta_i = \bigvee_{i \in I} \zeta_i = \bot_L$, which is strongly symmetrical. We have thus only to consider the case where $I \neq \emptyset$. Let $s, t \in S$ such that $s \leq \bigvee_{i \in I} \eta_i(t)$. As L is ISD, we get

$$s = s \wedge \Big(\bigvee_{i \in I} \eta_i(t) \Big) = \bigvee_{i \in I} \big(s \wedge \eta_i(t) \big) \quad ;$$

it follows then that there is some $j \in I$ such that $s \wedge \eta_j(t) \neq \bot$, so there is some $s' \in S\big[s \wedge \eta_j(t)\big]$. As $s' \leq s$ and ζ_j is increasing, we have

$$\zeta_j(s') \leq \zeta_j(s) \leq \bigvee_{i \in I} \zeta_i(s) \quad .$$

If η_j strongly mirrors ζ_j, as $s' \leq \eta_j(t)$, we get $t \leq \zeta_j(s')$, and hence

$$t \leq \zeta_j(s') \leq \bigvee_{i \in I} \zeta_i(s) \quad ;$$

thus $\bigvee_{i \in I} \eta_i$ strongly mirrors $\bigvee_{i \in I} \zeta_i$. If η_j weakly mirrors ζ_j, as $s' \leq \eta_j(t)$, we get $\zeta_j(s') \wedge t \neq \bot$, so that

$$\bot < \zeta_j(s') \wedge t \leq \Big(\bigvee_{i \in I} \zeta_i(s) \Big) \wedge t \quad ;$$

thus $\bigvee_{i \in I} \eta_i$ weakly mirrors $\bigvee_{i \in I} \zeta_i$.

3) Let $s, t \in S$ such that $s \leq \eta\eta'(t)$. Since $\eta'(t) = \bigvee S[\eta'(t)]$ and the dilation η distributes the supremum, we have $\eta\eta'(t) = \bigvee_{v \in S[\eta'(t)]} \eta(v)$. As L is ISD, we get

$$s = s \wedge \eta\eta'(t) = s \wedge \Big(\bigvee_{v \in S[\eta'(t)]} \eta(v) \Big) = \bigvee_{v \in S[\eta'(t)]} \big(s \wedge \eta(v) \big) \quad ;$$

it follows then that there is some $u \in S[\eta'(t)]$ such that $s \wedge \eta(u) \neq \bot$, so there is some $s' \in S\big[s \wedge \eta(u)\big]$. As $s' \leq s$ and ζ is increasing, we get $\zeta(s') \leq \zeta(s)$. As η weakly mirrors ζ and $s' \leq \eta(u)$, we get $\zeta(s') \wedge u \neq \bot$; take $u' \in S\big[\zeta(s') \wedge u\big]$; now $u' \leq \zeta(s') \leq \zeta(s)$ and ζ' is increasing, so $\zeta'(u') \leq \zeta'(\zeta(s')) \leq \zeta'(\zeta(s))$; thus $\zeta'(u') \leq \zeta'(\zeta(s))$.

If η' strongly mirrors ζ', as $u' \leq u \leq \eta'(t)$, we get $t \leq \zeta'(u')$; hence $t \leq \zeta'(u) \leq \zeta'(\zeta(s))$. Thus $s \leq \eta\eta'(t)$ implies $t \leq \zeta'\zeta(s)$, that is, $\eta\eta'$ strongly mirrors $\zeta'\zeta$.

If η' weakly mirrors ζ', as $u' \leq u \leq \eta'(t)$, we get $\zeta'(u') \wedge t \neq \bot$; but $\zeta'(u') \leq \zeta'(\zeta(s))$, so $\bot < \zeta'(u') \wedge t \leq \zeta'(\zeta(s)) \wedge t$. Thus $s \leq \eta\eta'(t)$ implies $\zeta'\zeta(s) \wedge t \neq \bot$, that is, $\eta\eta'$ weakly mirrors $\zeta'\zeta$. $\square$

Corollary 2.1. Assume that L is ISD. The family of strongly (*resp.*, weakly) symmetrical dilations is sup-closed and power-closed, and it contains $\perp_L$. The family of weakly symmetrical dilations contains also **id**.

Proof:
It is known [9] that the family of dilations is sup-closed and closed under composition (in L^L); in particular it contains $\perp_L$ and it is power-closed. Moreover, recall that every dilation is increasing.

Given a family δ_i ($i \in I$) of strongly (*resp.*, weakly) symmetrical dilations, $\bigvee_{i\in I} \delta_i$ is strongly (*resp.*, weakly) symmetrical by item 2 of Proposition 2.1, and it is a dilation by [9]. Now $\perp_L$ is strongly symmetrical by Proposition 2.1, and a dilation by [9].

Given a strongly (*resp.*, weakly) symmetrical dilation δ and an integer $n > 0$, δ^n is a dilation by [9], and we show by induction that δ^n is strongly (*resp.*, weakly) symmetrical. First, this is true for $n = 1$; second, given $n > 1$ such that the property is true for $n-1$, since δ and δ^{n-1} are strongly (*resp.*, weakly) symmetrical, by item 3 of Proposition 2.1, $\delta\delta^{n-1}$ strongly (*resp.*, weakly) mirrors $\delta^{n-1}\delta$, that is δ^n is strongly (*resp.*, weakly) symmetrical.

Finally **id** is weakly symmetrical by Proposition 2.1, and a dilation by [9]. □

Note that there is no result like item 1 of Proposition 2.1 for weakly mirroring functions. Consider for example the ISD complete lattice of numerical functions $\mathbf{R} \to \overline{\mathbf{R}}$. We define the structuring functions f_n (for every $n \in \mathbf{N}$) and f' by:

$$f_n(x) = \begin{cases} -nx & \text{if } -1 \leq x < 0, \\ 0 & \text{if } 0 \leq x \leq 1, \\ -\infty & \text{otherwise;} \end{cases} \qquad f'(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1, \\ -\infty & \text{otherwise.} \end{cases}$$

We have $\bigwedge_{n\in\mathbf{N}} f_n = f'$. For every $n \in \mathbf{N}$, since f_n has as support the symmetrical interval $[-1, 1]$ (it is weakly symmetrical according to (5)), the dilation $\delta_{f_n} : g \mapsto g \oplus f_n$ is weakly symmetrical. However $\psi = \bigwedge_{n\in\mathbf{N}} \delta_{f_n}$ is not weakly symmetrical. Indeed, for a generator (an impulse) $i_{(q,s)}$ we have:

$$\psi(i_{(q,s)}) = \bigwedge_{n\in\mathbf{N}} \delta_{f_n}(i_{(q,s)}) = \bigwedge_{n\in\mathbf{N}} (f_n)_{(q,s)} = \Big(\bigwedge_{n\in\mathbf{N}} f_n\Big)_{(q,s)} = f'_{(q,s)} \quad .$$

Thus another generator $i_{(p,t)}$ verifies $i_{(p,t)} \leq \psi(i_{(q,s)}) = f'_{(q,s)}$ iff $q \leq p \leq q+1$ and $t \leq s$; provided that $p > q$ we have $\psi(i_{(p,t)}) \wedge i_{(q,s)} = f'_{(p,t)} \wedge i_{(q,s)} = \perp$, because $q \notin [p, p+1] = \mathrm{supp}(f'_{(p,t)})$. Thus ψ is not weakly symmetrical by (12). We illustrate this example in Figure 4.

We will now study the properties of the closing generated by a strongly or weakly symmetrical dilation. The first thing is determining the expression for this closing:

Lemma 2.1. Let δ be a dilation. The least closing $\geq \delta$ is

$$\hat{\delta} = \bigvee_{i=0}^{\infty} \delta^i = \bigvee_{j=1}^{\infty} (\mathbf{id} \vee \delta)^j \quad , \tag{13}$$

and it is also a dilation.

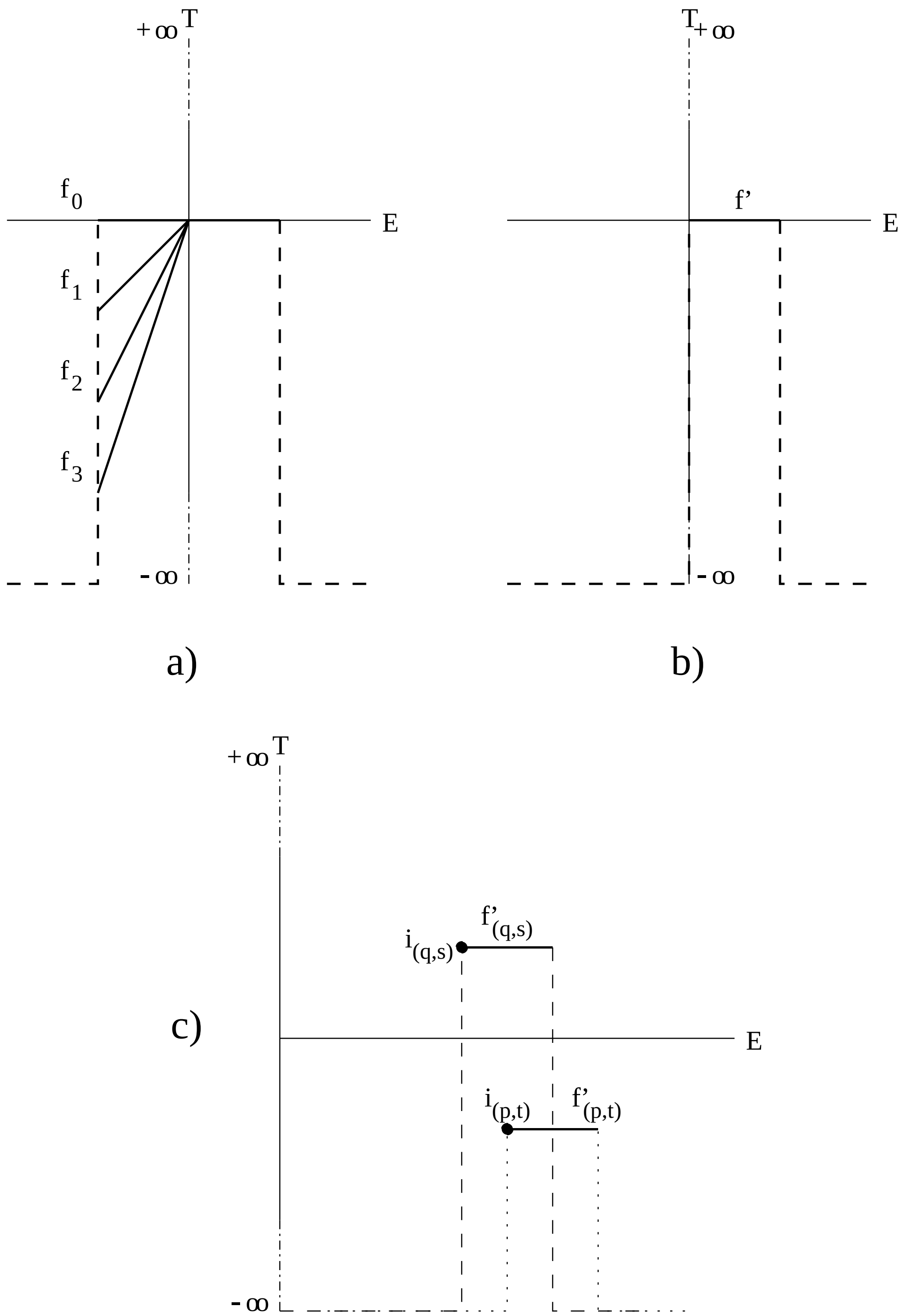

Figure 4. a) The weakly symmetrical function f_n ($n \in \mathbf{N}$). b) The infimum f' of all f_n ($n \in \mathbf{N}$). c) The operator ψ satisfying $\psi(i_{(q,s)}) = f'_{(q,s)}$ is not weakly symmetrical.

Proof:
For $i, j \in N$ we have $\delta^i \delta^j = \delta^{i+j}$, and since each δ^i is a dilation (i.e., it distributes the supremum operation), we get for every $I, J \subseteq \mathbf{N}$:

$$\Big(\bigvee_{i \in I} \delta^i\Big)\Big(\bigvee_{j \in J} \delta^j\Big) = \bigvee_{i \in I} \delta^i \Big(\bigvee_{j \in J} \delta^j\Big) = \bigvee_{i \in I} \bigvee_{j \in J} \delta^i \delta^j = \bigvee_{(i,j) \in I \times J} \delta^{i+j} \quad .$$

We show by induction that for $j > 0$, $(\mathbf{id} \vee \delta)^j = \bigvee_{i=0}^{j} \delta^i$. As $\delta^0 = \mathbf{id}$, this is clearly true for $j = 1$. Take $j > 1$ and assume that the property is true for $j - 1$; we get then

$$(\mathbf{id} \vee \delta)^j = (\mathbf{id} \vee \delta)^{j-1}(\mathbf{id} \vee \delta) = \Big(\bigvee_{i=0}^{j-1} \delta^i\Big)(\mathbf{id} \vee \delta) = \bigvee_{i=0}^{j-1} \big(\delta^i \vee \delta^{i+1}\big) = \bigvee_{i=0}^{j} \delta^i \quad .$$

It follows thus that

$$\bigvee_{j=1}^{\infty} (\mathbf{id} \vee \delta)^j = \bigvee_{j=1}^{\infty} \Big(\bigvee_{i=0}^{j} \delta^i\Big) = \bigvee_{i=0}^{\infty} \delta^i \quad .$$

Thus both expressions for $\hat{\delta}$ in (13) are equal.

We show now that $\hat{\delta}$ is the least closing $\geq \delta$. Since **id** appears in the decomposition of $\hat{\delta}$, the latter is extensive. As the family of dilations is closed under composition and supremum [9], $\hat{\delta}$ is a dilation; in particular it is increasing. Now since $\mathbf{N} = \{i + j \mid i, j \in \mathbf{N}\}$, we get:

$$\hat{\delta}\hat{\delta} = \Big(\bigvee_{i \in \mathbf{N}} \delta^i\Big)\Big(\bigvee_{i \in \mathbf{N}} \delta^i\Big) = \Big(\bigvee_{(i,j) \in \mathbf{N}^2} \delta^{i+j}\Big) = \bigvee_{i \in \mathbf{N}} \delta^i = \hat{\delta} \quad .$$

Hence $\hat{\delta}$ is also idempotent, and it is a closing. Since δ intervenes in the decomposition of $\hat{\delta}$, we have $\hat{\delta} \geq \delta$. Given a closing $\varphi \geq \delta$, the extensivity of φ gives $\varphi \geq \mathbf{id} = \delta^0$, and by induction every integer $i \geq 1$ gives $\varphi = \varphi^i \geq \delta^i$. Hence $\varphi \geq \bigvee_{i=0}^{\infty} \delta^i = \hat{\delta}$. $\Box$

We obtain thus an important result concerning the symmetry of $\hat{\delta}$:

Proposition 2.2. Assume that L is ISD. Let δ be a dilation.

1. If δ is extensive and strongly symmetrical, then $\hat{\delta}$ is strongly symmetrical.

2. If δ is weakly symmetrical, then $\hat{\delta}$ is weakly symmetrical.

Proof:
1) As δ is extensive, (13) gives $\hat{\delta} = \bigvee_{i=1}^{\infty} \delta^i$. By Corollary 2.1, each δ^i is strongly symmetrical, and hence $\hat{\delta}$ is strongly symmetrical.

2) Here (13) gives $\hat{\delta} = \mathbf{id} \vee \Big(\bigvee_{i=1}^{\infty} \delta^i\Big)$. By Corollary 2.1, **id** as well as each δ^i is weakly symmetrical, and hence $\hat{\delta}$ is weakly symmetrical. $\Box$

We will now introduce another property that allows us to get $\hat{\delta}$ strongly symmetrical for a weakly symmetrical δ:

Definition 2.2. An operator ψ is *climbing* if for any $s, t \in S$ such that $s \leq t$, there exist an integer $n \in \mathbf{N}$ and $u_0, \ldots, u_n \in S$ such that $u_0 = s$, $u_n = t$, and for each $i = 0, \ldots, n-1$ we have $u_i \leq u_{i+1} \leq \psi(u_i)$.

Consider for example the lattice of numerical functions $E \to T$ for $E = \mathbf{R}^n$ or $\mathbf{Z}^n$ and $T = \overline{\mathbf{R}}$ or $\overline{\mathbf{Z}}$. Given a structuring function f, the dilation δ_f by f is climbing iff $f(o) > 0$ (where o is the origin, or null vector, in E). Indeed, a generator $i_{(p,s)}$ gives $\delta_f(i_{(p,s)}) = f_{(p,s)}$, whose value at p is $s + f(o)$. Given another generator $i_{(q,t)}$, we have $i_{(p,s)} < i_{(q,t)}$ iff $p = q$ and $s < t$; thus for $f(o) \leq 0$ we cannot have $i_{(p,t)} \leq \delta_f(i_{(p,s)})$ with $t > s$, and so δ_f is not climbing. On the other hand, if $f(o) > 0$, we have $i_{(p,s+f(o))} \leq \delta_f(i_{(p,s)})$, $i_{(p,s+2f(o))} \leq \delta_f(i_{(p,s+f(o))})$, $i_{(p,s+3f(o))} \leq \delta_f(i_{(p,s+2f(o))})$, etc., and so δ_f is climbing. See again Figure 3.b.

We give now the properties of climbing operators:

Proposition 2.3.

1. Given two operators η, ζ such that $\eta \leq \zeta$ and η is climbing, ζ will be climbing.
2. Every climbing increasing operator is extensive.
3. Every extensive and strongly symmetrical operator is climbing.
4. A closing is strongly symmetrical iff it is climbing and weakly symmetrical.

Proof:
1) From $u_i \leq u_{i+1} \leq \eta(u_i)$ (for $i = 0, \ldots, n-1$) we derive $u_i \leq u_{i+1} \leq \zeta(u_i)$.

2) Let ψ be climbing and increasing. Let $x \in L$; for $s \in S[x]$, taking $t = s$, we have $s = u_0 \leq \psi(u_0) = \psi(s)$, and as ψ is increasing, $s \leq x$ implies $\psi(s) \leq \psi(x)$; hence $s \leq \psi(s) \leq \psi(x)$ for all $s \in S[x]$ from which we derive that $x = \bigvee S[x] \leq \psi(x)$, i.e., ψ is extensive.

3) Let ψ be extensive and strongly symmetrical. Let $s, t \in S$ such that $s \leq t$. As ψ is extensive, $t \leq \psi(t)$, so that $s \leq \psi(t)$, and as ψ is strongly symmetrical, this gives $t \leq \psi(s)$. Hence the property of Definition 2.2 is verified with $n = 1$, $u_0 = s$, and $u_1 = t$.

4) Let φ be a climbing and weakly symmetrical closing. Let $s, t \in S$ such that $s \leq \varphi(t)$. As φ is weakly symmetrical, $\varphi(s) \wedge t \neq \perp$, from which we derive that there is some $t' \in S[t]$ such that $t' \leq \varphi(s)$. As φ is climbing, we have $n \in \mathbf{N}$ and $u_0, \ldots, u_n \in S$ such that $u_0 = t'$, $u_n = t$, and for each $i = 0, \ldots, n-1$ we have $u_i \leq u_{i+1} \leq \varphi(u_i)$. We show by induction on i that $u_i \leq \varphi(s)$ for $i = 0, \ldots, n$. As $u_0 = t' \leq \varphi(s)$, this is true for $i = 0$. Suppose that the property is true for $i < n$, and let us show it for $i+1$: we have $u_i \leq \varphi(s)$ and $u_{i+1} \leq \varphi(u_i)$, and since φ is increasing and idempotent, we get $u_{i+1} \leq \varphi(u_i) \leq \varphi(\varphi(s)) = \varphi(s)$. Therefore $t = u_n \leq \varphi(s)$ by induction hypothesis. We have thus proved that for every $s, t \in S$ verifying $s \leq \varphi(t)$, we must have $t \leq \varphi(s)$, in other words φ is strongly symmetrical.

Conversely, let φ be a strongly symmetrical closing; φ is extensive, so it is climbing by item 3; also φ being strongly symmetrical, it is certainly weakly symmetrical. □

We end this section with a final result that will be used in the next section for building connected components:

Theorem 2.1. Assume that L is ISD. Given a climbing weakly symmetrical dilation δ, $\hat{\delta}$ is strongly symmetrical.

Proof:
As $\delta \leq \hat{\delta}$, $\hat{\delta}$ is climbing by item 1 of Proposition 2.3. By item 2 of Proposition 2.2, $\hat{\delta}$ is weakly symmetrical. By item 4 of Proposition 2.3, the climbing weakly symmetrical closing $\hat{\delta}$ is strongly symmetrical. $\square$

3. Geodesic operations and connectivity

We will now see what happens with the above properties of strong and weak symmetry, and climbing, when we consider the geodesic restriction of operators to the lattice $L[m]$ for some $m \in L$. We will also consider geodesic reconstruction from a geodesic dilation. Then we will explain how such a geodesic reconstruction from a weakly symmetrical climbing dilation can give rise to a connection on the lattice L.

Definition 3.1. Let $m \in L$ be called a *mask*. The *geodesic restriction to* m of an operator $\psi : L \to L$ is the operator $\psi_m : L[m] \to L[m]$ defined by

$$\forall x \in L[m], \qquad \psi_m(x) = \psi(x) \wedge m \ .$$

Note that $L[m]$ is a complete lattice for the ordering by $\leq$, with the same supremum operation $\bigvee$ and least element $\perp$ as in L; it has also the same *non-empty* infimum operation $\bigwedge$ as L, but its greatest element, or empty infimum $\bigwedge \emptyset$, is different: it is m instead of $\top$. It follows that when L is ISD, so is $L[m]$. Moreover $L[m]$ has $S[m]$ as sup-generating family.

We will write $\mathbf{id}_m$ for the identity operator on $L[m]$, this notation is unambiguous, because $\mathbf{id}_m$ is indeed the geodesic restriction to m of $\mathbf{id}$. The least and greatest operators on $L[m]$ are respectively $\perp_{L[m]} : x \mapsto \perp$ and $\top_{L[m]} : x \mapsto m$. For $\eta \in L^L$, we had defined above $\eta^0 = \mathbf{id}$, so we define $(\eta_m)^0$ to be $\mathbf{id}_m$.

Let us now see how the properties of operators considered in the previous section are preserved by geodesic restriction:

Proposition 3.1. Let $m \in L$ be a mask. The geodesic restriction $\psi \mapsto \psi_m$ maps $\perp_L$, $\top_L$, and $\mathbf{id}$ on $\perp_{L[m]}$, $\top_{L[m]}$, and $\mathbf{id}_m$ respectively, and it is:

- increasing, that is: $\eta \leq \zeta \implies \eta_m \leq \zeta_m$;
- compatible with the infimum operation, that is: for $\mathcal{F} \subseteq L^L$, $\left(\bigwedge_{\psi \in \mathcal{F}} \psi\right)_m = \bigwedge_{\psi \in \mathcal{F}} (\psi_m)$;
- when L is ISD, compatible with the supremum operation, that is: for $\mathcal{F} \subseteq L^L$, $\left(\bigvee_{\psi \in \mathcal{F}} \psi\right)_m = \bigvee_{\psi \in \mathcal{F}} (\psi_m)$.

For any operators $\eta, \zeta \in L^L$, the following properties are inherited by their geodesic restrictions η_m, ζ_m:

1. that η strongly (*resp.*, weakly) mirrors ζ;
2. that η is climbing;
3. that η is increasing;

4. that η is extensive;

5. that η is a closing;

6. when L is ISD, that η is a dilation.

Proof:
The geodesic restriction of $\bot_L$, $\top_L$, and **id** map $x \in L[m]$ respectively on $\bot \wedge m = \bot$, $\top \wedge m = m$, and **id**$(x) \wedge m = x \wedge m = x$, so they are $\bot_{L[m]}$, $\top_{L[m]}$, and **id**$_m$.

If $\eta \leq \zeta$, then for $x \in L[m]$ we have $\eta(x) \leq \zeta(x)$, so that $\eta_m(x) = \eta(x) \wedge m \leq \zeta(x) \wedge m = \zeta_m(x)$.

Let $\mathcal{F} \subseteq L^L$. If $\mathcal{F} = \emptyset$, we get

$$\Big(\bigwedge_{\psi \in \emptyset} \psi\Big)_m = \big(\top_L\big)_m = \top_{L[m]} = \bigwedge_{\psi \in \emptyset} (\psi_m)$$

and

$$\Big(\bigvee_{\psi \in \emptyset} \psi\Big)_m = \big(\bot_L\big)_m = \bot_{L[m]} = \bigvee_{\psi \in \emptyset} (\psi_m) \quad .$$

Assume now that $\mathcal{F} \neq \emptyset$. For $x \in L[m]$ we have

$$\begin{aligned} \Big(\bigwedge_{\psi \in \mathcal{F}} \psi\Big)_m(x) &= \Big(\bigwedge_{\psi \in \mathcal{F}} \psi\Big)(x) \wedge m = \Big(\bigwedge_{\psi \in \mathcal{F}} \psi(x)\Big) \wedge m \\ &= \bigwedge_{\psi \in \mathcal{F}} \big(\psi(x) \wedge m\big) = \bigwedge_{\psi \in \mathcal{F}} \psi_m(x) = \Big(\bigwedge_{\psi \in \mathcal{F}} \psi_m\Big)(x) \quad . \end{aligned}$$

Thus $\big(\bigwedge_{\psi \in \mathcal{F}} \psi\big)_m = \bigwedge_{\psi \in \mathcal{F}} (\psi_m)$. If L is ISD, we get also

$$\begin{aligned} \Big(\bigvee_{\psi \in \mathcal{F}} \psi\Big)_m(x) &= \Big(\bigvee_{\psi \in \mathcal{F}} \psi\Big)(x) \wedge m = \Big(\bigvee_{\psi \in \mathcal{F}} \psi(x)\Big) \wedge m \\ &= \bigvee_{\psi \in \mathcal{F}} \big(\psi(x) \wedge m\big) = \bigvee_{\psi \in \mathcal{F}} \psi_m(x) = \Big(\bigvee_{\psi \in \mathcal{F}} \psi_m\Big)(x) \quad . \end{aligned}$$

Thus $\big(\bigvee_{\psi \in \mathcal{F}} \psi\big)_m = \bigvee_{\psi \in \mathcal{F}} (\psi_m)$.

1) We take equations (11,12) which define strong and weak mirroring in Definition 2.1; if we restrict ourselves to $s, t \in S[m]$, having $s, t \leq m$, then first $s \leq \eta(t) \iff s \leq \eta(t) \wedge m = \eta_m(t)$, second $t \leq \zeta(s) \iff t \leq \zeta(s) \wedge m = \zeta_m(s)$, and third $\zeta(s) \wedge t = \zeta(s) \wedge m \wedge t = \zeta_m(s) \wedge t$. Hence the two equations which define strong and weak mirroring are preserved by the geodesic restriction of η, ζ to m.

2) We take Definition 2.2 and restrict ourselves to $s, t \in S[m]$. Since $u_0 \leq \cdots \leq u_n = t \leq m$, we get that $u_i \in S[m]$ for $i = 0, \ldots, n$; now for $i < n$ we have $u_i \leq u_{i+1} \leq \eta(u_i)$ and $u_{i+1} \leq m$, from which we derive that $u_i \leq u_{i+1} \leq \eta(u_i) \wedge m = \eta_m(u_i)$. Thus η_m is climbing.

3) If η is increasing, for every $x, y \in L[m]$, $x \leq y$ implies that $\eta(x) \leq \eta(y)$, and so $\eta_m(x) = \eta(x) \wedge m \leq \eta(y) \wedge m = \eta_m(y)$; thus η_m is increasing.

4) If η is extensive, for every $x \in L[m]$ we have $x \leq \eta(x)$, and as $x \leq m$, we get $x \leq \eta(x) \wedge m = \eta_m(x)$; thus η_m is extensive. (We can also remark that **id** $\leq \eta$, so that **id**$_m \leq \eta_m$).

5) If η is a closing, it is increasing, extensive and idempotent. Then η_m is increasing and extensive by items 3 and 4; we have thus only to show that it is idempotent. For every $x \in L[m]$ we have $x \leq \eta_m(x) \leq \eta(x)$, and applying the increasing and idempotent operator η to this inequality, we get

$\eta(x) \leq \eta(\eta_m(x)) \leq \eta(\eta(x)) = \eta(x)$, that is $\eta(\eta_m(x)) = \eta(x)$. The definition of $\eta_m : y \mapsto \eta(y) \wedge m$ gives then

$$\eta_m(\eta_m(x)) = \eta(\eta_m(x)) \wedge m = \eta(x) \wedge m = \eta_m(x) \ ,$$

that is $\eta_m(\eta_m(x)) = \eta_m(x)$, and so η_m is idempotent. It is thus a closing.

6) If η is a dilation, η distributes the supremum operation, and as L is ISD, for every non-empty family x_i ($i \in \mathcal{I}$) of elements of $L[m]$ we have:

$$\begin{aligned} \eta_m\Big(\bigvee_{i\in\mathcal{I}} x_i\Big) &= \eta\Big(\bigvee_{i\in\mathcal{I}} x_i\Big) \wedge m &= \Big(\bigvee_{i\in\mathcal{I}} \eta(x_i)\Big) \wedge m \\ &= \bigvee_{i\in\mathcal{I}}\big(\eta(x_i) \wedge m\big) &= \bigvee_{i\in\mathcal{I}} \eta_m(x_i) \ . \end{aligned}$$

Thus η_m distributes non-empty suprema. Now the dilation η preserves the empty supremum $\perp = \bigvee \emptyset$, and we have then

$$\eta_m(\perp) = \eta(\perp) \wedge m = \perp \wedge m = \perp \ ,$$

that is η_m preserves the empty supremum $\perp$. Therefore η_m is a dilation on $L[m]$. □

Note that when η is anti-extensive, for $x \in L[m]$ we have $\eta(x) \leq x \leq m$, so $\eta_m(x) = \eta(x)$; in other words η_m is the restriction of η to $L[m]$. In particular when η is an opening on L, its restriction η_m will be an opening on $L[m]$.

Although being an opening or a closing are preserved by geodesic restriction, note that idempotence is generally *not* a property preserved by geodesic restriction, even for an increasing operator. Take for example $L = \mathcal{P}(\mathbf{Z})$, let $Y = \{-1, 0\}$, and define η by $\eta(X) = \emptyset$ if X is empty or a singleton, while $\eta(X) = Y$ if X has at least two elements; clearly η is increasing and idempotent; however for the mask $M = \{0, 1\}$, we have $\eta_M(M) = M \cap Y = \{0\}$ and $\eta_M(M \cap Y) = \emptyset$, so η_M is not idempotent.

For $\eta, \zeta \in L^L$, every $x \in L[m]$ gives $(\eta\zeta)_m(x) = \eta\zeta(x) \wedge m$; if ζ preserves $L[m]$, that is $\zeta(x) \in L[m]$ for $x \in L[m]$, then $\eta\zeta(x) \wedge m = \eta(\zeta(x)) \wedge m = \eta_m(\zeta(x)) = \eta_m\zeta(x)$ and we have we have $(\eta\zeta)_m = \eta_m\zeta$. However $(\eta\zeta)_m$ is generally different from $\eta_m\zeta_m$. When η is increasing, for $x \in L[m]$ we have $\zeta_m(x) = \zeta(x) \wedge m \leq \zeta(x)$, so $\eta(\zeta_m(x)) \leq \eta(\zeta(x))$, and we get then $\eta_m\zeta_m(x) = \eta_m(\zeta_m(x)) = \eta(\zeta_m(x)) \wedge m \leq \eta(\zeta(x)) \wedge m = \eta\zeta(x) \wedge m = (\eta\zeta)_m(x)$; thus $\eta_m\zeta_m \leq (\eta\zeta)_m$. This inequality is in general sharp.

Consider now a dilation δ on L. We will define from δ two operators on $L[m]$, and we will show that when the complete lattice L is ISD, these operators are both closings and dilations on $L[m]$.

For the first operator, we first apply Lemma 2.1 to δ, and obtain $\hat{\delta}$, the least closing on L which is $\geq \delta$; we have $\hat{\delta} = \bigvee_{i=0}^{\infty} \delta^i$, and $\hat{\delta}$ is also a dilation. Second, we take the geodesic restriction $\hat{\delta}_m$ of $\hat{\delta}$ to m:

$$\hat{\delta}_m : L[m] \to L[m] : x \mapsto \Big(\bigvee_{i=0}^{\infty} \delta^i(x)\Big) \wedge m \ , \tag{14}$$

where δ^0 is the identity on L. By item 5 of Proposition 3.1, $\hat{\delta}_m$ is a closing on $L[m]$. When L is ISD, we have

$$\hat{\delta}_m(x) = \bigvee_{i=0}^{\infty} (\delta^i)_m(x) \ ,$$

where $(\delta^i)_m(x) = \delta^i(x) \wedge m$, and by item 6 of Proposition 3.1, $\hat{\delta}_m$ is a dilation on $L[m]$.

For the second operator, we first take the geodesic restriction δ_m of δ to m, defined by $\delta_m(x) = \delta(x) \wedge m$ for every $x \in L[m]$. Assuming that L is ISD, by item 6 of Proposition 3.1, δ_m is a dilation on $L[m]$. Second, we apply Lemma 2.1 to δ_m, and obtain $\widehat{\delta_m}$, the least closing on $L[m]$ which is $\geq \delta_m$, and $\widehat{\delta_m}$ is also a dilation on $L[m]$; we have

$$\widehat{\delta_m} : L[m] \to L[m] : x \mapsto \bigvee_{i=0}^{\infty} (\delta_m)^i(x) \quad , \tag{15}$$

where $(\delta_m)^0$ is the identity on $L[m]$, and $\delta_m(z) = \delta(z) \wedge m$ for $z \in L[m]$.

These two closings are generally not equal:

Proposition 3.2. Let δ be a dilation. The two operators $\hat{\delta}_m$ and $\widehat{\delta_m}$ on $L[m]$ defined in (14,15), satisfy the inequality $\hat{\delta}_m \geq \widehat{\delta_m}$, and $\hat{\delta}_m$ is a closing. When the complete lattice L is ISD, they are both closings and dilations on $L[m]$.

Proof:
We explained above (using Lemma 2.1 and items 5 and 6 of Proposition 3.1) that $\hat{\delta}_m$ is a closing, and that for L ISD, $\widehat{\delta_m}$ is also a closing, and both $\hat{\delta}_m$ and $\widehat{\delta_m}$ are dilations on $L[m]$.

Since $\hat{\delta} \geq \delta$, we get $\hat{\delta}_m \geq \delta_m$, and we know that $\hat{\delta}_m$ is a closing on $L[m]$; the extensivity of $\hat{\delta}_m$ gives then $\hat{\delta}_m \geq (\delta_m)^0$, while its idempotence gives $\hat{\delta}_m \geq (\delta_m)^i$ for every $i > 0$, so we get $\hat{\delta}_m \geq \widehat{\delta_m}$. □

In fact for L ISD, $\hat{\delta}_m$ and $\widehat{\delta_m}$ are the greatest and least elements in a family of operators on $L[m]$ which are both closings and dilations, each operator of this family takes the form

$$x \mapsto \Big(\bigvee_{i=0}^{\infty} \psi_i(x) \Big) \wedge m = \bigvee_{i=0}^{\infty} (\psi_i)_m(x) \quad ,$$

where ψ_0 is the identity, and for $i > 0$, ψ_i is a composition (in any order) of i times the dilation δ and any number of times the restriction $x \mapsto x \wedge m$, with the condition that for $i, j > 0$ we have $(\psi_i)_m(\psi_j)_m \geq (\psi_{i+j})_m$. For example we can take a fixed integer $n > 0$, then define $\psi_i = (\delta_m)^i$ for $i \leq n$ and $\psi_i = \delta^{i-n}(\delta_m)^n$ for $i > n$.

Note that this inequality $\hat{\delta}_m \geq \widehat{\delta_m}$ is in general sharp. For example take $L = \mathcal{P}(\mathbf{Z}^2)$, the lattice of subsets of the digital plane; for a mask $M \subseteq \mathbf{Z}^2$, we have $L[M] = \mathcal{P}(M)$. Let δ be the translation by one pixel to the left. Then $\hat{\delta}$ adds to a set $Y \subseteq \mathbf{Z}^2$ all pixels to the left of Y. Now $\hat{\delta}_M$ adds to a subset X of M all pixels of M which are to the left of X, while $\widehat{\delta_M}$ adds to X all pixels p of M such that there is a horizontal line segment included in M, having p as left end, and whose right end is in X. We illustrate this example in Figure 5. Note that for every $i > 0$, $(\delta^i)_m$ is strictly greater than $(\delta_m)^i$.

3.1. Geodesic reconstruction

The above example gives a practical indication that, although the first closing $\hat{\delta}_m$ defined in (14) can be interesting, it is the second closing $\widehat{\delta_m}$ defined in (15) that really gives what one would expect from a geodesic reconstruction, namely the propagation of the marker inside a connected component of the mask.

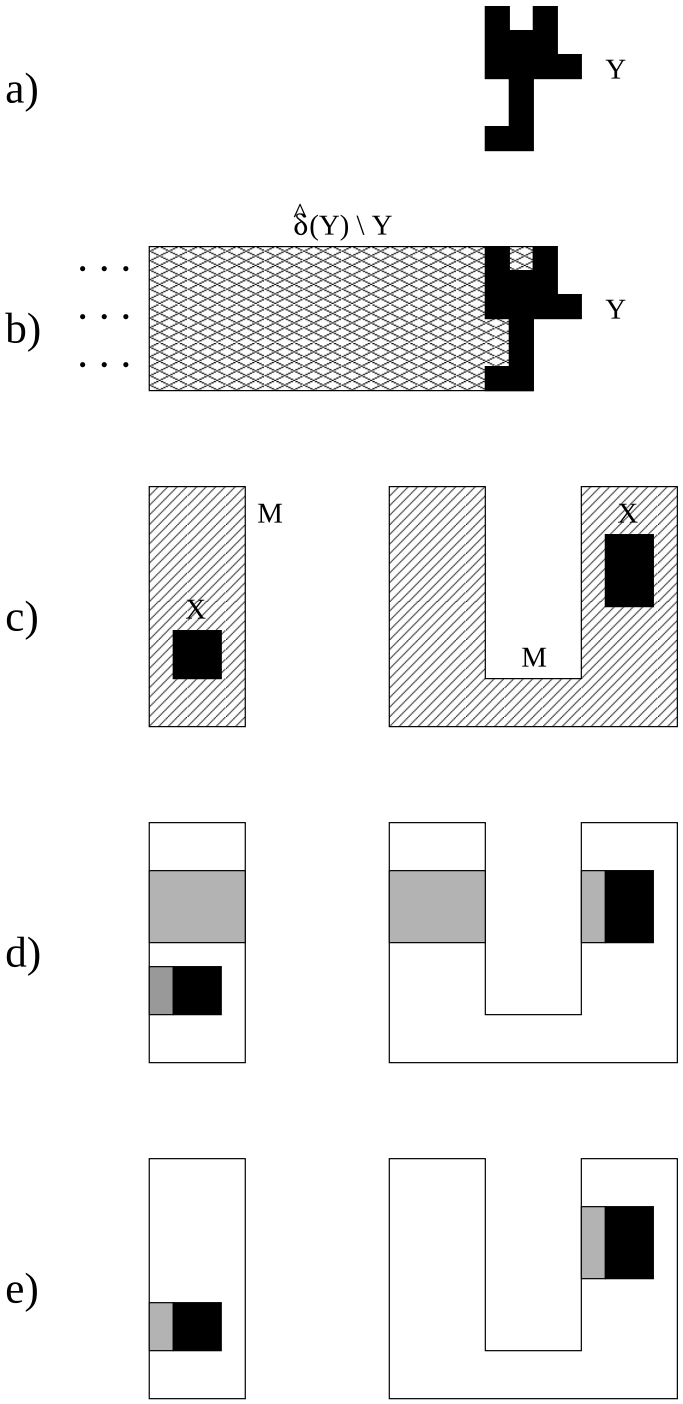

Figure 5. The dilation δ in $\mathcal{P}(\mathbf{Z}^2)$ translates each figure by one pixel to the left. a) $Y \subseteq \mathbf{Z}^2$. b) $\hat{\delta}(Y)$. c) A mask $M \subseteq \mathbf{Z}^2$ and a subset X of M. d) $\hat{\delta}_M(X)$. e) $\widehat{\delta_M}(X)$.

Let us give general criteria for geodesic reconstruction from a marker x inside a mask m. We assume temporarily that $x \leq m$; we will see later what to do when $x \not\leq m$. Writing $\rho(m)(x)$ for this geodesic reconstruction from x inside m, we can state the following:

- the marker must be inside the mask: $\rho(m)(x)$ is defined for $x \leq m$;
- the geodesic reconstruction contains the marker and is inside the mask: $x \leq \rho(m)(x) \leq m$;
- the geodesic reconstruction increases with the marker: $x \leq y \leq m$ implies $\rho(m)(x) \leq \rho(m)(y)$;
- the geodesic reconstruction increases with the mask: $x \leq n \leq m$ implies $\rho(n)(x) \leq \rho(m)(x)$;
- replacing the marker by the geodesic reconstruction leads to the same geodesic reconstruction: $\rho(m)\big(\rho(m)(x)\big) = \rho(m)(x)$;
- replacing the mask by the geodesic reconstruction leads to the same geodesic reconstruction: $\rho\big(\rho(m)(x)\big)(x) = \rho(m)(x)$.

This gives the following definition:

Definition 3.2. A *geodesic reconstruction system* on L is a map $\rho : L \to \bigcup_{m \in L} L[m]^{L[m]}$ associating to every $m \in L$ an operator $\rho(m) : L[m] \to L[m]$, such that:

- for every $m \in L$, $\rho(m)$ is a closing on $L[m]$;
- for $x \in L$, $\rho(\cdot)(x) : L^*[x] \to L^*[x] : m \mapsto \rho(m)(x)$ is an opening on $L^*[x]$.

In $\rho(m)(x)$ (where $x \in L[m]$), one calls m the *mask*, x the *marker*, and $\rho(m)(x)$ the *geodesic reconstruction from x inside m*.

Note that since $\rho(m)$ is a closing on $L[m]$, given $x, y \in L[m]$ such that $x \leq y \leq \rho(m)(x)$, we must have $\rho(m)(y) = \rho(m)(x)$. Also, since $\rho(\cdot)(x)$ is an opening on $L^*[x]$, for $m, n \in L^*[x]$ such that $\rho(m)(x) \leq n \leq m$, we have $\rho(n)(x) = \rho(m)(x)$. In particular, if $x \leq y \leq \rho(m)(x) \leq n \leq m$, then $\rho(m)(y) = \rho(m)(x) = \rho(n)(x) = \rho(n)(y)$.

We will see that the above two closings $\hat{\delta}_m$ and $\widehat{\delta_m}$ give indeed geodesic reconstruction systems:

Proposition 3.3. The following choices of $\rho(m)$ ($m \in L$) give geodesic reconstruction systems on L:

1. $\rho(m) = \varphi_m$, for a closing φ on L;
2. when L is ISD, $\rho(m) = \widehat{\delta_m}$ for a dilation δ on L, cfr. (15);
3. $\rho(m) = \hat{\delta}_m$ for a dilation δ on L, cfr. (14).

Proof:
1) By item 5 of Proposition 3.1, φ_m is a closing on $L[m]$ for every $m \in L$. For $x \leq n \leq m$ we have $\varphi_n(x) = \varphi(x) \wedge n \leq \varphi(x) \wedge m = \varphi_m(x)$. If $\varphi_m(x) \leq n \leq m$, we get:

$$\varphi_n(x) = \varphi(x) \wedge n = \varphi(x) \wedge (m \wedge n) = (\varphi(x) \wedge m) \wedge n = \varphi_m(x) \wedge n = \varphi_m(x) \ .$$

Thus $n = \varphi_m(x)$ implies $\varphi_n(x) = \varphi_m(x)$, and so for a fixed x, $\varphi_m(x)$ acts as an opening on the argument m.

2) By Proposition 3.2, $\widehat{\delta_m}$ is a closing on $L[m]$ for every $m \in L$.

Suppose that $x \leq n \leq m$; for any $y \in L[n]$ we have $\delta_n(y) = \delta(y) \wedge n \leq \delta(y) \wedge m = \delta_m(y)$. We show by induction that we have $(\delta_n)^i(x) \leq (\delta_m)^i(x)$ for each $i \geq 0$. Indeed this is obviously true for $i = 0$, and if it is true for $i \geq 0$ we derive that it is true for $i+1$: the inequality $\delta_n(y) \leq \delta_m(y)$ with $y = (\delta_n)^i(x)$ gives $(\delta_n)^{i+1}(x) = \delta_n\big((\delta_n)^i(x)\big) \leq \delta_m\big((\delta_n)^i(x)\big)$; as $(\delta_n)^i(x) \leq (\delta_m)^i(x)$ and δ_m is an increasing operator on $L[m]$, we get $\delta_m\big((\delta_n)^i(x)\big) \leq \delta_m\big((\delta_m)^i(x)\big) = (\delta_m)^{i+1}(x)$; combining both inequalities gives $(\delta_n)^{i+1}(x) \leq (\delta_m)^{i+1}(x)$. Hence by (15) we obtain $\widehat{\delta_n}(x) \leq \widehat{\delta_m}(x)$.

Suppose now that $\widehat{\delta_m}(x) \leq n \leq m$. For any $y \leq \widehat{\delta_m}(x)$, as $\widehat{\delta_m}$ is a closing on $L[m]$ which is $\geq \delta_m$, we have

$$\delta_m(y) \leq \widehat{\delta_m}(y) \leq \widehat{\delta_m}\big(\widehat{\delta_m}(x)\big) = \widehat{\delta_m}(x) \leq n \quad ,$$

and as $n \leq m$, we get:

$$\delta_n(y) = \delta(y) \wedge n = \delta(y) \wedge (m \wedge n) = (\delta(y) \wedge m) \wedge n = \delta_m(y) \wedge n = \delta_m(y) \quad .$$

Thus each $y \leq \widehat{\delta_m}(x)$ verifies $\delta_n(y) = \delta_m(y)$. Now for every $i \geq 0$ we have $(\delta_m)^i(x) \leq \widehat{\delta_m}(x)$, so that by induction we get $(\delta_n)^i(x) = (\delta_m)^i(x)$. Hence by (15) we obtain $\widehat{\delta_n}(x) = \widehat{\delta_m}(x)$. Thus $n = \widehat{\delta_m}(x)$ implies $\widehat{\delta_n}(x) = \widehat{\delta_m}(x)$, and so for a fixed x, $\widehat{\delta_m}(x)$ acts as an opening on the argument m.

3) By Lemma 2.1, $\hat{\delta}$ is a closing, so we apply item 1 with $\varphi = \hat{\delta}$. □

A well-known case of geodesic reconstruction system is given for $L = \mathcal{P}(\mathbf{Z}^2)$, and for $X \subseteq M \subseteq E$, $\rho(M)(X)$ is the union of all 4-connected components of M having a nonvoid intersection with X. Here $\rho(M)$ is obtained according to item 2, by taking for δ the map which adds to a set its 4-neighbourhood. The same holds with 8-connectivity and 8-neighbourhoods.

Remark 3.1. In the definition of a geodesic reconstruction system, the mask m and the marker x play dual roles. More precisely, from the map $\rho : L \to \bigcup_{m \in L} L[m]^{L[m]}$ we define the map $\rho^* : L \to \bigcup_{m \in L} L^*[m]^{L^*[m]}$ by $\rho^*(x)(y) = \rho(y)(x)$, and then ρ is a geodesic reconstruction system on $(L, \leq)$ iff ρ^* is a geodesic reconstruction system on the dual lattice $(L, \geq)$. For ρ^*, the dual marker (equal to the mask for ρ) is above the dual mask (equal to the marker for ρ). Let us illustrate this for the geodesic reconstructions given in Proposition 3.3. We say that the lattice L is *infinite infimum distributive* (in brief, *IID*) if the binary join operation $\vee$ distributes the infimum operation $\bigwedge$, that is

$$\forall x \in L,\ \forall y_i \in L\ (i \in I), \qquad x \vee \Big(\bigwedge_{i \in I} y_i\Big) = \bigwedge_{i \in I} (x \vee y_i) \quad .$$

For an erosion ε, the least opening $\leq \varepsilon$ is the erosion

$$\tilde{\varepsilon} = \bigwedge_{i=0}^{\infty} \varepsilon^i = \bigwedge_{j=1}^{\infty} (\mathbf{id} \wedge \varepsilon)^j \quad .$$

The dual geodesic restriction to m of an operator $\psi : L \to L$ is is the operator $\psi^m : L^*[m] \to L^*[m]$ defined by $\psi^m(x) = \psi(x) \vee m$ for every $x \in L^*[m]$. The dual version of Proposition 3.3 states thus that the following choices of $\rho(m)(x)$ ($m \in L$, $x \in L[m]$) give geodesic reconstruction systems on L:

1. $\rho(m)(x) = \gamma^x(m) = \gamma(m) \vee x$, for an opening γ on L;
2. when L is ISD, $\rho(m)(x) = \widetilde{\varepsilon^x}(m)$ for an erosion ε on L;
3. $\rho(m)(x) = \tilde{\varepsilon}^x(m)$ for an erosion ε on L.

The second case corresponds to the classical geodesic reconstruction by erosion.

We will now extend each $\rho(m)$ to an operator $L \to L[m]$. This can be done in two ways. For every $m \in L$ we define the *minoration* map $\mu_m : L \to L[m]$ by setting for each $x \in L$:

$$\mu_m(x) = \begin{cases} x & \text{if } x \leq m, \\ \bot & \text{if } x \not\leq m. \end{cases} \tag{16}$$

The other map $L \to L[m]$ that we consider is the *meet* map $x \mapsto x \wedge m$. Note that for $m \neq \top, \bot$, μ_m is not increasing; on the other hand the meet map is always increasing, and when L is ISD, it is a dilation. Also, both maps induce the identity on $L[m]$, and only on it: for $x \in L$ we have:

$$x \in L[m] \iff \mu_m(x) = x \iff x \wedge m = x \quad .$$

Finally, we always have $\mu_m(x) \leq x \wedge m$.

We can thus consider the following extensions to L of the geodesic reconstruction inside a marker m: the *minoration geodesic reconstruction*

$$\rho_\mu(m) : L \to L[m] : x \mapsto \rho(m)(\mu_m(x)) = \begin{cases} \rho(m)(x) & \text{if } x \leq m, \\ \rho(m)(\bot) & \text{if } x \not\leq m. \end{cases} \tag{17}$$

and the *meet geodesic reconstruction*

$$\rho_\wedge(m) : L \to L[m] : x \mapsto \rho(m)(x \wedge m) \quad . \tag{18}$$

When restricted to $x \in L[m]$, both $\rho_\mu(m)$ and $\rho_\wedge(m)$ coincide with $\rho(m)$. Note also that since the meet map $x \to x \wedge m$ and $\rho(m)$ are increasing, $\rho_\wedge(m)$ is increasing. When L is ISD and $\rho(m) = \widehat{\delta_m}$ for a dilation δ on L (cfr. item 3 of Proposition 3.3), $\rho_\wedge(m)$ will be a dilation. Finally, since every $x \in L$ satisfies $\mu_m(x) \leq x \wedge m$, and $\rho(m)$ is increasing, we have $\rho_\mu(m) \leq \rho_\wedge(m)$.

Although $\rho_\mu(m)$ and $\rho_\wedge(m)$ are defined as maps $L \to L[m]$, they can more generally be considered as operators $L \to L$, and from this point of view they are idempotent. Indeed, since $\rho_\mu(m)(x) \in L[m]$, we have $\rho_\mu(m)(\rho_\mu(m)(x)) = \rho(m)(\rho_\mu(m)(x)) = \rho(m)(\rho(m)(\mu_m(x))) = \rho(m)(\mu_m(x)) = \rho_\mu(m)(x)$, thanks to the idempotence of of $\rho(m)$; a similar argument holds for $\rho_\wedge(m)$.

We illustrate in Figure 6 these two mappings in the case where $L = \mathcal{P}(\mathbf{Z}^2)$, and for $X \subseteq M \subseteq E$, $\rho(M)(X)$ is the union of all 4-connected components of M having a nonvoid intersection with X.

An interesting fact is that both $\rho_\mu(m)$ and $\rho_\wedge(m)$ lead to geodesic openings on L:

Theorem 3.1. Let ρ be a geodesic reconstruction system on L. For a fixed $x \in L$, the two operators on L

$$\gamma_x : m \mapsto \rho_\mu(m)(x) \tag{19}$$

and

$$\gamma'_x : m \mapsto \rho_\wedge(m)(x) \tag{20}$$

are openings, and we have $\gamma_x \leq \gamma'_x$.

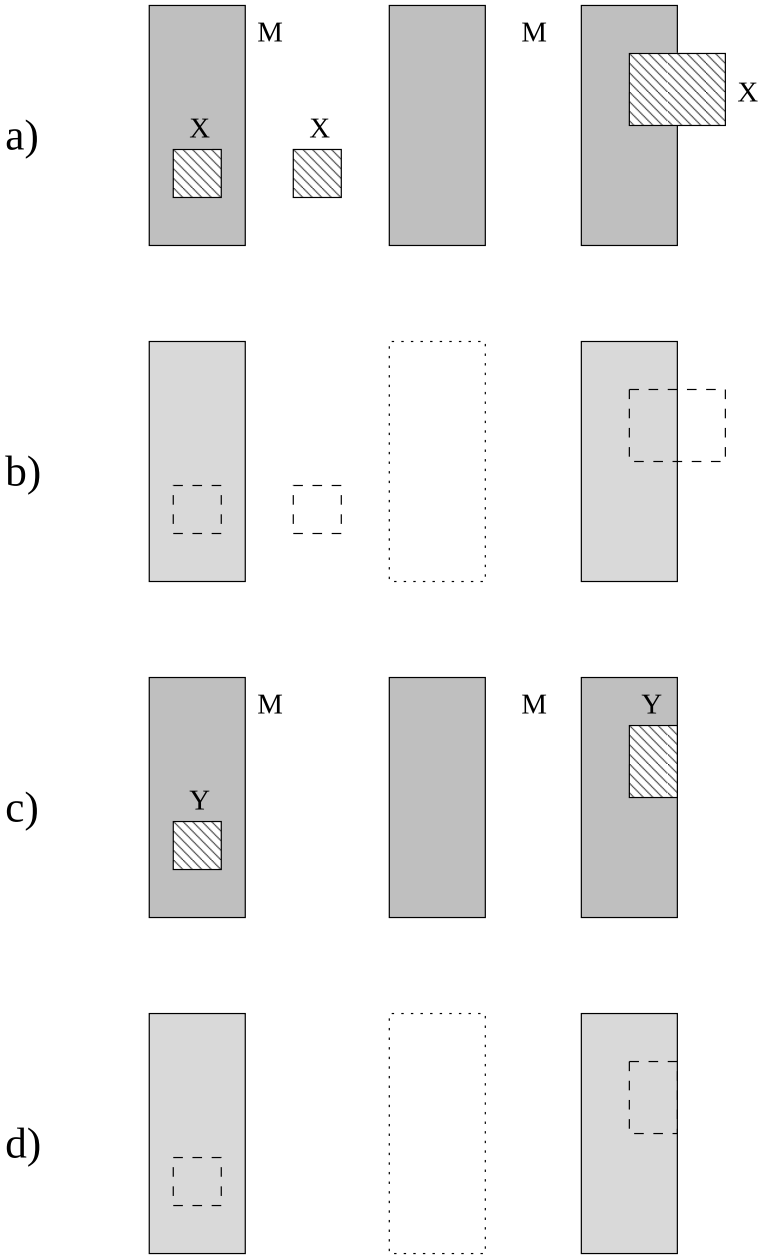

Figure 6. $L = \mathcal{P}(\mathbf{Z}^2)$ and for $X \subseteq M \subseteq E$, $\rho(M)(X)$ is the union of all 4-connected components of M intersecting X. a) X (hatched) not contained in M (in grey). b) $\rho_\wedge(m)(X) = \rho_M(X \cap M)$ (in light grey) consists of all 4-connected components of M intersecting X, while $\rho_\mu(m)(X)$ is empty. c) Y (hatched) contained in M (in grey). d) $\rho_\mu(m)(Y) = \rho_\wedge(m)(Y) = \rho(M)(Y)$ (in light grey) consists of all 4-connected components of M intersecting Y.

Proof:
By the definitions (17,18) of $\rho_\mu(m)$ and $\rho_\wedge(m)$, we have $\gamma_x(m) = \rho_\mu(m)(x) = \rho(m)(\mu_m(x)) \in L[m]$ and $\gamma'_x(m) = \rho_\wedge(m)(x) = \rho(m)(x \wedge m) \in L[m]$; thus $\gamma_x(m) \leq m$ and $\gamma'_x(m) \leq m$, hence γ_x and γ'_x are anti-extensive.

Let us now show that γ_x and γ'_x are increasing. Let $n \leq m$; by Definition 3.2, every $y \in L[n]$ gives $\rho(n)(y) \leq \rho(m)(y)$. For γ_x we have three cases:

- $x \leq n$. Then $x \leq m$, so by (17) we have $\gamma_x(n) = \rho_\mu(n)(x) = \rho(n)(x)$ and $\gamma_x(m) = \rho_\mu(m)(x) = \rho(m)(x)$. As $x \in L[n]$, $\rho(n)(x) \leq \rho(m)(x)$, so $\gamma_x(n) \leq \gamma_x(m)$.

- $x \not\leq n$ but $x \leq m$. Here (17) gives $\gamma_x(n) = \rho_\mu(n)(x) = \rho(n)(\perp)$ while $\gamma_x(m) = \rho_\mu(m)(x) = \rho(m)(x)$. Now as $\perp \in L[n]$, $\rho(n)(\perp) \leq \rho(m)(\perp)$, and as $\rho(m)$ is increasing on $L[m]$, we have $\rho(m)(\perp) \leq \rho(m)(x)$; hence $\gamma_x(n) = \rho(n)(\perp) \leq \rho(m)(x) = \gamma_x(m)$.

- $x \not\leq m$. Then $x \not\leq n$ and here (17) gives $\gamma_x(n) = \rho_\mu(n)(x) = \rho(n)(\perp)$ and $\gamma_x(m) = \rho_\mu(m)(x) = \rho(m)(\perp)$. But as $\perp \in L[n]$, $\rho(n)(\perp) \leq \rho(m)(\perp)$, and so $\gamma_x(n) \leq \gamma_x(m)$.

Thus in all three cases $n \leq m$ gives $\gamma_x(n) \leq \gamma_x(m)$, that is γ_x is increasing.

On the other hand (18) gives $\gamma'_x(n) = \rho_\wedge(n)(x) = \rho(n)(x \wedge n)$ and similarly $\gamma'_x(m) = \rho(m)(x \wedge m)$. As $x \wedge n \in L[n]$, we have $\rho(n)(x \wedge n) \leq \rho(m)(x \wedge n)$; since $x \wedge n \leq x \wedge m$ and $\rho(m)$ is increasing on $L[m]$, $\rho(m)(x \wedge n) \leq \rho(m)(x \wedge m)$. Therefore we get $\gamma'_x(n) = \rho(n)(x \wedge n) \leq \rho(m)(x \wedge m) = \gamma'_x(m)$, and hence γ'_x is increasing.

Let us now prove the idempotence of γ_x. For $m \in L$, we set $n = \gamma_x(m)$, and we must show that $n = \gamma_x(n)$. Note that $n \leq m$, because γ_x is anti-extensive. We have two cases:

- $x \leq m$. Here $n = \gamma_x(m) = \rho_\mu(m)(x) = \rho(m)(x)$. As $\rho(m)$ is extensive on $L[m]$, we have $x \leq \rho(m)(x) = n$; then $\gamma_x(n) = \rho_\mu(n)(x) = \rho(n)(x)$. As $x \leq \rho(m)(x) = n \leq m$, by Definition 3.2 we have $\rho(m)(x) = \rho(n)(x)$; hence $\gamma_x(n) = n$.

- $x \not\leq m$; then $x \not\leq n$. Here $n = \gamma_x(m) = \rho_\mu(m)(x) = \rho(m)(\perp)$, and $\gamma_x(n) = \rho_\mu(n)(x) = \rho(n)(\perp)$. As $\perp \leq \rho(m)(\perp) = n \leq m$, by Definition 3.2 we have $\rho(m)(\perp) = \rho(n)(\perp)$; hence $\gamma_x(n) = n$.

Therefore in both cases the equality $\gamma_x(n) = n$ holds for $n = \gamma_x(m)$, and so γ_x is idempotent.

We now prove that γ'_x is idempotent. Similarly as above, we take $m \in L$, set $n = \gamma'_x(m)$, and we must show that $n = \gamma'_x(n)$. By the anti-extensivity of γ'_x, we have $n \leq m$. Here (18) gives $n = \gamma'_x(m) = \rho_\wedge(m)(x) = \rho(m)(x \wedge m)$. As $\rho(m)$ is extensive on $L[m]$, we have $x \wedge m \leq \rho(m)(x \wedge m) = n$, and as $n \leq m$, we get $x \wedge n = x \wedge m$. Hence $\gamma'_x(n) = \rho_\wedge(n)(x) = \rho(n)(x \wedge n) = \rho(n)(x \wedge m)$. As $x \wedge m \leq \rho(m)(x \wedge m) = n \leq m$, by Definition 3.2 we have $\rho(m)(x \wedge m) = \rho(n)(x \wedge m)$. Therefore $n = \rho(m)(x \wedge m) = \rho(n)(x \wedge m) = \gamma'_x(n)$.

Being anti-extensive, increasing and idempotent, γ_x and γ'_x are openings. Also for $m \in L$, since every $x \in L$ satisfies $\mu_m(x) \leq x \wedge m$, and $\rho(m)$ is increasing, we have

$$\gamma_x(m) = \rho_\mu(m)(x) = \rho(m)(\mu_m(x)) \leq \rho(m)(x \wedge m) = \rho_\wedge(m)(x) = \gamma'_x(m) \quad ,$$

so that $\gamma_x \leq \gamma'_x$. ☐

Note that by (17,18), for $x \leq m$, $\gamma_x(m) = \rho(m)(x) = \gamma'_x(m)$, while for $x \not\leq m$, we have $\gamma_x(m) = \rho(m)(\perp) = \gamma_\perp(m)$, but $\gamma'_x(m) = \rho(m)(x \wedge m) = \gamma_{x \wedge m}(m) = \gamma'_{x \wedge m}(m)$.

For $x \in L$, we will call γ_x the *minoration geodesic opening* for x, and γ'_x the *meet geodesic opening* for x. We will see in the next subsection that under certain circumstances the minoration geodesic opening can lead to a connection.

3.2. Connection from geodesic openings

We will give here the axioms for connectivity on a complete lattice L with a sup-generating family S, then show how such a connectivity can arise from the minoration geodesic openings when the closings of the geodesic reconstruction system are strongly symmetric and preserve $\perp$; this will in particular be the case when these closings arise from weakly symmetrical climbing dilations.

After we wrote the first version of this paper in 1999, Braga-Neto and Goutsias obtained independently (see [5], especially its Section III) some results on the relation between connection and geodesic reconstruction similar to some of ours in this section. However their point of view is different from ours: they start with connections, and characterize related geodesic reconstructions, while we start from a geodesic reconstruction system and see whether it leads to a connection. Hence their exposition is different from ours, and in some cases ours is more detailed. Whenever there is such a similarity, we will indicate before the statement of our result which one of Braga-Neto and Goutsias is related to it, and if necessary, we will briefly discuss the differences.

Connectivity on complete lattices was studied in [17], and we will consider the variant of it with "canonic markers" (the sup-generating family S), which was briefly dealt with in Section 2.3 of that paper. Indeed this variant follows more closely the set-theoretical version investigated previously in [16], but most of all it allows to express more clearly the bijection between connections and systems of connectivity openings. We recall here the two equivalent axiomatics for this type of connectivity. The first one is, according to [17]:

Definition 3.3. Given a complete lattice L with sup-generating family S, a *connection* on L is a class $C \in \mathcal{P}(L)$ satisfying the following three conditions:

1. $\perp \in C$;
2. $S \subseteq C$;
3. given $X \subseteq C$ such that $\bigwedge X \neq \perp$, we have $\bigvee X \in C$.

The family of connections on L is written $\mathcal{C}(L)$.

The family $\mathcal{C}(L)$ of connections on L is ordered by inclusion, and closed under intersection; the greatest connection is L. It follows that to every $P \subseteq L$ corresponds the *connection generated by* P, which is the least connection on L containing P, and we write it $C(P)$; furthermore $\mathcal{C}(L)$ is a complete lattice for the ordering by inclusion: a family of connections C_i ($i \in I$) has as infimum its intersection $\bigcap_{i \in I} C_i$ and as supremum the connection generated by its union, $C\big(\bigcup_{i \in I} C_i\big)$.

While the above set of axioms describes the family of "connected objects" in L, the second axiomatics explicits the notion of "connected component" of an object containing a marker. We adopt the version given in Section 2.3 of [17], where the markers are restricted to S:

Definition 3.4. Given a complete lattice L with sup-generating family S, a *system of connection openings* on L is a map $S \to L^L$ associating to every $s \in S$ an opening γ_s, such that for every $s, t \in S$ and $x \in L$ we have:

4. $\gamma_s(s) = s$;
5. $\gamma_s(x) \wedge \gamma_t(x) \neq \perp \implies \gamma_s(x) = \gamma_t(x)$;
6. $s \not\leq x \implies \gamma_s(x) = \perp$.

The family of systems of connection openings on L is written $\Gamma(L)$.

Let us give some consequences of these axioms. For every $s \in S$, by item 4 and the fact that γ_s is increasing, for $s \leq x$ we have $s = \gamma_s(s) \leq \gamma_s(x)$; combining with item 6, this gives:

$$\gamma_s(x) \neq \perp \iff s \leq x \iff s \leq \gamma_s(x) \ . \tag{21}$$

Now for $t \in S$, if $t \leq \gamma_s(x)$, by the anti-extensivity of γ_s we have $t \leq x$, so that $t \leq \gamma_t(x)$, and item 5 gives then $\gamma_t(x) = \gamma_s(x)$; in other words

$$\forall t \in S\big[\gamma_s(x)\big], \qquad \gamma_t(x) = \gamma_s(x) \ . \tag{22}$$

The family $\Gamma(L)$ of systems of connection openings on L can be ordered as follows: given two such systems γ_s and γ'_s ($s \in S$), we write $(\gamma_s,\ s \in S) \leq (\gamma'_s,\ s \in S)$ iff for each $s \in S$ we have $\gamma_s \leq \gamma'_s$. Then the greatest system of connection openings is given by setting:

$$\forall s \in S,\ x \in L, \qquad \gamma_s(x) = \begin{cases} x & \text{if } s \leq x, \\ \perp & \text{if } s \not\leq x. \end{cases}$$

The equivalence between the above two concepts was incompletely analysed in [17]. Another version of it is given in Theorem 3.1 of [5]. We give here a more precise statement of this equivalence. The proof is relatively straightforward, but nevertheless interesting for its logical clarity.

Proposition 3.4. There is a bijection between the family $\mathcal{C}(L)$ of connections on L and the family $\Gamma(L)$ of systems of connection openings on L. A connection C and the corresponding system of connection openings γ_s, $s \in S$, define each other by the following two equivalent relations:

- For $x \in L$ and $s \in S$, $\gamma_s(x) = \bigvee\{c \in C \mid s \leq c \leq x\}$; in other words $\gamma_s(x) = \perp$ if $s \not\leq x$, while for $s \leq x$ it is the greatest $c \in C$ such that $s \leq c \leq x$.
- $C = \{\gamma_s(x) \mid s \in S,\ x \in L\}$.

Furthermore, this bijection preserves ordering, and so it is an isomorphism between the partially ordered sets $(\mathcal{C}(L), \subseteq)$ and $(\Gamma(L), \leq)$, in particular $\Gamma(L)$ is a complete lattice.

Proof:
We have to show that if G is the map $\mathcal{C}(L) \to \Gamma(L)$ building γ_s, $s \in S$, from C, and K is the map $\Gamma(L) \to \mathcal{C}(L)$ building C from γ_s, $s \in S$, then:

a) for every $C \in \mathcal{C}(L)$, $G(C) \in \Gamma(L)$;
b) for every $(\gamma_s,\ s \in S) \in \Gamma(L)$, $K(\gamma_s,\ s \in S) \in \mathcal{C}(L)$;
c) for every $C \in \mathcal{C}(L)$, $C = K(G(C))$;
d) for every $(\gamma_s,\ s \in S) \in \Gamma(L)$, $(\gamma_s,\ s \in S) = G(K(\gamma_s,\ s \in S))$;
e) for every $C, C' \in \mathcal{C}(L)$, $C \subseteq C' \implies G(C) \leq G(C')$;
f) for every $(\gamma_s,\ s \in S), (\gamma'_s,\ s \in S) \in \Gamma(L)$, $(\gamma_s,\ s \in S) \leq (\gamma'_s,\ s \in S) \implies K(\gamma_s,\ s \in S) \subseteq K(\gamma'_s,\ s \in S)$.

a) For a connection C, $G(C)$ is given by setting for every $s \in S$ and $x \in L$: $\gamma_s(x) = \bigvee\{c \in C \mid s \leq c \leq x\}$. Clearly $\gamma_s(x) = \bigvee \emptyset = \perp$ for $s \not\leq x$; thus axiom 6 holds. On the other hand for $s \leq x$, the set $\{c \in C \mid s \leq c \leq x\}$ is not empty, since it contains s, and by axiom 3 its supremum belongs to C; thus $\gamma_s(x) \in C$ and obviously $s \leq \gamma_s(x) \leq x$; therefore $\gamma_s(x)$ is the greatest $c \in C$ such that $s \leq c \leq x$. In particular, $\gamma_s(x) \in C$ for all $s \in S$ and $x \in L$.

Obviously $\gamma_s(x) \leq x$, and for $x \leq y$, the set of $c \in C$ with $s \leq c \leq x$ is included in the set of $c \in C$ with $s \leq c \leq y$, so we have $\gamma_s(x) \leq \gamma_s(y)$. For $s \not\leq x$ we have $\gamma_s(x) = \perp$, so that $\gamma_s(\gamma_s(x)) = \perp$ also; on the other hand for $s \leq x$, as $\gamma_s(x) \in C$ and $s \leq \gamma_s(x)$, we get $\gamma_s(\gamma_s(x)) = \gamma_s(x)$. Therefore γ_s is an opening.

For $s \in S$, as $s \in C$, we get $\gamma_s(s) = s$, so axiom 4 holds. Given $s, t \in S$ such that $\gamma_s(x) \wedge \gamma_t(x) \neq \perp$, we have $\gamma_s(x), \gamma_t(x) \in C$, so axiom 3 gives $\gamma_s(x) \vee \gamma_t(x) \in C$, and as $s \leq \gamma_s(x) \vee \gamma_t(x) \leq x$ and $\gamma_s(x)$ is the greatest element of C between s and x, we must have $\gamma_s(x) \vee \gamma_t(x) = \gamma_s(x)$, in other words $\gamma_t(x) \leq \gamma_s(x)$; we obtain similarly $\gamma_s(x) \leq \gamma_t(x)$, so the equality $\gamma_s(x) = \gamma_t(x)$ follows, that is axiom 5 holds. Hence $G(C) \in \Gamma(L)$.

b) Let γ_s, $s \in S$ be a system of connection openings, and let $C = K(\gamma_s,\ s \in S) = \{\gamma_s(x) \mid s \in S,\ x \in L\}$. For $s \in S$, as γ_s is anti-extensive, we have $\gamma_s(\perp) = \perp$, so $\perp \in C$, and C verifies axiom 1. Now axiom 4 gives $\gamma_s(s) = s$, so that $s \in C$, and C satisfies axiom 2. Finally, take $X \subseteq C$ such that $\bigwedge X \neq \perp$; let $s \in S$ such that $s \leq \bigwedge X$. Given $c \in X$, we have $c = \gamma_t(x)$ for some $t \in S$ and $x \in L$; by (22), $c = \gamma_s(x)$. As γ_s is an opening, the family of all $\gamma_s(x)$, $x \in L$, is sup-closed (see [13]); as each $c \in X$ belongs to that family, so does $\bigvee X$, and we have some $y \in L$ with $\bigvee X = \gamma_s(y)$. Therefore $\bigvee X \in C$, and C verifies axiom 3. We have thus shown that $C = K(\gamma_s,\ s \in S) \in \mathcal{C}(L)$.

c) Let $C \in \mathcal{C}(L)$ and $C' = K(G(C))$; we show that $C = C'$. For $c \in C$ and $s \in S[c]$, we have $c = \gamma_s(c)$, so $c \in C'$; hence $C \subseteq C'$. Let $y \in C'$, we have $y = \gamma_s(x)$ for some $x \in L$ and $s \in S$, and $\gamma_s(x) \in C$, as explained at the end of the first paragraph of a); hence $C' \subseteq C$. The equality $C = C'$ follows from the double inclusion.

d) Take $(\gamma_s,\ s \in S) \in \Gamma(L)$, and let $(\gamma'_s,\ s \in S) = G(K(\gamma_s,\ s \in S))$; we show that both systems of connection openings are equal, that is $\gamma_s = \gamma'_s$ for every $s \in S$. For every $s \in S$ and $x \in L$ we have:

$$\gamma'_s(x) = \bigvee \{\gamma_t(y) \mid t \in S,\ y \in L, \text{ and } s \leq \gamma_t(y) \leq x\} \ .$$

If $s \not\leq x$, then clearly $\gamma'_s(x) = \bigvee \emptyset = \perp = \gamma_s(x)$. Suppose now that $s \leq x$. For any such $\gamma_t(y)$ between s and x, (22) gives $\gamma_t(y) = \gamma_s(y)$, and as $\gamma_s(y) \leq x$ and γ_s is an opening, we get $\gamma_s(y) = \gamma_s(\gamma_s(y)) \leq \gamma_s(x)$; thus $\gamma_t(y) \leq \gamma_s(x)$. On the other hand $s \leq \gamma_s(x) \leq x$, so we have shown that $\gamma_s(x)$ is the greatest $\gamma_t(y)$ such that $s \leq \gamma_t(y) \leq x$, and hence $\gamma'_s(x) = \gamma_s(x)$.

e) If $C \subseteq C'$, then for $s \in S$ and $x \in L$, $\{c \in C \mid s \leq c \leq x\} \subseteq \{c \in C' \mid s \leq c \leq x\}$, so that $\gamma_s(x) \leq \gamma'_s(x)$ for $G(C) = (\gamma_s,\ s \in S)$ and $G(C') = (\gamma'_s,\ s \in S)$, that is $G(C) \leq G(C')$.

f) Let $(\gamma_s,\ s \in S) \leq (\gamma'_s,\ s \in S)$; thus $\gamma_s \leq \gamma'_s$ for each $s \in S$, and by [13] we have $\gamma'_s\gamma_s = \gamma_s$.

Any element of $K(\gamma_s,\ s \in S)$ takes the form $\gamma_s(x)$ for some $s \in S$ and $x \in L$; now $\gamma_s(x) = \gamma'_s\gamma_s(x) = \gamma'_s(\gamma_s(x))$, and so it is also an element of $K(\gamma'_s,\ s \in S)$. Therefore $K(\gamma_s,\ s \in S) \subseteq K(\gamma'_s,\ s \in S)$.

As the partially ordered sets $(\mathcal{C}(L), \subseteq)$ and $(\Gamma(L), \leq)$ are isomorphic, and $(\mathcal{C}(L), \subseteq)$ is a complete lattice, $(\Gamma(L), \leq)$ is also a complete lattice. □

The form of connectivity given in Definition 3.4 allows us to obtain a connection from the openings γ_x defined in Theorem 3.1, and when L is ISD, to obtain in this way every connection.

Definition 3.5. The geodesic reconstruction system ρ (on L) is called *connecting* if for all $m \in L$, $\rho(m)$ is strongly symmetrical and for $m \neq \top$, $\rho(m)(\bot) = \bot$. Write $\mathcal{G}(L)$ for the family of connecting geodesic reconstruction systems on L.

Our next result is similar to Theorem 3.3 of [5]. However, given the difference of point of view, there are some differences between the two, which will be detailed after the proof. Also our statement is more precise:

Theorem 3.2. The construction of γ_s, $s \in S$, according to (17,19), gives a map from the family $\mathcal{G}(L)$ of connecting geodesic reconstruction systems on L to the family $\Gamma(L)$ of systems of connection openings on L, and when L is ISD, this map is surjective. More precisely:

1. Let ρ be a geodesic reconstruction system on L. The openings γ_s, $s \in S$, defined according to (17,19), namely
$$\forall s \in S,\ \forall m \in L, \qquad \gamma_s(m) = \rho_\mu(m)(s) = \rho(m)(\mu_m(s)) \ , \tag{23}$$
constitute a system of connection openings on L, if and only if ρ is connecting. We have then
$$\forall s \in S,\ \forall m \in L, \qquad \gamma_s(m) = \begin{cases} \rho(m)(s) & \text{if } s \leq m, \\ \bot & \text{if } s \not\leq m. \end{cases} \tag{24}$$

2. Assume that L is ISD. Let γ_s, $s \in S$, be a system of connection openings on L, and define the map $\rho : L \to \bigcup_{m \in L} L[m]^{L[m]}$ by
$$\rho(m)(x) = \bigvee_{s \in S[x]} \gamma_s(m) \ . \tag{25}$$
Then ρ is a connecting geodesic reconstruction system on L, $\rho(\top)(\bot) = \bot$, and (23) holds.

Proof:
1) Let ρ be a geodesic reconstruction system on L, and let γ_s, $s \in S$ be given by(23).

We know from Theorem 3.1 that the γ_s are openings. Let us show that they always verify axiom 4 of Definition 3.4: as $s \leq s$, we have $\gamma_s(s) = \rho(s)(s)$, and as $\rho(s)$ is an extensive operator on $L[s]$, we have $s \leq \rho(s)(s) \in L[s]$, that is $\gamma_s(s) = \rho(s)(s) = s$.

In order for the γ_s, $s \in S$, to constitute a system of connection openings, we need to verify axioms 5 and 6; we show that they hold iff ρ is connecting, that is for every $m \in L$, $\rho(m)$ is strongly symmetrical and for $m \neq \top$, $\rho(m)(\bot) = \bot$.

a) Suppose that for every $m \in L$, $\rho(m)$ is strongly symmetrical and for $m \neq \top$, $\rho(m)(\bot) = \bot$. We show axioms 5 and 6, as well as (24).

Note that (17,19) gives $\gamma_s(m) = \rho(m)(s)$ if $s \leq m$, and $\gamma_s(m) = \rho(m)(\bot)$ for $s \not\leq m$; however for $s \not\leq m$ we must have $m \neq \top$, and we assume here that $\rho(m)(\bot) = \bot$, we get then $\gamma_s(m) = \bot$, so that (24) holds in this case.

Let $m \in L$ and $s, t \in S$ such that $\gamma_s(m) \wedge \gamma_t(m) \neq \bot$. In particular we have $\gamma_s(m), \gamma_t(m) \neq \bot$, so by (24) we have $s, t \leq m$, and also $\gamma_s(m) = \rho(m)(s)$ and $\gamma_t(m) = \rho(m)(t)$. Take $r \in S[\gamma_s(m) \wedge \gamma_t(m)]$. We have $r \leq \gamma_s(m) = \rho(m)(s)$ and $r \leq \gamma_t(m) = \rho(m)(t)$, and as $\rho(m)$ is strongly symmetrical, we get $s, t \leq \rho(m)(r)$. As $\rho(m)$ is increasing and idempotent on $L[m]$, the four inequalities

$$r \leq \rho(m)(s), \quad r \leq \rho(m)(t), \quad s \leq \rho(m)(r), \quad t \leq \rho(m)(r) \ ,$$

imply that

$$\begin{array}{rrcccl} & \rho(m)(r) & \leq & \rho(m)(\rho(m)(s)) & = & \rho(m)(s) \ , \\ & \rho(m)(r) & \leq & \rho(m)(\rho(m)(t)) & = & \rho(m)(t) \ , \\ & \rho(m)(s) & \leq & \rho(m)(\rho(m)(r)) & = & \rho(m)(r) \ , \\ \text{and} & \rho(m)(t) & \leq & \rho(m)(\rho(m)(r)) & = & \rho(m)(r) \ , \end{array}$$

in other words $\rho(m)(r) = \rho(m)(s) = \rho(m)(t)$. Hence $\gamma_s(m) = \rho(m)(s) = \rho(m)(t) = \gamma_t(m)$. We have thus shown that axiom 5 is verified.

Given $m \in L$ and $s \in S$, if $s \not\leq m$, then $\gamma_s(m) = \bot$ by (24), so that axiom 6 holds.

b) Suppose that axioms 5 and 6 hold. We show that for every $m \in L$, $\rho(m)$ is strongly symmetrical and for $m \neq \top$, $\rho(m)(\bot) = \bot$.

Let $m \in L$ and $s, t \in S[m]$ such that $s \leq \rho(m)(t)$. As $s, t \leq m$, (23) gives $\gamma_s(m) = \rho(m)(s)$ and $\gamma_t(m) = \rho(m)(t)$. So $s \leq \gamma_t(m)$, and by (22) we have $\gamma_s(m) = \gamma_t(m)$; hence $\rho(m)(s) = \rho(m)(t)$, and as $t \leq \rho(m)(t)$ by the extensivity of $\rho(m)$, we deduce that $t \leq \rho(m)(s)$. Hence $\rho(m)$ is strongly symmetrical.

If $m \neq \top$, then there is $s \in S$ such that $s \not\leq m$. By axiom 6, we have $\gamma_s(m) = \bot$; by (23) we have $\gamma_s(m) = \rho(m)(\bot)$; combining both, we get $\rho(m)(\bot) = \bot$.

2) Let γ_s, $s \in S$, be a system of connection openings, and let ρ be defined by (25). We show first that ρ satisfies the requirements of Definition 3.2, next that $\rho(m)(\bot) = \bot$ for all $m \in L$ (in particular for $m = \top$), and finally that (23) holds. Then it will follow from item 1 that for every $m \in L$, $\rho(m)$ is strongly symmetrical. Hence ρ will be a connecting geodesic reconstruction system.

Let $m \in L$ and $x \in L[m]$. For $s \in S[x]$, as γ_s is anti-extensive, $\gamma_s(m) \leq m$, and as $s \leq x \leq m$, by (21) we have $s \leq \gamma_s(m)$; we deduce that

$$x = \bigvee S[x] \leq \bigvee_{s \in S[x]} \gamma_s(m) \leq m \ ,$$

that is $x \leq \rho(m)(x) \in L[m]$. Hence $\rho(m)$ is an extensive operator on $L[m]$.

Given $y \in L[m]$ such that $x \leq y$, $S[x] \subseteq S[y]$, so that

$$\rho(m)(x) = \bigvee_{s \in S[x]} \gamma_s(m) \leq \bigvee_{s \in S[y]} \gamma_s(m) = \rho(m)(y) \ ,$$

that is $\rho(m)$ is increasing.

Let $t \in S\big[\rho(m)(x)\big]$; by (25) we have $t \leq \bigvee_{s \in S[x]} \gamma_s(m)$. By ISD we get

$$t = t \wedge \bigvee_{s \in S[x]} \gamma_s(m) = \bigvee_{s \in S[x]} \big(t \wedge \gamma_s(m)\big) \ ,$$

from which we deduce that there is some $s' \in S[x]$ such that $t \wedge \gamma_{s'}(m) \neq \perp$. There is thus some $t' \in S[t]$ such that $t' \leq \gamma_{s'}(m)$. As $t \leq m$, (21) gives $t \leq \gamma_t(m)$; since $t' \leq t \leq \gamma_t(m)$ and $t' \leq \gamma_{s'}(m)$, axiom 5 implies that $\gamma_t(m) = \gamma_{s'}(m)$. Therefore every $t \in S\big[\rho(m)(x)\big]$ verifies $\gamma_t(m) \leq \bigvee_{s \in S[x]} \gamma_s(m)$, so that

$$\rho(m)\big(\rho(m)(x)\big) = \bigvee_{t \in S\left[\rho(m)(x)\right]} \gamma_t(m) \leq \bigvee_{s \in S[x]} \gamma_s(m) = \rho(m)(x) \ ;$$

as $\rho(m)$ is extensive, we have the reverse inequality $\rho(m)(x) \leq \rho(m)\big(\rho(m)(x)\big)$, and the equality follows. Hence $\rho(m)$ is idempotent.

Suppose now that $x \leq n \leq m$. For every $s \in S[x]$, as γ_s is increasing, we have $\gamma_s(n) \leq \gamma_s(m)$, and we deduce that

$$\rho(n)(x) = \bigvee_{s \in S[x]} \gamma_s(n) \leq \bigvee_{s \in S[x]} \gamma_s(m) = \rho(m)(x) \ .$$

Suppose finally that $x \leq \rho(m)(x) \leq n \leq m$. By (25), for every $s \in S[x]$, $\gamma_s(m) \leq \rho(m)(x)$, so that $\gamma_s(m) \leq n \leq m$. As γ_s is increasing and idempotent, we get $\gamma_s(n) = \gamma_s(m)$; as this holds for all $s \in S[x]$, by (25) we deduce that $\rho(n)(x) = \rho(m)(x)$.

We have thus shown that ρ is a geodesic reconstruction system on L. Now we show that for all $m \in L$, $\rho(m)(\perp) = \perp$; indeed

$$\rho(m)(\perp) = \bigvee_{s \in S[\perp]} \gamma_s(m) = \bigvee \emptyset = \perp \ .$$

Let us finally show that (23) holds, that is, for every $s \in S$ and $m \in L$, $\gamma_s(m) = \rho(m)(\mu_m(s))$. Suppose first that $s \leq m$. For $t \in S[s]$, $t \leq s \leq m$, so by (21,22), $\gamma_t(m) = \gamma_s(m)$. Hence

$$\rho(m)(\mu_m(s)) = \rho(m)(s) = \bigvee_{t \in S[s]} \gamma_t(m) = \gamma_s(m) \ .$$

Suppose next that $s \not\leq m$. Then by axiom 6, $\gamma_s(m) = \perp$, and (17) gives $\rho(m)(\mu_m(s)) = \rho(m)(\perp) = \perp$; thus $\rho(m)(\mu_m(s)) = \gamma_s(m)$.

As ρ is a geodesic reconstruction system on L and (23) holds, it follows from item 1 that $\rho(m)$ is strongly symmetrical for every $m \in L$. $\square$

Note that this surjection $\mathcal{G}(L) \to \Gamma(L)$ is not always a bijection. Indeed the system of connection openings γ_s, $s \in S$, determines the value of $\rho(m)(s)$ for $m \in L$ and $s \in S[m]$ (namely, $\rho(m)(s) = \gamma_s(m)$), but the values of $\rho(m)(x)$ for $x \in L[m] \setminus S$ are not determined. Take for example $L = \mathcal{P}(E)$ with a connection $\mathbf{C}$ such that E has at least 3 connected components for $\mathbf{C}$; we define ρ' as follows:

$$\forall M \subseteq E,\ \forall X \subseteq M, \qquad \rho'(M)(X) = \begin{cases} \bigcup_{p \in X} \gamma_p(M) & \text{if } \exists Y \in \mathbf{C},\ X \subseteq Y, \\ M & \text{otherwise}. \end{cases}$$

In other words, $\rho'(M)(X)$ is as in (25) if X lies in a connected component of E, but it gives M otherwise. Then ρ' is a geodesic reconstruction system which gives through (23) the connection $\mathbf{C}$, as does the geodesic reconstruction system ρ defined in (25); this happens because $\rho'(M)(p) = \rho(M)(p)$ for any point p. However $\rho' \neq \rho$; indeed taking $X = E_1 \cup E_2$ for two connected components E_1 and E_2 of E, we have $\rho'(E)(X) = E$, but $\rho(E)(X) = X$.

Remark 3.2. Theorem 3.3 of [5] is restricted to the case where L is ISD (which we assumed only for the surjectivity in item 2). Moreover, they consider a restricted class of connecting geodesic reconstruction systems. Indeed, they assume (under another terminology) a geodesic reconstruction system ρ such that for all $m \in L$, $\rho(m)$ is strongly symmetrical, and for every $x \in L$ we have

$$\rho(m)(x) = \bigvee_{s \in S[x]} \rho(m)(s) \ . \tag{26}$$

Clearly this implies that $\rho(m)(\perp) = \perp$ for all $m \in L$, so ρ is connecting. Note that this equation is satisfied when $\rho(m)$ is a dilation. Write $\mathcal{G}^*(L)$ for the family of connecting geodesic reconstruction systems on L which satisfy (26). As Braga-Neto pointed out to us in a private communication, we have then a bijection between $\mathcal{G}^*(L) \to \Gamma(L)$ given by (23,24), whose inverse is given by (25). This bijection implies that $\mathcal{G}^*(L)$ has the same the lattice-theoretical structure as $\Gamma(L)$.

A consequence of the above theorem is that we get a connection from the geodesic restriction of a strongly symmetrical closing, or of a climbing weakly symmetrical dilation:

Theorem 3.3. The γ_s, $s \in S$, given in (23) constitute a system of connection openings on L for the following choices of $\rho(m)$, $m \in L$:

1. $\rho(m) = \varphi_m$, for a strongly symmetrical closing φ on L satisfying $\varphi(\perp) = \perp$;
2. when L is ISD, $\rho(m) = \widehat{\delta_m}$, for a climbing weakly symmetrical dilation δ on L;
3. when L is ISD, $\rho(m) = \hat{\delta}_m$ for a climbing weakly symmetrical dilation δ on L.

Proof:
By Proposition 3.3, all three choices of $\rho(m)$ constitute a geodesic reconstruction system on L. If $\varphi(\perp) = \perp$, then for each $m \in L$ we have $\rho(m)(\perp) = \varphi_m(\perp) = \varphi(\perp) \wedge m = \perp$. As δ is a dilation, $\hat{\delta}$ is also a dilation by Lemma 2.1, so $\hat{\delta}(\perp) = \perp$, and hence $\hat{\delta}_m(\perp) = \hat{\delta}(\perp) \wedge m = \perp$. When L is ISD, by Proposition 3.2 $\widehat{\delta_m}$ is a dilation on $L[m]$, so that we have $\widehat{\delta_m}(\perp) = \perp$.

If φ is strongly symmetrical, then by item 1 of Proposition 3.1, φ_m is also strongly symmetrical. Assume now that L is ISD. If δ is climbing and weakly symmetrical, then by Theorem 2.1, $\hat{\delta}$ is strongly symmetrical. By item 1 of Proposition 3.1, $\hat{\delta}_m$ is also strongly symmetrical. Furthermore by items 1 and 2 of Proposition 3.1, δ_m is also climbing and weakly symmetrical; by Theorem 2.1, $\widehat{\delta_m}$ is strongly symmetrical.

Hence all three choices of $\rho(m)$ give $\rho(m)(\perp) = \perp$ and $\rho(m)$ strongly symmetrical. Applying Theorem 3.2, the γ_s, $s \in S$, constitute a system of connection openings on L. $\square$

Note that when the elements of S are not all atoms, we cannot obtain a connection by using γ'_s (see (20)) instead of γ_s, because axiom 6 in Definition 3.4 will not hold. Indeed for $s \in S$ not an atom, there is $m \in L$ such that $s \not\leq m$ and $s \wedge m \neq \perp$; then by (18,20) we have $\gamma'_s(m) = \rho(m)(s \wedge m) \geq s \wedge m > \perp$, contradicting axiom 6. On the other hand, for every atom a we have always $\mu_m(a) = a \wedge m$ for all $m \in L$, so that $\gamma_a = \gamma'_a$; if all elements of S are atoms, we have thus $\gamma_s = \gamma'_s$ for all $s \in S$.

Note also that the dual geodesic reconstruction systems given in Remark 3.1, in particular $\rho(m)(x) = \gamma(m) \vee x$ for an opening γ, do not lead to connections.

When L is ISD, as $\hat{\delta}_m \geq \widehat{\delta_m}$ by Proposition 3.2, the connected component $\gamma_s(m)$ of m containing s will be larger with $\hat{\delta}_m$ than with $\widehat{\delta_m}$. In other words, the connection corresponding to $\widehat{\delta_m}$ will be a subset of that corresponding to $\hat{\delta}_m$. Similarly, if we have a family of dilations $\delta[\lambda]$ depending on a parameter λ, such that $\delta[\lambda]$ increases with λ, then $\widehat{\delta[\lambda]_m}$ will increase with λ, and the same holds for $\hat{\delta}[\lambda]_m$, so that the corresponding connection will also increase with λ. In particular the connection openings $\gamma_{s,\lambda}$ with varying parameter λ, corresponding to the geodesic dilations $\widehat{\delta[\lambda]_m}$, will form a *granulometry*.

For example with $L = \mathcal{P}(\mathbf{Z}^2)$, taking for each integer $n > 0$, $\delta[n]$ to be the dilation by the $(2n+1) \times (2n+1)$-square centered about the origin (in other words, the set of pixels at 8-distance at most n from the origin), clearly $\delta[n]$ is strongly symmetrical and climbing, and it increases with n; here the connection built from the geodesic dilations $\widehat{\delta[n]_M}$ for all masks $M \subseteq \mathbf{Z}^2$, is made of the following "connected"sets: all $X \subseteq \mathbf{Z}^2$ such that for every $x, y \in X$ there is a sequence $x = x_0, \ldots, x_r = y$ with x_i at 8-distance at most n from x_{i+1} for $i = 0, \ldots, r-1$. Obviously this connection increases with n.

4. Examples

We will describe here several examples of connections corresponding to the connection openings γ_x arising from a connecting geodesic reconstruction system according to (23).

The most interesting case is when the geodesic reconstruction system is given by $\rho(m) = \widehat{\delta_m}$ for a climbing weakly symmetrical dilation δ on L, following item 2 of Theorem 3.3. Here the lattice L must be ISD. This will be illustrated in Subsection 4.1 by examples with binary, grey-level, and colour images.

Non-ISD lattices, especially partitions, will be considered in Subsection 4.2. There we will also briefly discuss flat zones and connected operators for grey-level images; we will explain that these concepts do not correspond to a connection on the lattice of grey-level images, because they rely on a non-climbing dilation for the geodesic reconstruction. However the "object oriented" approach of Agnus [1, 2], breaking the numerical order on grey-levels and considering them simply as labels, could lead to a connection consisting of all flat zones, which can be obtained by geodesic reconstruction.

4.1. Sets, numerical and multivalued functions

A straightforward case is taking $L = \mathcal{P}(E)$ for a set E. Here the notions of strong and weak symmetry of an operator are equivalent; so we will simply say that an operator is (or is not) symmetrical. Also an increasing operator is climbing iff it is extensive. Suppose that we have a symmetrical relation $\sim$ on E, which we call *adjacency* (see [14] for more details); we define the dilation δ adding to a set X all points which are adjacent to a point of X:

$$\delta(X) = X \cup \{y \in E \mid \exists x \in X,\ x \sim y\} \ .$$

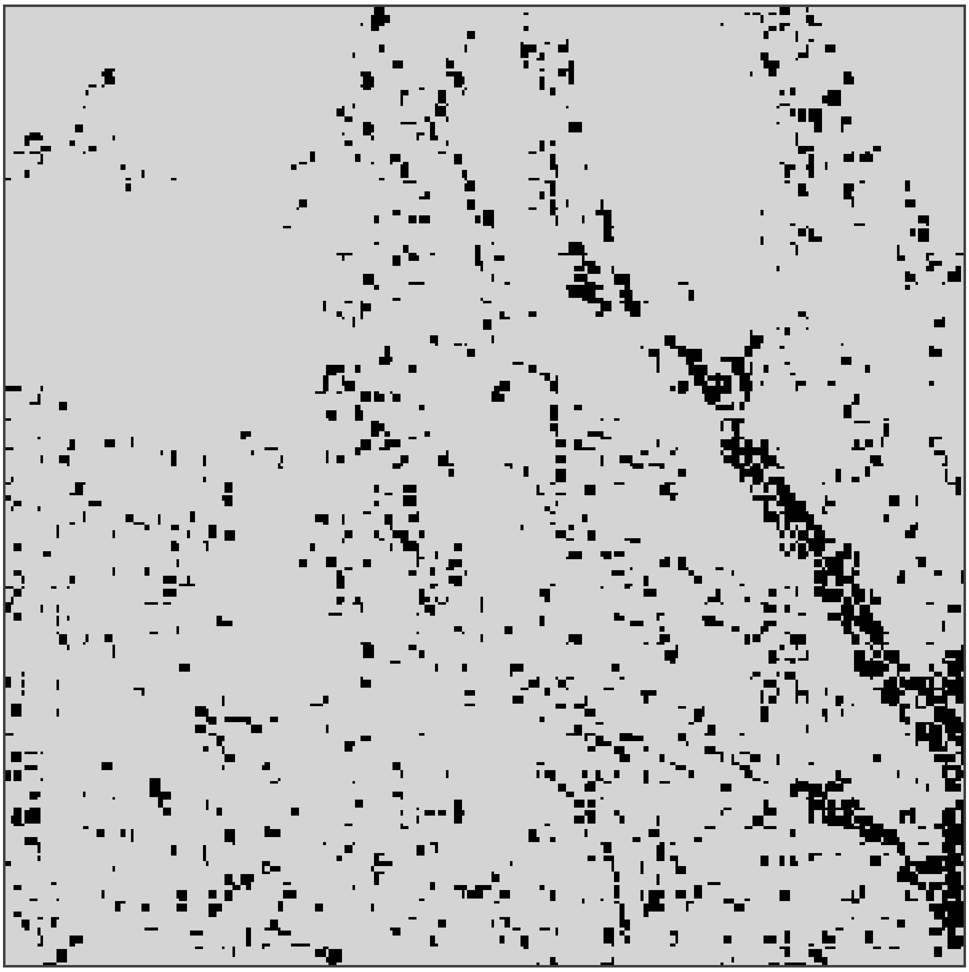

Figure 7. Partial map of the city of Nice.

Then, given a mask $M \subseteq E$, for any marker $X \subseteq M$, $\widehat{\delta_M}(X)$ is the union of all connected components of M having a non-empty intersection with X; here the connectivity obtained from $\widehat{\delta_M}$ is the one arising from the adjacency relation $\sim$ in the graph-theoretical sense.

Generally one considers $E = \mathbf{Z}^n$ or $\mathbf{R}^n$. In the last example of the previous section, we have considered a dilation by a structuring element which is connected in some usual sense. But our results do not require such an assumption. For instance, taking $E = \mathbf{Z}^2$, the dilation by *any* symmetrical structuring element containing the origin is symmetrical and extensive (thus, climbing); hence it leads to a connection. We illustrate this point by an example. From a city map of Nice, in France (Figure 7), one would like to extract the alignments

- of a certain thickness e,
- of houses from a given distance d apart (a characteristic of the type of settlement),

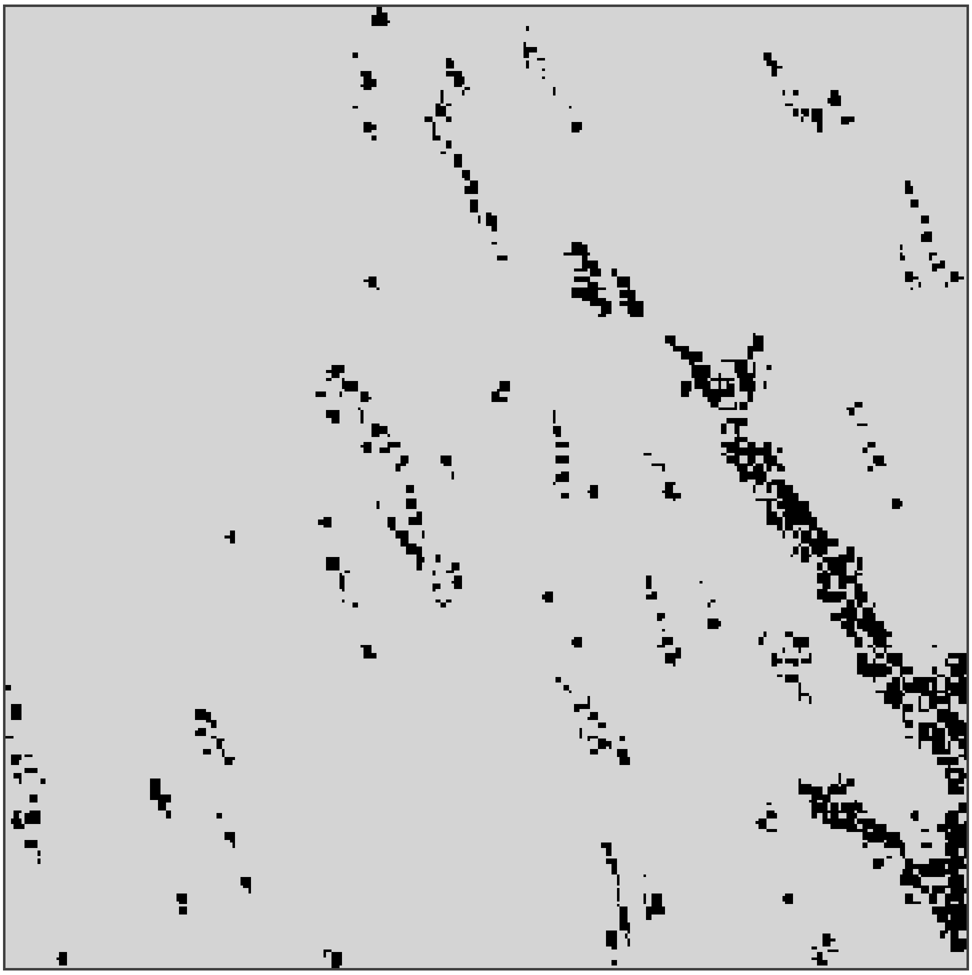

Figure 8. Alignments in the main direction extracted from Figure 7.

- and in the main direction α of the zone investigated.

The second condition on distance is fulfilled by taking a structuring element B made of a triplet of small discs from d apart, and the third requirement by orienting the triplet in direction α. The origin is located at the central point of B, which makes the dilation by B $\delta_B : \mathcal{P}(\mathbf{Z}^2) \to \mathcal{P}(\mathbf{Z}^2) : X \mapsto X \oplus B$ extensive and symmetrical.

Building connected components with the connectivity openings defined in (23), where the geodesic reconstruction arises from the geodesic dilation as indicated in item 3 of Theorem 3.3, we obtain the image shown in Figure 8, which shows the actual alignments. From Theorem 3.3 it follows that a connection has been generated, i.e., that the alignments *segment* the set under study. A final reconstruction after erosion of size e has eliminated the narrow components, fulfilling thus the first condition.

Take now E to be a metric space, with distance d. For any $r \geq 0$ and $x \in E$, we define $B_r(x)$, the *closed ball of radius r centered about x*, by

$$B_r(x) = \{y \in E \mid d(x,y) \leq r\} \ ;$$

then the *dilation of radius* r is the map $\delta_r : \mathcal{P}(E) \to \mathcal{P}(E)$ defined by

$$\delta_r(X) = \bigcup_{x \in X} B_r(x) \ .$$

Note that $\delta_r(\{x\}) = B_r(x)$; as the distance is symmetrical (that is, $d(x,y) = d(y,x)$), the dilation δ_r is symmetrical (recall that for sets, strong and weak symmetry are equivalent); also δ_r is extensive, thus climbing. The connection C_r built from the geodesic restriction of δ_r consists of all *r-connected* sets; here we say that $X \subseteq E$ is r-connected if for every $x, y \in X$, there exists an integer $n \geq 0$ and a sequence $x_0, \ldots, x_n$, such that $x_0 = x$, $x_n = y$, and $d(x_i, x_{i+1}) \leq r$ for all $0 \leq i \leq n-1$. For $r = 0$, C_r consists only of singletons and the empty set; thus we restrict ourselves to the case $r > 0$. The intersection $\bigcap_{r>0} C_r$ is a connection, whose members are the so-called *well-linked* sets (a set is well-linked iff it is r-connected for every $r > 0$, see [6], Chapter 1, Section 19). How does this compare with connectedness in the usual topological sense, where a set is connected iff it cannot be partitioned by its intersection with two open sets? We have the following:

Proposition 4.1. Every (topologically) connected subset of E is well-linked. Every compact well-linked set is connected.

Proof:
Let X be a connected subset of E. We show that for every $r > 0$, X is r-connected. This is obvious for $X = \emptyset$, so we assume $X \neq \emptyset$. Indeed, let X_1 be a non-void r-connected component of X, and let $X_2 = X \setminus X_1$; X_2 is the union of all other r-connected components of X. For $x \in X$, every $y \in X$ such that $d(x,y) \leq r$ must be in the same r-connected component of X as x. Hence for $x \in X_1$, $B_r(x) \cap X \subseteq X_1$, so X_1 is open in the topology on X; similarly all other r-connected components of X are open in X, so X_2 is open in the topology on X. As X, being connected, cannot be partitioned into two sets which are open for the topology on X, we have $X_2 = \emptyset$ and $X_1 = X$, that is, X is r-connected. Being r-connected for all $r > 0$, X is well-linked.

Let X be a compact well-linked subset of X. If $X = \emptyset$, obviously X is connected, so we assume $X \neq \emptyset$. Suppose that X is not connected; there is thus some $X_1 \subseteq X$ and $X_2 = X \setminus X_1$ which are both non-void and open for the topology on X. By complementarity, they are closed for the topology on X, and as X is compact in E, X_1 and X_2 are compact subsets of E. For $r > 0$, as X is well-linked, there is a sequence $x_0, \ldots, x_n$ such that $x_0 \in X_1$, $x_n \in X_2$, and $d(x_i, x_{i+1}) \leq r$ for all $0 \leq i \leq n-1$; taking the largest i such that $x_i \in X_1$, we have $x_{i+1} \in X_2$ and $d(x_i, x_{i+1}) \leq r$. Hence $d(X_1, X_2) \leq r$ for every $r > 0$, so that $d(X_1, X_2) = 0$, and as X_1 and X_2 are compact in E, this means that $X_1 \cap X_2 \neq \emptyset$, a contradiction. Hence our supposition is false, and X must be connected. □

Note that without the compactness assumption, a well-linked set can be disconnected: in $\mathbf{R}^2$ (with the Euclidean metric), the set X of points (x,y) such that $x \neq 0 \neq y$ (X consists of $\mathbf{R}^2$ minus the x and y axes) is well-linked, but not connected; its connected components are the four quadrants enclosed by the x and y axes, and they are at distance 0 from each other. Another example is given by the closed set X consisting of all points (x, y) such that $x > 0$ and $|y| \geq 1/x$; it has two connected components (given by the inequations $y \geq 1/x$ and $y \leq -1/x$ respectively) at distance 0 from each other, so it is well-linked.

Let us now discuss the generality of our approach to the construction of a connection from geodesic operations. Following Theorem 3.3 we built connected components with a reconstruction from a dilation δ. Indeed, the strong symmetry of $\hat{\delta}_m$ and $\widehat{\delta_m}$ follows from the fact that the family of strongly (or weakly) symmetrical dilations is power-closed, as shown in Corollary 2.1; this result relies on item 3 of Proposition 2.1. Would it be possible to obtain a similar result by starting from an extensive and increasing operator which is not a dilation? The answer is negative, we give an example in $\mathcal{P}(\mathbf{Z}^2)$ with a symmetrical increasing and extensive operator ψ such that ψ^n for $n > 1$ and $\hat{\psi} = \bigvee_{i=0}^{\infty} \psi^i$ are not symmetrical; the same will be true of ψ_M and $\widehat{\psi_M}$ for a mask M. Thus item 3 of Proposition 2.1 and Corollary 2.1 require indeed the assumption that we have a dilation, and Theorem 3.3 requires also dilations for the connection from geodesic reconstruction.

We consider subsets of $\mathbf{Z}^2$. Let the structuring elements A and B consist of the origin and its left and right neighbours, with B containing in addition the top neighbour of the origin (see Figure 9.a). We define the operator ψ on $\mathcal{P}(\mathbf{Z}^2)$ as follows: for $X \subseteq \mathbf{Z}^2$ we have

$$\psi(X) = \begin{cases} X \oplus A & \text{if } |X| \leq 1, \\ X \oplus B & \text{if } |X| > 1, \end{cases}$$

in other words $\psi(\emptyset) = \emptyset$, $\psi(\{p\}) = A_p$ for a point $p \in \mathbf{Z}^2$, while $\psi(X) = X \oplus B$ for a set $X \subseteq \mathbf{Z}^2$ containing at least two points. Since both A and B contain the origin, ψ is extensive; as ψ uses Minkowski addition with a structuring element (A or B) which increases with the set, ψ is increasing; finally ψ is symmetrical because the structuring element A is symmetrical: for $p, q \in \mathbf{Z}^2$, we have $p \in A_q \iff q \in A_p$. For every integer $n > 0$, let us write nB for the Minkowski sum of B taken n times; in other words $1B = B$ and $nB = B \oplus (n-1)B$ for $n > 1$. It is easily seen by induction that ψ^n verifies for every $X \subseteq \mathbf{Z}^2$:

$$\psi^n(X) = \begin{cases} X \oplus A \oplus (n-1)B & \text{if } |X| \leq 1, \\ X \oplus nB & \text{if } |X| > 1. \end{cases}$$

We illustrate $\psi^2(\{p\}) = (A \oplus B)_p$ and $\psi^3(\{p\}) = (A \oplus 2B)_p$ in Figure 9.b. As the structuring element $A \oplus (n-1)B$ is not symmetrical for $n > 1$, ψ^n is not symmetrical: take $q \in \big(A \oplus (n-1)B\big)_p = \psi^n(\{p\})$ lying above p, and then $p \notin \big(A \oplus (n-1)B\big)_q = \psi^n(\{q\})$; this is shown for $n = 2$ and $n = 3$ in Figure 9.b.

Now we define $\hat{\psi} = \bigvee_{i=0}^{\infty} \psi^i$; it is easily verified that for every $X \subseteq \mathbf{Z}^2$ we have $\hat{\psi}(X) = X \oplus H$, where H is the digital half-plane made of all points of $\mathbf{Z}^2$ lying above the origin. As $H = H \oplus H$, $\hat{\psi}$ is an idempotent dilation, it is thus the least closing $\geq \psi$. Again, $\hat{\psi}$ is not symmetrical, because H is not symmetrical: for q above p, $q \in \hat{\psi}(\{p\})$ but $p \notin \hat{\psi}(\{q\})$, so if we wanted to define a connection from ψ, q would be in the connected component of $\mathbf{Z}^2$ containg p, while p would not be in the connected component of $\mathbf{Z}^2$ containing q, contradicting (22).

Let us now give an example with numerical functions (grey-level images); for the sake of simplicity, we consider functions $\mathbf{Z}^2 \to \overline{\mathbf{Z}}$, although most of what we say can be extended to functions $E \to T$, where E is an arbitrary set, and $T = \overline{\mathbf{Z}}$ or $\overline{\mathbf{R}}$. Here S consists of the "impulse" function $i_{(p,t)}$ ($p \in \mathbf{Z}^2$,

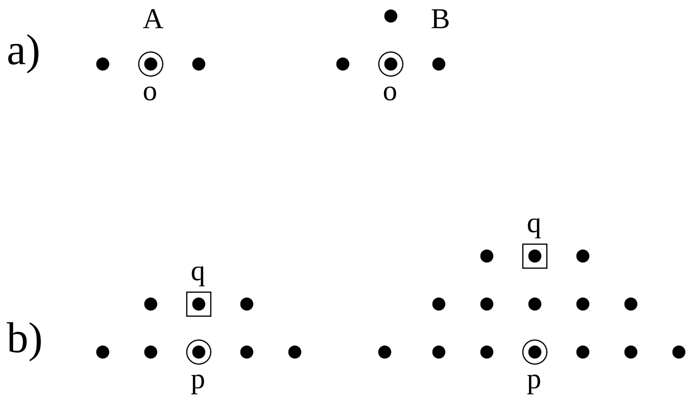

Figure 9. a) The two structuring elements A and B; here o designates the origin (shown circled). b) $\psi^2(\{p\}) = (A \oplus B)_p$ and $\psi^3(\{p\}) = (A \oplus 2B)_p$ for a point $p \in \mathbf{Z}^2$ (shown circled); there is a point q (shown surrounded by a square) such that $q \in \psi^2(\{p\})$ but $p \notin \psi^2(\{q\})$ (respectively: $q \in \psi^3(\{p\})$ but $p \notin \psi^3(\{q\})$).

$t \in \mathbf{Z}$) defined in (1). Recall that the *support* supp(f) of a function f consists of all points p such that $f(p) > -\infty$, and then we have: supp$(f \wedge g) =$ supp$(f) \cap$supp(g) and supp$(f \oplus g) =$ supp$(f) \oplus$supp(g).

Let the structuring function v have as support the 5-pixel cross V in $\mathbf{Z}^2$ made of the origin o and its 4 neigbours in the vertical and horizontal directions. We assume that $v(o) > 0$. We consider the dilation $\delta : f \mapsto f \oplus v$ by v. As supp(v) is symmetrical, δ is weakly symmetrical (see Subsections 1.1 and 1.2). As $v(o) > 0$, δ is climbing. The connection whose connected components are obtained from the geodesic reconstruction $\rho(m) = \widehat{\delta_m}$ using the geodesic restriction of δ, consists in all functions having a 4-connected support. Thus, for a mask function m and an impulse $i_{(p,t)} \leq m$, the connected component of m containing $i_{(p,t)}$ is the function g whose support is the 4-connected component of supp(m) containing p, and we have $g(x) = m(x)$ for $x \in$ supp(g).

Let us explain this concretely. Given a mask function m with support M, for every function $f \leq m$, the geodesic restriction to m of the dilate of f by v, namely $\delta_m(f) = (f \oplus v) \wedge m$, has support

$$\mathrm{supp}\big((f \oplus v) \wedge m\big) = \big(\mathrm{supp}(f) \oplus \mathrm{supp}(v)\big) \cap \mathrm{supp}(m) = \big(\mathrm{supp}(f) \oplus V\big) \cap M \quad .$$

As V consists of the origin and its 4-neighbourhood, the dilation $X \mapsto X \oplus V$ by V adds to a set all points 4-adjacent to it. Thus, iterating δ_m, we get that supp$\big(\widehat{\delta_m}(f)\big)$ consists of all 4-connected components of supp(m) which intersect supp(f). As δ is climbing, on this support the function $\widehat{\delta_m}(f)$ will have the same value as m; indeed, after n iterations of δ_m, a point q is reached in the support supp$\big(\delta_m^n(f)\big)$ with value y on q; as $f(o) > 0$, every application of δ increases grey-levels by at least $f(o)$, and there is some integer $k \geq 0$ such that such that $y + k \cdot f(o) \geq m(q)$, so iterating δ_m k more times, we will get $\delta_m^{n+k}(f)(q) = m(q)$. If we start with $f = i_{(p,t)}$, as supp$(i_{(p,t)}) = \{p\}$, we get the function whose support is the 4-connected component of supp(m) containing p, and having the same values as m on this support.

If we took as support of v the 3×3 square centered about the origin, this would have given as connection the family of functions whose support is 8-connected. The same construction works for any connectivity on $\mathbf{Z}^2$ arising from an adjacency relation: we take then as support of v the set consisting of the origin and the pixels adjacent to it.

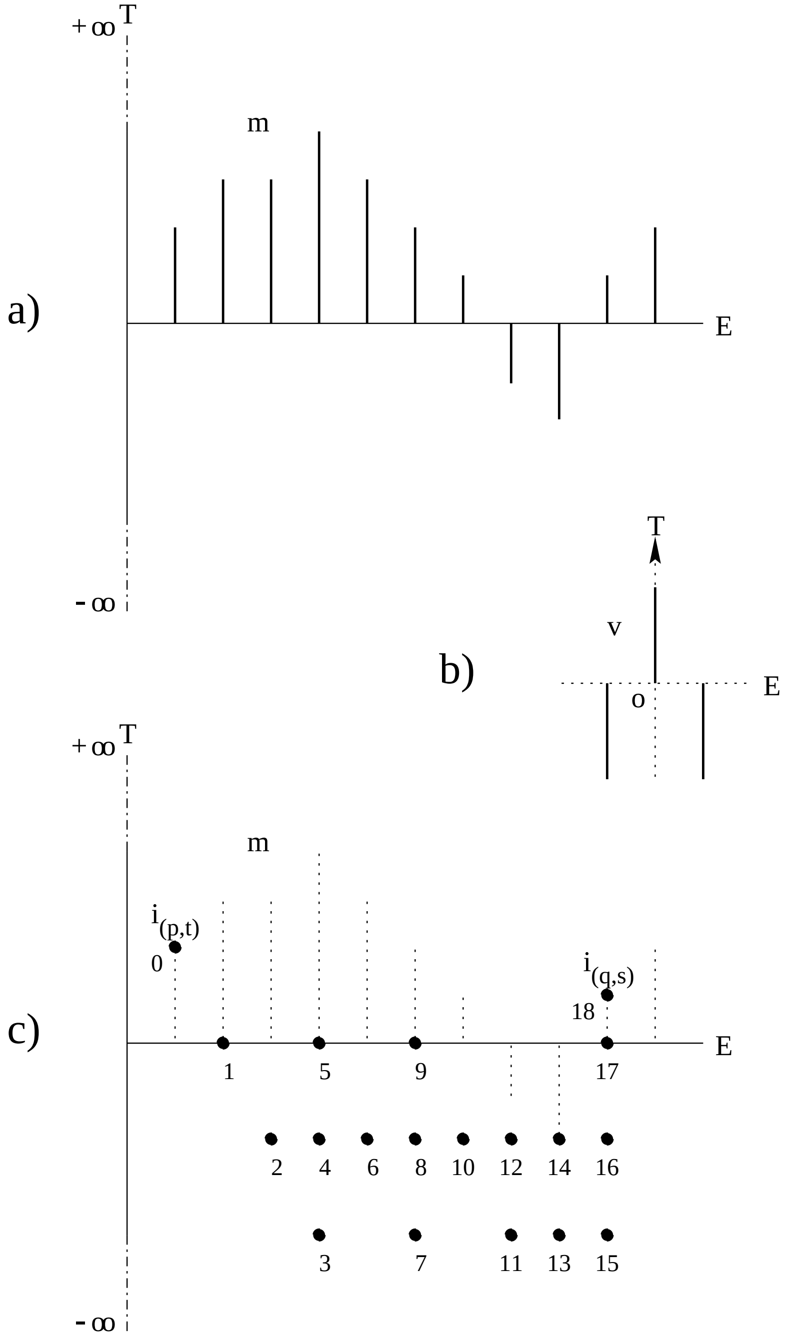

Figure 10. We consider functions $\mathbf{Z} \to \overline{\mathbf{Z}}$, and the connectivity class on $\mathcal{P}(\mathbf{Z})$ made of all integer traces of convex subsets of $\mathbf{R}$. a) The mask function m. b) The structuring function v; here o designates the origin. b) Starting from an impulse $i_{(p,t)}$ below m, and iterating the geodesic dilation by v with mask m, one will progressively reach any impulse $i_{(q,s)}$ in the connected component of m containing $i_{(p,t)}$; here the numbers indicate how many iterations are needed to reach an impulse.

We illustrate in Figure 10 the analogue of this construction for functions $\mathbf{Z} \to \overline{\mathbf{Z}}$, whith the connection on $\mathcal{P}(\mathbf{Z})$ based on the adjacency relation $\sim$ defined by $x \sim y$ iff $|x - y| = 1$; here connected sets correspond to integer convex sets.

In Subsection 4.2 of both [17, 19], Serra considered such a type of connection, namely the family of all functions $E \to T$ whose support belongs to a given connection (connectivity class) on $\mathcal{P}(E)$, such as the family of all 4-connected (*resp.*, 8-connected) sets.

We finally give some examples of connections on colour images which can be defined from geodesic dilations. Here we assume a 3-dimensional colour space, so a colour image is a vector-valued function $f : E \to T^3 : p \mapsto \big(f_0(p), f_1(p), f_2(p)\big)$, or it is represented as a triplet $f = (f_0, f_1, f_2)$ of numerical functions $f_j : E \to T$ ($j = 0, 1, 2$), called *channels*; here E is an arbitrary set, and $T = \overline{\mathbf{Z}}$ or $\overline{\mathbf{R}}$. The set L of colour images is then an ISD lattice for the product ordering w.r.t. the ordering on each channel. We take as generators the single-channel impulses $i^j_{(p,t)}$ ($j = 0, 1, 2$, $p \in E$, $t \in T \setminus \{\pm\infty\}$) defined by

$$i^j_{(p,t)} = (f_0, f_1, f_2) \quad \text{with} \quad f_k = \begin{cases} i_{(p,t)} & \text{if } k = j, \\ \perp & \text{if } k \neq j. \end{cases}$$

The support of a colour image is defined as the set of points where at least one of the channels gives a value $> -\infty$, in other words it is the union of the supports in the 3 individual channels: $\mathrm{supp}(f) = \mathrm{supp}(f_0) \cup \mathrm{supp}(f_1) \cup \mathrm{supp}(f_2)$.

It follows from the exposition in [9] of morphology on product spaces that a dilation on colour images can be represented as a 3×3 matrix of dilations for numerical functions. More precisely, given a dilation δ, for $f = (f_0, f_1, f_2)$ we have $\delta(f) = (g_0, g_1, g_2)$, where $g_j = \bigvee_{k=0}^{2} \delta_{jk}(f_k)$ for $j = 0, 1, 2$, and each δ_{jk} is a dilation on numerical functions; here δ_{jk} gives the contribution of channel k in the input image to channel j in the output image. It can be seen that δ is weakly/strongly symmetrical iff each δ_{jk} weakly/strongly mirrors δ_{kj}, and that δ is climbing iff each δ_{jj} is climbing.

Let us describe some connections that can be obtained from geodesic dilations. As in the previous example we take $E = \mathbf{Z}^2$ with 4-connectivity on $\mathcal{P}(E)$, and $T = \overline{\mathbf{Z}}$. We consider again a numerical structuring function $v : E \to T$ whose support is the 5-pixel cross V in $\mathbf{Z}^2$ made of the origin o and its 4 neigbours in the vertical and horizontal directions, and such that $v(o) > 0$.

In our first example, we take for δ a diagonal matrix: δ_{jj} is the dilation by v: $f_j \mapsto f_j \oplus v$, while for $j \neq k$, δ_{jk} is constant $\perp$: $f_k \mapsto \perp$. Given a mask image $m = (m_0, m_1, m_2)$, the geodesic restriction to m of δ is

$$\delta_m : (f_0, f_1, f_2) \mapsto (f'_0, f'_1, f'_2) \quad \text{with} \quad f'_j = (f_j \oplus v) \wedge m_j \ .$$

So, starting from a single-channel impulse $i^j_{(p,t)}$, and applying the infinite iteration $\widehat{\delta_m}$, we will get a geodesic reconstruction on channel j, and $\perp$ in other channels. Thus the connected component of m containing $i^j_{(p,t)}$ is a single-channel function g, such that $g_k = \perp$ for $k \neq j$, and g_j is the restriction of m_j to the 4-connected component of $\mathrm{supp}(m_j)$ containing p. Thus the connection consists of all colour images having on one channel a numerical function with 4-connected support, and the other two channels reduced to $\perp$.

In our second example, we take for δ the constant matrix: for all j, k, δ_{jk} is the dilation by v: $f_k \mapsto f_k \oplus v$. For the mask image $m = (m_0, m_1, m_2)$, we get as geodesic dilation:

$$\delta_m : (f_0, f_1, f_2) \mapsto (f'_0, f'_1, f'_2) \quad \text{with} \quad f'_j = \big((f_0 \vee f_1 \vee f_2) \oplus v\big) \wedge m_j \ .$$

Here the dilation behaves as if all 3 channels were identical, and the geodesic dilation behaves somewhat as a grey-level operator. Starting from $i^j_{(p,t)}$, $\widehat{\delta_m}$ will reconstruct impulses of m in all 3 channels together, spreading them successively through 4-adjacency on the base points, continuing as long as one still has impulses (i.e., one does not get outside of the support of m). Thus the connected component of m containing $i^j_{(p,t)}$ is the restriction of m to the 4-connected component of the support of m (that is, $\text{supp}(m) = \text{supp}(m_0) \cup \text{supp}(m_1) \cup \text{supp}(m_2)$) containing p; then the connection consists of all colour images with 4-connected support.

In our third example, δ_{jj} is the dilation by v: $f_j \mapsto f_j \oplus v$, while for $j \neq k$, δ_{jk} is the identity operator. Applying δ to a single-channel impulse, this impulse is dilated by v in its own channel, and simply copied into the other two channels. Here the pattern of geodesic reconstruction of $\widehat{\delta_m}$ is to alternatively spread impulses in one channel through 4-adjacency, and switch to another channel while keeping the same position. Then the connected component of m containing $i^j_{(p,t)}$ is the restriction of m to the subset V of $\text{supp}(m)$ consisting of all points q such that there is a path $p = z_0, \ldots, z_n = q$ (with $n \geq 0$), where for every $i = 0, \ldots, n-1$, z_i is 4-adjacent to z_{i+1} and there is a channel k with $z_i, z_{i+1} \in \text{supp}(m_k)$.

The 3 choices of δ in the above examples can be ordered: the first one is the least, the second one is the greatest, and the third one is intermediate. The same ordering applies then to the corresponding connections.

4.2. Weighted partitions, flat zones and related topics

Image segmentation takes the form of a partition of the space on which the image is defined. The family of partitions of a set E has a fine to coarse ordering ($\mathbf{P} \leq \mathbf{Q}$ means that partition $\mathbf{P}$ is finer than partition $\mathbf{Q}$, in other words every class of $\mathbf{P}$ is included in a class of $\mathbf{Q}$), and forms a lattice which is not distributive. The least (finest) partition has the singletons as classes, while the greatest (coarsest) partition has the single class E.

Given a connection $\mathcal{C}$ on E, the family of partitions with connected classes is closed under the supremum, and it contains the least and greatest partitions. It is thus a complete lattice.

In [17, 19] Serra defined a grey-level extension of partitions called *weighted partitions*. We give here a simplified definition of them, which corresponds nonetheless to the same concept. We assume a metric space (E, d) and consider functions $E \to T$, where T is a closed part of $\overline{\mathbf{R}} = \mathbf{R} \cup \{+\infty, -\infty\}$. Let φ be a function $\mathbf{R}^+ \to \mathbf{R}^+$ which is continuous, increasing and sublinear (that is, $\varphi(a+b) \leq \varphi(a) + \varphi(b)$). A function $f : E \to T$ is called *φ-continuous* if

$$\forall p, q \in E, \qquad |f(p) - f(q)| \leq \varphi\big(d(p,q)\big) \ .$$

Here φ is called the *modulus of continuity*. As φ is continuous at the origin, a φ-continuous function is uniformly continuous. Well-known examples are Lipschitz functions (for $\varphi(t) = t$) and constant functions (for $\varphi(t) = 0$). Note that the set of φ-continuous functions is a complete sublattice of the lattice T^E of functions, in other words the (pointwise) infimum (resp., supremum) of a family of φ-continuous functions is again a φ-continuous function. A *weighted partition* is an ordered pair $(f, \mathbf{P})$, where $\mathbf{P}$ is a partition of E and f is a function $E \to T$ such that for every class C of $\mathbf{P}$, the restriction f_C of f to C is φ-continuous. We take the product ordering on weighted partitions: $(f, \mathbf{P}) \leq (g, \mathbf{Q})$ iff $f \leq g$ and $\mathbf{P} \leq \mathbf{Q}$. For this ordering the family of weighted partitions is a complete lattice: for a family $(f_i, \mathbf{P}_i)$ ($i \in I$) of ordered partitions, its infimum is $\big(\bigwedge_{i \in I} f_i, \bigwedge_{i \in I} \mathbf{P}_i\big)$, and its supremum is

$\left(f, \bigvee_{i\in I} \mathbf{P}_i\right)$, where f is the least function $\geq \bigvee_{i\in I} f_i$ such that f_C is φ-continuous for every class C of $\bigvee_{i\in I} \mathbf{P}_i$. (We used here the fact that the φ-continuous functions form a complete sublattice of T^E.)

We can also consider the weighted partitions with connected classes, that is weighted partitions $(f, \mathbf{P})$ where the partition $\mathbf{P}$ has connected classes for a connection $\mathcal{C}$ on E. They form a complete lattice.

For $X \subseteq E$, write X_P for the partition whose classes are X and all singletons included in $E \setminus X$. Write $\perp$ for the least element of T. For a function $f : E \to T$, write $f_{C,\perp}$ for the function defined by

$$f_{C,\perp}(p) = \begin{cases} f(p) & \text{if } p \in C, \\ \perp & \text{if } p \in E \setminus C. \end{cases}$$

Given a connected $C \subseteq E$, a *cylinder of base* C is a weighted partition (f, C_P) such that $f_{E\setminus C} = \perp$. Serra showed [19] that cylinders of connected base form a connection in the lattice of weighted partitions with connected classes. For a weighted partition $(f, \mathbf{P})$ with connected classes, its connected components are all the cylinders $(f_{C,\perp}, C_P)$ whose bases C are all the classes of P. When $\varphi(t) = 0$, this corresponds into a decomposition of f into flat zones (the so-called *flat zone connection* of [19]). When we have $\varphi(x) = \max(x, c)$ for a constant $c > 0$, the cylinders form the so-called *jump connection* of [19]; the generators of the lattice are the cylinders whose base is a singleton.

The segmentation of a function f amounts to finding a maximal weighted partition $(f, \mathbf{P})$, in other words a maximal partition $\mathbf{P}$ such that f_C is φ-continuous for every class C of $\mathbf{P}$. We cannot use the results of the previous sections concerning geodesical dilations, because the lattice is not ISD, so in fact we circumvent the structure of the lattice by working separately on each class (which is a set). We start with the finest partition (with singletons as classes), and each cylinder of singleton base $(f_{\{p\},\perp}, \{p\}_P)$ (for $p \in E$) is iteratively grown into a maximal cylinder $(f_{C,\perp}, C_P)$. For the flat zone connection, there is a unique maximal segmentation, namely the decomposition of E into flat zones, in other words the connected components of all level sets $f^{-1}(t)$, $t \in T$. For the jump connection, there is no such unicity, because for $\varphi \neq 0$, we can have two intersecting parts C, D of E such that f_C and f_D are φ-continuous, but $f_{C\cup D}$ is not.

Connections on functions and weighted partitions have not been used often in practice. A more popular generalization of connectivity on sets is the notion of flat zones. Assume a connection $\mathcal{C}$ on the space E. Given a numerical function $f : E \to T$, where T is the space of numerical values (grey-levels), a *flat zone* of f is a maximal subset Z of E which is connected ($Z \in \mathcal{C}$) and on which f has constant value ($\exists t \in T$, $\forall p \in Z$, $f(p) = t$). The flat zones of f form a partition of E. In the case of binary images $f : E \to \{0, 1\}$, flat zones correspond to the connected components of the figure $f^{-1}(1)$ and of the background $f^{-1}(0)$.

Then an operator ψ on numerical functions is called *connected* if for every function f, the flat zone partition of $\psi(f)$ is coarser than that of f, in orther words for every flat zone Z of f, $\psi(f)$ has a constant value on Z. An interesting fact is that the best known connected operators are geodesic reconstructions (by dilation and by erosion) with a "symmetrical" flat structuring function. Assume that the connection $\mathcal{C}$ arises from a translation-invariant adjacency relation (e.g., the 4- and 8-connectivities), and let B be the neighbourhood of the origin o for that adjacency. Here B is a symmetrical structuring element. We extend B into a flat structuring function f given by $f(p) = 0$ for $p \in B$ and $f(p) = -\infty$ for $p \notin B$. Here f satisfies the following *flat symmetry* requirements, which should be compared to the strong and weak symmetry for functions defined in Subsection 1.1:

1. $\mathrm{supp}(f)$ *is a symmetrical set, and*

2. *for every* $p \in \mathrm{supp}(f)$, $f(p) = 0$.

Writing δ for the dilation by f, then for every mask function m and marker function $w \leq m$, the flat zone partition associated to $\widehat{\delta_m}(w)$ is coarser than the one associated to m, so the map $m \mapsto \widehat{\delta_m}(w)$ is a connected opening on the family of functions $\geq w$.

If we compare the dilation by f and the geodesic reconstruction using the dilation by f with the analysis made in Subsection 3.2, we see an important difference: δ is weakly symmetrical, but not climbing. Hence $\widehat{\delta_m}$ will not be strongly symmetrical. In fact we have the following type of "symmetry" for $\widehat{\delta_m}$: given two impulses $i_{(p,t)}$ and $i_{(q,s)}$, if $i_{(q,s)} \leq \widehat{\delta_m}(i_{(p,t)})$, then $s \leq t$, $i_{(p,s)} \leq \widehat{\delta_m}(i_{(q,s)})$, and $i_{(q,s)} \leq \widehat{\delta_m}(i_{(p,s)})$. If we turn to the openings γ_s defined in (23), we see that they do not satisfy requirement 5 of a system of connection openings (Definition 3.4): two distinct "connected components" can overlap. What happens in practice is the following: from an impulse $i_{(p,t)} \leq m$ (that is, $t \leq m(p)$), the flat zone Z containing p gets the grey-level t in the reconstruction, and from a flat zone Z having grey-level t in the reconstruction, a neighbouring flat zone Z' will get in the reconstruction the grey-level $\min(t, m(Z'))$.

It might be possible to devise an axiomatic for "generalized connected components", where the requirement 5 about the absence of overlap between distinct connected components would be replaced by something weaker. Here the geodesic reconstruction $\widehat{\delta_m}$ would transform an impulse under m into the "connected component" of m containing it. The corresponding symmetry for the closing $\varphi = \widehat{\delta_m}$ would be:

$$\forall s, t \in S, \qquad s \leq \varphi(t) \implies t \wedge \varphi(s) \in S \text{ and } s \leq \varphi\big(t \wedge \varphi(s)\big).$$

Such a geodesic reconstruction does not isolate flat zones. From a given flat zone, neighbouring flat zones with a lower grey-level are added in the geodesic reconstruction, until the whole space is covered.

Agnus [1, 2], under the terminology of "object-oriented morphological operators", introduced (somewhat informally) the idea of considering grey-levels as mere labels associated to pixels, without any numerical ordering between them, except with the least and greatest grey-levels $\perp$ and $\top$: $\perp \leq t \leq \top$ for every $t \in T$ (so that T is a complete lattice). Under this framework he defined "object-oriented" anti-extensive erosions and the geodesic reconstruction by a neighbourhood dilation. The first author is working with Agnus on morphological operators in the function lattice $L = T^E$ with this ordering (which is not distributive for $|T| \geq 5$), and it seems that our theory can be applied in this framework: there is a connection made of flat zones (cylinders of connected base with constant grey-level on it), and for a function $m : E \to T$ such that $m(p) \neq \top$ for every $p \in E$, its connected components are its flat zones, which can be obtained by geodesic reconstruction from markers. This work will be the subject of an incoming publication.

5. Conclusion

We have given a new theory of geodesic operations on a complete lattice, in particular we studied extensively geodesic reconstruction systems and the associated openings, and then the generation of connectivity from such a geodesic reconstruction when the latter uses symmmetrical dilations. We found that several known cases of connections on sets or numerical functions arise from a geodesic reconstruction system obtained by iterating a geodesic dilation. We gave also a practical example of a new connection on sets constructed in that way, that can be used to detect alignments in maps. Finally, through geodesic dilations, we could define several connections on colour images.

This work is an illustration of the use of recursion in theoretical computer science. Indeed, geodesic reconstruction is by nature a recursive operation defined from a dilation, and the fact that dilation distributes the supremum operation allows us to transform it into a countable iteration (cfr. Lemma 2.1); for finite images, it becomes a finite iteration. In [11] we had also highlighted the relation between the design of idempotent filters in mathematical morphology, and the theory of abstract interpretation of programming, where both use Tarski's fixpoint theorem and its generalizations.

Practical algorithms could be devised for the implementation of the construction of connected components through such iterated geodesic dilations, for example using hierarchical queues.

Our main problem has been the clarification of the notion of a symmetrical operator, which is straightforward and unique for sets ($x \in \psi(\{y\}) \iff y \in \psi(\{x\})$ for all points x, y). We saw that for general complete lattices, in particular the one of numerical functions, at least two notions of symmetry must be considered: the weak and strong ones. In the case of a translation-invariant dilation for numerical functions, these two symmetries can be expressed in terms of a form of symmetry of the structuring function f: for weak symmetry, f must have symmetrical support, and strong symmetry requires furthermore that $f(p) = +\infty$ on that symmetrical support.

In the case of numerical functions, there is a third form of symmetry that has been used for constructing annular openings [13]: here the symmetry of the structuring function f means that f has a symmetrical support and satisfies $f(p) + f(-p) \geq 0$ on that support. We called it here *annular symmetry*. In Subsection 4.2 we have also considered the particular case where f has a symmetrical support and satisfies $f(p) = 0$ on that support; the corresponding "flat symmetrical" dilation leads through geodesic reconstruction to a connected filter (in terms of flat zones), but not to a connection on the lattice of numerical functions. The axioms satisfied by such "flat" geodesic reconstructions from arbitrary markers are worth investigating; they are weaker than those for a connection.

As can be seen from the study of annular filters on complete lattices [14], this third notion of annular symmetry can take two slightly different forms in a general complete lattice (the two forms are equivalent for numerical functions). This hints that the two forms of symmetry considered here (weak and strong) could admit some variants in a general lattice (and the same problem would arise for the generalization of flat symmetry to the lattice-theoretical framework). This question will be dealt with in future papers.

The mathematical theory of connectivity on abstract pictorial objects is a difficult research topic. From the set-theoretical axioms in [16], further studied in [8, 10, 12], the corresponding concepts and axioms for lattices were derived [17]. This has led [4, 5, 18, 19] to some theoretical developments, together with examples of practical applications for some lattices, in particular the one of numerical functions (grey-level images), and the one of partitions (image segmentations). As can be seen in this paper (and also [5]), many results on geodesic reconstruction require the lattice to be ISD; this requirement is met by the ones of sets (binary images), numerical or multivalued functions (grey-level or colour images). It would be interesting to see what can be obtained in a lattice which is not ISD, like the one of closed sets (which is distributive, but not ISD), or the one of convex sets and the one of partitions (which are both not distributive).

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