EQUICONTINUOUS RANDOM FUNCTIONS

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The paper proposes a class of Random Functions which turns out to model the multidimensional scenes (microscopy, macroscopy, video sequences, etc.) in a particularly adequate way. In the triplet (Ω, σ, P) that defines a random function, the σ-algebra σ, here, is that introduced by G. Matheron in his theory of the upper (or lower) semi-continuous functions from a topological space E into $\overline{\mathbb{R}}$. On the other hand, the set Ω of the mathematical objects is the class L_φ of the equicontinuous functions of a given modulus, φ say, that map E, supposed to be metric, into $\overline{\mathbb{R}}$ or $\left(\overline{\mathbb{R}}\right)^n$. For a comprehensive member of metrics on $\overline{\mathbb{R}}$, class L_φ is a compact subset of the u.s.c. functions $E \to \overline{\mathbb{R}}$, on which the topology reduces to that of the pointwise convergence. In addition, class L_φ is closed under the usual dilations, erosions and morphological filters, as well as for convolutions g such that $\int |g\,(dx)| = 1$. Examples of the soundness of the model are given.

1 Introduction and reminders

1.1 Introduction

In what follows, the purpose is to construct a class of random functions that correctly model the functions involved in the visual world. These functions, ambiguously called "images", may be scalar or multivalued (e.g. grey or color images), may concern projections or sections (e.g. SEM versus optical microscopy), may involve motion or not, etc.

Such images are always processed by means of filtering, supremum and infimum operations, rescaling, convolution, edge correction, sum and difference, and so on. Therefore, when a mathematical status is proposed to model them, it has to fit sufficiently with all the mappings. For example, is the model closed under supremum, under convolution? Are the most common mappings continuous for the topology of the model? Is the difference of two random functions a random function? etc.

After having reminded and criticized the classical random model used in Mathematical Morphology, namely that of the upper semi-continuous functions, we will remind the basic concepts of φ-continuity. The second part will then be devoted to the algebraic properties of φ-continuity, and the third part to the topological ones. Finally, the stochastic version of such classes (part four) will become self-evident.

1.2 Reminder on the upper-semi-continuous random functions

In 1969, G. Matheron extended his random sets theory to the upper semi-continuous functions from an L.C.D space E into $\overline{\mathbb{R}}$ by considering them, via their subgraphs, as closed sets in $E \otimes \overline{\mathbb{R}}$. In this approach, the familly $\mathcal{C}$ of those sets $C \in \mathcal{F}\left(E \otimes \overline{\mathbb{R}}\right)$ that satisfy the two conditions

i/ $C \supset E_{-\infty}$

ii/ $\forall x \in E, \forall t \in \overline{\mathbb{R}}, (x,t) \in C \Rightarrow \{x\} \otimes [-\infty, t] \subset C$

is identified with the class F of the u.s.c. functions $E \rightarrow \overline{\mathbb{R}}$, and it is proved to be a *compact family* in $\mathcal{F}\left(E \otimes \overline{\mathbb{R}}\right)$. The topology on F is obtained as the restriction to $\mathcal{C}$ of the topology on $\mathcal{F}(E \otimes \overline{\mathbb{R}})$. Consequently, the open sets in F are generated by the parts of F whose elements f satisfy the two conditions:

$$\overline{X}_f\left(G\right) = \sup\left\{f(x), x \in G\right\} > b \quad \text{and} \quad \inf\left\{\overline{X}_f\left(G\right), G \supset K\right\} < a \qquad (1)$$

as G spans the open sets of E, and K its compact sets $(a, b \in \overline{\mathbb{R}})$. This results in the following criterion of convergence [theorem 3.2.1 in [4]]

Proposition 1 *A sequence f_n converges towards f in F if and only if it satisfies the two following conditions: 1/ for all $x \in E$, there exists a sequence $x_n \rightarrow x$ in E such that the sequence $f_n(x_n) \rightarrow f(x)$ in $\overline{\mathbb{R}}$. 2/ If a sequence x_{n_k} converges towards x in E, then the sequence $f_{n_k}(x_{n_k})$ satisfies $\overline{\lim} f_{n_k}(x_{n_k}) \leq f(x)$.*

The next step consists in equipping F with the σ-algebra generated by its topology, i.e. by the events $\overline{X}_f\left(G\right)$ introduced in rel.(1). Finally a random u.s.c. function f is defined by providing the measurable space (F, σ) with a probability P. The compactness of set F ensures that there actually exist probabilities on σ.

Just as a random variable is characterized by its distribution function, a random function $f \in (F, \sigma, P)$ is completely determined by the joint probabilities

$$\Pr\left\{\sup\left\{f\left(x\right), x \in B_1\right\} < \lambda_1 \quad ; \ldots \quad \sup\{f\left(x\right), x \in B_n\} < \lambda_n\right\} \qquad (2)$$

for every finite sequence $B_1...B_n$ of compact sets in E and of real values $\lambda_1...\lambda_n$. Formula (2) expresses a general theorem on random sets due to G. Choquet [1] and G. Matheron [5], which is interpreted here for random functions [8].

Discussion

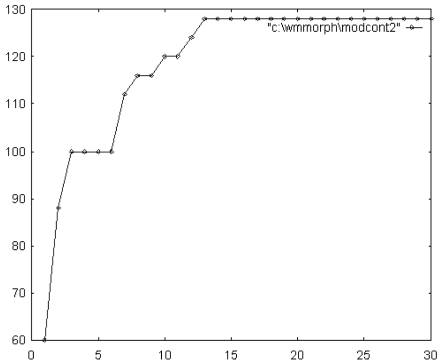
Prior to the u.s.c. model we have just sketched, random functions used to be described by means of their spatial laws, an approach that could not enable to construct models exhibiting discontinuities (vertical cliffs,...) or to calculate notions such as an average number of maxima. The u.s.c. random functions have opened the way to such possibilities. The reader will find a substantial panel of models and of applications in various works performed at the Centre de Morphologie Mathématique [see in particular the studies in this field due to D. Jeulin [3], J. Serra [10], F. Prêteux and M. Schmitt [7]].

These functions always involve some dissymmetry between foreground and background (typically: suprema of compact cylinders, or pulses of different heights located at Poisson points). Moreover in image analysis, it is rarely the raw image which is to be modelled, but rather some filtered version, or a residual, i.e. the difference between initial image and filtered one. Since the upper, or lower, semi-continuous mappings from F into itself are measurable, the Minkowski additions $\delta(f) = f \oplus K$ w.r. to a compact set K, which are continuous, and the Minkowski subtractions $\varepsilon(f) = f \ominus K$, which are upper semi-continuous, define random functions, as well as their products.

However, the limits of such an extension are rapidly reached. Class F is not closed under difference, and does not allow to model the residuals. Also, in lattice F, the infinum $\bigwedge f_i$ is identical to numerical inf, whereas the supremum $\bigvee f_i$ is the *topological closure* of the numerical sup. This is the reason why Minkowski addition, but not subtraction, is continuous. Now one cannot design an experiment able to bring to the fore such a distinction. In practice, one passes from a dilation to an erosion of function f by replacing it by $-f$, or by $m-f$, and negation is continuous. Is a continuity that no experiment will never distinguish from semi-continuity a worthwhile property of the model? Finally, F is not a vector space, and this is a pity, for numerous techniques in image analysis are of barycentric type (e.g. convolution). But is it possible to construct a function lattice, sufficiently regular and which should accept some linear operations?

These three critics holds on space F, as a deterministic structure and not on the σ-algebra. Therefore in the following, we will attempt to overcome them by modifying F (in fact it will be a restriction), without breaking into the probabilization method.

a)



b)

Figure 1: *a) Original image (left) b) its modulus of continuity (right)*

1.3 Reminder on φ-continuity

A presentation of φ-continuity may be found in [2]. We remind here some of its major features. With any numerical function f from a metric space E into $\overline{\mathbb{R}}$ one can always associate a second function, say φ, from $\mathbb{R}_+$ into $\overline{\mathbb{R}_+}$, as follows:

$$\varphi(h) = \sup\{|f(x) - f(y)|, x, y \in E, d(x, y) \leq h\}. \tag{3}$$

Equation (3) is absolutely general, and results in an increasing positive function. Now, if in addition f is supposed to be *uniformly continuous*, then, equivalently, we have

$$Lim_{h \to 0}\ \varphi(h) = 0$$

i.e. φ is continuous near the origin. In this case, φ is called *modulus of continuity* and describes, in some sense, the degree of regularity of the function f under study. This approach can be compared with the use of the variogram, or the covariance, in the order two analysis. But, unlike the variogram, which provides the *quadratic mean* of the variation between two points, here we take into account the *maximum* absolute difference. For example, by adding to f a point impulse of maximum intensity, we do not change the variogram (in the Euclidean case) whereas the modulus is transformed into its horizontal asymptote, and becomes a Heaviside function.

This said, in practice modulus φ turns out to be a meaningful descriptor. One can see from Fig. 1a and 1b how the range of φ reflects the sizes of the features in f, and its asymptote their dynamics.

Also, it is noticeable that Eq. (3) applies to both continuous or discrete spaces. Actually, it will allow to bridge the gap between the two cases. Therefore, E will denote an arbitrary metric space of distance d ($\mathbb{R}^n, \mathbb{Z}^n$, their cond. versions, etc.)

The modulus of continuity turns out to be a tool for classifying the functions, and for grouping them into families that exhibit nice common properties. Here, the notion of equicontinuity w.r. to modulus φ, or more briefly, of φ-continuity, is used as a starting point.

Definition 1 *A function $f : E \to \overline{\mathbb{R}}$ is φ-continuous when, given a modulus of continuity φ we have for all $x : y \in E$,*

$$|f(x) - f(y)| \leq \varphi\left[d(x,y)\right]$$

Similarly, the family of all functions $f : E \to \overline{\mathbb{R}}$ which satisfy the above inequality, for a given modulus φ, is called φ-continuous, and denoted by L_φ.

Hence, the members of the φ-continuous classes are uniformly continuous functions, and

$$\varphi_1 \leq \varphi_2 \Rightarrow L_{\varphi_1} \subseteq L_{\varphi_2}.$$

In particular, the constant functions are the only elements common to all φ-continuous classes.

2 Lattices of equi-continuous functions

Given a lattice L, a subset $L' \subset L$ is said to be a *sub-lattice* of L when L' is closed under the supremum and the infinum of L, and admits the same two extrema as L. Then all concepts or mappings defined for L, and which involve uniquely sup and inf, have a meaning over L'. We will see now that such a nice property is satisfied by the equi-continuous families.

Theorem 2 *For every modulus of continuity φ, the class L_φ of the φ-continuous mappings from E into $\overline{\mathbb{R}}$ is a sublattice of $\overline{\mathbb{R}}^E$.*

Proof. Let $\{f_i, i \in I\}$ be a family in L_φ ; put $f = \bigvee f_i$. If $f(x) = +\infty$ for some $x \in E$, then $f = +\infty$, hence is φ-continuous. If, $f(x) < +\infty$, then $f(y)$, which is bounded by

$$\varphi\left[d\left(x,y\right)\right] + f\left(x\right)$$

is finite, and we can write, for all $x, y \in E$:

$$f\left(y\right) - f\left(x\right) \leq \varphi\left[d\left(x,y\right)\right] \quad \text{and as well} \quad f\left(x\right) - f\left(y\right) \leq \varphi\left[d\left(x,y\right)\right].$$

A similar result may also be obtained for $\bigwedge f_i$, which achieves the proof. Q.E.D.

Comments

1/ Theorem 2 is a classical one [see for ex. a proof in [2] for the Lipschitz case]. However, the property it states owes more to the compacity of $\overline{\mathbb{R}}$ than to its complete ordering, and this point is less known. As a counter example, take $0 < a < b < 1$ and the lattice $L = \{x : x \in \mathbb{R}, -\infty \leq x < a \quad \text{or} \quad b \leq x \leq 1\}$. Consider the family of Lipschitz functions $f_i :]-\infty, a] \rightarrow L$ defined by $f_i(x) = x - \varepsilon_i$ with $\varepsilon_i \downarrow 0$. For $x = a$ and $y = a - \alpha, (\alpha > 0)$, we have $d(x, y) = \alpha$ but

$$\bigvee f_i(x) = b, \bigvee f_i(y) = a - x \quad \text{hence} \quad \left| \bigvee f_i(x) - \bigvee f_i(y) \right| = b - a + \alpha > \alpha!$$

2/ The theorem refers to a double status of $\overline{\mathbb{R}}$ which is compact (for the order topology) and also equipped with the Euclidean distance. Since according to the situation to be described several distances may be chosen, it should be wise to distinguish explicitely between the topology and an additional distance.

2.1 CCO-lattices and robustness

These comments suggest to resort to the notion of *compact with closed order lattices* (in short CCO-lattices), introduced by G. Matheron [6]. Given a lattice T, provide it with a topology that

i/ makes T compact

ii/ closes the ordering of T, i.e. if two families x_i and y_i $i \in I$ are filtered by a same base of filters $\mathcal{B}$ and satisfy $x_i \leq y_i$, $x_i \rightarrow x$, $y_i \rightarrow y$ in T, then $x \leq y$.

Such a lattice T is said to be CCO. Independently of its topology, introduce an ecart d_T on T, such that $x_i \rightarrow x$ in T implies $d_T(x_i, x) \rightarrow 0$. This ecart will be said to be *robust* for T, when for all pair $\{a_i\}$ and $\{b_i\}$, $i \in I$ of elements of T, the two inequalities

$$d\{\bigvee a_i, \bigvee b_i\} \leq \sup\{d(a_i, b_i)\}$$
$$d\{\bigwedge a_i, \bigwedge b_i\} \leq \sup\{d(a_i, b_i)\}$$

are true [11]. The two notions of compacity with closed ordering and of robustness yield a generalization of the above theorem 2.

Lemma 3 *Let $\{T_i, i \in I\}$ be a family of lattices, each T_i being robust for the ecart d_i. Then the product T of the T_i is a lattice robust for the ecart $d(x, y) = \sup\{d_i(x_i, y_i), i \in I\}$ [12].*

This result may be interpreted in terms of multi-valued lattice, as well as function lattices. In the first case, the family I labels several colors, for example, whereas in the second one I turns out to be a space E, all the T_i are identical to a T_0 and $T = (T_0)^E$. In the latter case, x and y are functions from E into

T_0 and the ecart on T is that of the uniform convergence topology. Finally, by combining the two interpretations we see that the proposition still applies to multivalued functions.

Theorem 4 *Let T be a lattice provided with a robust ecart d_T , and E be a metric space. Then the family L_φ of the φ-continuous functions $f : E \to T$ is a sub-lattice of T^E , and is robust for the ecart of the uniform convergence topology associated with d_T.*

[proof given in [12]].

This generalization of theorem 2 can be completed by the following criterion, which provides a comprehensive class of robust ecarts [12]:

Proposition 5 *Let T be a totally ordered CCO-lattice, and d_T be an ecart on T such that $x_i \to x$ in T implies $d_T(x_i \to x) \to 0$. If*

$$a \leq x \leq y \leq b \quad in \quad T \quad \Rightarrow d_T(x, y) \leq d_T(a, b)$$

then T is robust for the ecart d_T.

Proposition 5 assures the robustness of a comprehensive class of ecarts in the case of the totally ordering, and the above lemma 3, extends such a robustness to the multivalued cases.

2.2 Dilations on $\overline{\mathbb{R}}^E$ and structuring functions

In any lattice, the two basic families of operations are those which preserve either the sup, or the inf. The former are called "dilations", the latter "erosions". Consider for example the dilations that act on the lattice $\overline{\mathbb{R}}^E$ of the numerical functions from E into $\overline{\mathbb{R}}$. L is *sup-generated* by the pulse functions $u_{x,t},\ x \in E, t \in \overline{\mathbb{R}}$:

$$u_{z,t}(y) = t \quad \text{when} \quad y = z \quad \text{and} \quad u_{z,t} = -\infty \quad \text{otherwise.}$$

Indeed, every function $f \in \mathbb{R}^E$ admits a decomposition

$$f = \bigvee \{u_{z,t} \quad , \quad z \in E \quad , \quad t \leq f(z)\}.$$

Let $\delta : \mathbb{R}^E \to \mathbb{R}^E$ be a dilation on $\mathbb{R}^E$. The transforms $\delta(u_{z,t})$ of the pulses are in turn sup-generators in the space image $\delta\left(\mathbb{R}^E\right)$ since

$$\delta f = \bigvee \{\delta(u_{z,t}) \quad , \quad z \in E \quad , \quad t \leq f(z)\} \qquad f \in \mathbb{R}^E$$

It is more convenient, here, to introduce the *structuring functions* $g_{y,t}$, by taking the reciprocal of the $\delta(u_{z,t})'$s, namely

$$g_{y,t}(z) = \delta(u_{z,t}(y)) \qquad y, z \in E$$

In the following, we shall focus on the dilations whose structuring functions commute with translation on $\overline{\mathbb{R}}$, i.e. such that

$$g_{z,t} = g_{z,0} + t \qquad (\text{with } g_z = g_{z,0})$$

In such cases, the general expression reduces, and gives, at point $y \in E$:

$$(\delta f)(y) = \bigvee \{g_y(z) + f(z) \quad , \quad z \in E\} \tag{4}$$

All the dilations encountered in practice are particular cases of the last representation (4).

2.3 Dilations on L_φ

We now focus on the sub lattice L_φ of the φ-continuous functions on $\overline{\mathbb{R}}$, and we wonder about the image $\delta(L_\varphi)$ of L_φ under a dilation of the type Eq. (4). Pertinent results are obtained when the variation of the g_x's over space E is provided with a certain regularity, that we will formalize when space E is metric.

Proposition 6 *Let $\mathcal{G}$ be a family of numerical functions over a metric space E,*

i/ which admit a common finite upper bound

ii/ whose cross sections

$$X_t(g) = \{y : g(y) \geq t\} \qquad g \in \mathcal{G}$$

are compact, for all $t \in \overline{\mathbb{R}} \setminus \{-\infty\}$.

If g_ρ stands for the dilate of g by a circular cylinder of radius ρ and height $k\rho$, i.e.

$$g_\rho(z) = \sup\{g(y), \quad y \in B_\rho(z)\} \; + \; k\rho$$

Then the quantity

$$h(g, g') = \inf\left\{\rho : g \leq g'_\rho \quad , \quad g' \leq g_\rho\right\} \qquad g, g' \in \mathcal{G}$$

is a Hausdorff type distance on $\mathcal{G}$.

[easy proof].

Consider now a structuring family $\{g_x, \quad x \in E\}$, which is supposed to satisfy the two conditions of proposition 6, and whose variation over E is governed by a modulus of continuity φ', i.e.

$$h(g_x, g_y) \leq \varphi'[d(x, y)] \qquad x, y \in E \tag{5}$$

we may state:

Theorem 7 *Let E be a metric space, and $\delta : \overline{\mathbb{R}}^E \to \overline{\mathbb{R}}^E$ be a dilation on the lattice $\overline{\mathbb{R}}^E$, whose structuring functions $\{g_x, x \in E\}$ admit a modulus of continuity φ' (i.e. satisfy 5). Then δ maps the sub-lattice L_φ of the φ-continuous functions in the sub-lattice $L_{(\varphi+k)\circ\varphi'}$ of the $(\varphi + k) \circ \varphi'$-continuous functions.*

Proof. Let $f \in L_\varphi$. Put $h\,(g_x, g_y) = h$. At point y, we have:

$$(\delta f)\,(y) = \sup\{f(z) + g_y(z), z \in E\}.$$

But $g_y(z) \leq \sup\{g_x(u), u \in B_h(z)\} \quad + kh$ (Hausdorff distance) and $f(z) \leq f(u) + \varphi(d(z,u))$. Hence, we have

$$\begin{array}{rcl} (\delta f)(y) & \leq & \sup\{f(u) + g_x(u) + \varphi\,(d\,(z,u)) \quad ; \quad z \in B_h(u), u \in E\} + \;kh \\ & \leq & \sup\{f(u) + g_x(u), +u \in E\} + \varphi\,(h) + kh = (\delta f)(x) + \varphi(h) + kh \end{array}$$

and the similar inequality, by interverting x and y. Finally:

$$|(\delta f)(y) - (\delta f)(x)| \leq \varphi(h) + k(h) \leq (\varphi + k) \circ \varphi'(d(x,y))$$

Q.E.D.

Particular cases:

1. Suppose E to be affine and take for g_x the translate by vector x of the structuring function g associated with the origin. Then $h(g_x\,, g_y) = d(x,y)$ and $(\varphi + k) \circ \varphi' = \varphi$. The dilations that are translation invariant preserve all equicontinuous lattices L_φ.

2. Take for g_x a *flat* structuring function, of compact support K_x, i.e.

$$\begin{array}{rclcl} g_x\,(y) & = & 0 & \text{when} & y \in K_x \\ g_x\,(y) & = & -\infty & \text{when} & \text{not} \end{array}$$

Then the expression 4 of a dilation reduces to

$$(\delta f)\,(y) = \bigvee\{f\,(z)\,, z \in K_y\} \tag{6}$$

where the geometrical role of the (variable) structuring elements $\{K_y, y \in E\}$ appears clearly. The dilations of the type 6, which are said to be flat, exhibit a number of remarkable features. Among others:

i/ They map into itself each quasi sub-lattice of φ-continuous functions: $f : E \to [a,b]$, with $a, b \in \overline{\mathbb{R}}, a < b$. In digital terms, this means that the transform of an image with n grey levels comprises at most n grey levels: there is no overflow.

ii/ They commute with the anamorphoses on $\overline{\mathbb{R}}$ (e.g. the dilate of the log of an image equals the log of the dilate). Again, such a property is important in practice, since it makes the space operations independent of the grey tone calibrations.

For a flat dilation δ of structuring elements $\{K_x, x \in E\}$, with

$$h(K_x, K_y) \leq \varphi'[d(x,y)],$$

where h is the set-oriented Hausdorff distance, the theorem proves that any φ-continuous function is transformed into a $\varphi \circ \varphi'$-continuous one. In particular, when $\varphi' \leq$ Identity, δ maps every L_φ into itself. This case occurs for example when E is affine and $K_x = K_0 + x, x \in E$ (translation invariance), or also when $K_x = \{K_0 + x\} \cap Z$ where Z is a rectangular window.

Theorem 7, which has been stated for dilations admits by duality a similar version for erosions, and of course extends to any inf of dilations which have the same modulus φ'. Another instructive feature concerns the structuring functions, for which no continuity is required. For example, the two conditions of proposition 6 may be satisfied by upper semi continuous functions.

3 Topologies on the $\mathbf{L}_\varphi$ lattices

In G. Matheron's theory of the compact lattices [6], the continuity for $\bigvee$ and $\bigwedge$ operators is pointed out as one of the major goals to achieve. It is a matter of ensuring the continuity of the mapping $F \to \bigvee F$ (or $F \to \bigwedge F$) from $\mathcal{F}(T)$ into T when T is a topological lattice. Here F stands for a closed family of elements of T and $\bigvee F$ (resp $\bigwedge F$) for their supremum (resp their infimum). In the usual CCO lattices, when the mapping $F \to \bigvee F$ is continuous, then $F \to \bigwedge F$ is u.s.c. only (e.g. closed sets of $\mathbb{R}^n$, u.s.c. functions from $\mathbb{R}^n$ into $\overline{\mathbb{R}}$). The double continuity is thus an exceptionally strong property. Therefore the following criterion (No 6.1. in [6]) turns out to be a corner stone for the whole theory:

Proposition 8 *(From Matheron) An algebraic lattice T admits a necessarily unique CCO topology such that* $\bigvee$ and $\bigwedge$ *are continuous if and only if for all x and all y in T, with $y \not\leq x$, one can find two elements x' and y' which satisfy the three conditions*

$$x \not\in M_{y'} \quad ; \quad y \not\in M^{x'} \quad ; \quad M^{x'} \cup M_{y'} = T$$

$$\textit{where} \quad M_{y'} = \{z : z \in T, \quad z \leq y'\} \quad \textit{and} \quad M^{x'}\{z : z \in T, \quad z \geq x'\}$$

Remarkably, proposition 8 does not demand any topological prerequisite. It will not allow us to derive the $\bigvee$ and $\bigwedge$ continuities for the L_φ lattices when

$T = \overline{\mathbb{R}}$, and next $T = \left(\overline{\mathbb{R}}\right)^n$, n a finite positive integer.

Theorem 9 *Let E be a metric space, φ be a modulus of continuity and L_φ the lattice of the φ-continuous functions from E into $\overline{\mathbb{R}}$. Then the unique topology that makes L_φ CCO, with continuous $\bigvee$ and $\bigwedge$ is the topology of the pointwise convergence.*

Proof. Consider two distinct functions f and g of L_φ. There exists at least one point $x \in E$ and a real member a with (for example) the strict inequalities

$$g_x < a < f(x)$$

Introduce the two following elements f_0 and g_0 of L_φ:

$$f_0(y) = a - \varphi\left[d\left(x,y\right)\right] \qquad g_0\left(y\right) = a + \varphi\left[d\left(x,y\right)\right] \qquad \forall y \in E \tag{7}$$

Function f does not belong to the lower bounds of g_0 , since $f(x) > a$, i.e. $f \not\in M^{g_0}$. Similarly, we have $g \not\in M_{f_0}$. Moreover, any function $s \in L_\varphi$ is either $\leq g_0$ (when $s(x) \leq a$) or $\geq f_0$ (when $s(x) \geq a$), so we can write $M^{g_0} \cup M_{f_0} = L_\varphi$. Therefore proposition 8 applies, and lattice L_φ is CCO with continuous $\bigvee$ and $\bigwedge$ for a certain topology. One can find out this topology by means of a general characterization [6], but in the present case, it suffices to observe that L_φ is a compact sub-lattice of the upper semi continuous functions, sub-lattice on which both topologies of Matheron and of the pointwise convergence coincide. Now, the $\bigvee$ is continuous for the first one, hence also in the pointwise sense. Similartly, L_φ is a compact sublattice of the lower semi continuous function, hence $\bigwedge$ is continuous in the pointwise sense, which achieves the proof. Q.E.D.

This result generalizes to φ-continuous functions $E \rightarrow \overline{\mathbb{R}}$ a theorem already established by G. Matheron [theorem 6.5 in [6]]. The extension may be pursued further. First, space $\overline{\mathbb{R}}$ may be replaced by any compact segment $S \subset \overline{\mathbb{R}}$. Clearly, the φ-continuous functions from E into S form a compact quasi sub-lattice of L_φ ("quasi" because the extreme elements are not preserved). The proof may be reproduced integrally for them. Also, $\overline{\mathbb{Z}}$ may be substituted for $\overline{\mathbb{R}}$ and any subset of $\overline{\mathbb{Z}}$ for S. Second, the theorem extends to product lattices.

Corollary 10 *Theorem 9 remains true when $\overline{\mathbb{R}}$ is replaced by any product $T = \Pi\left\{T_j, \quad j \in J\right\}$ of closed subsets T_j of $\overline{\mathbb{R}}$ or of $\overline{\mathbb{Z}}$.*

Proof. As previously, consider two distinct functions f and g of T, i.e. $f = \{f_j, \quad j \in J\}$ and $g = \{g_j, \quad j \in J\}$. There exists at least one label $v \in J$ such that $f_v \neq g_v$, with $f_v\left(x\right) > g_v\left(x\right)$, strictly, for a point $x \in E$. Lattice T_v enters the framework of theorem 9, which determines two distinct functions f_{v_0} and g_{v_0}

from equations 7. Let then f_0 be the function $E \rightarrow T$ whose label v is equal to f_{v_0}, and whose all other components $f_j, \quad j \in J, j \neq v$ coincide with the inf in the corresponding lattice T_j. Similarly, define g_0 to be the function equal to g_{v_0} for $j = v$ and equal to the sup in T_j for all $j \not= v$. The criterion of proposition 8 is still satisfied for f_0 and g_0, which results in the corollary. Q.E.D.

The consequences of theorem 9, and of its corollary, on dilation, and more generally on increasing mappings are considerable. For the sake of pedagogy, we will treat the "flat" case only, which the most used in applications.

Proposition 11 *Let E be a metric space (distance d), $K : E \rightarrow \mathcal{K}(E)\backslash\emptyset$ be a structuring element such that*

$$h\left[K(x), K(y)\right] \leq \varphi'\left[d(x,y)\right]$$

(h, Hausdorff distance) for some modulus φ', and let $\delta : E \rightarrow \overline{\mathbb{R}}$ be the dilation of structuring element K. Then, for each modulus φ, the mapping $\delta : L_{\varphi \rightarrow} L_{\varphi \circ \varphi'}$ is continuous.

Proof. Given an arbitrary point $x \in E$, consider a family f_n in L_φ with $f_n \rightarrow f$ for the pointwise convergence. We draw from theorem 9 that

$$\bigvee \left\{ f_x\left(y\right), y \in K_n\left(x\right) \right\} \rightarrow \bigvee \left\{ f\left(y\right), y \in K\left(x\right) \right\}.$$

Since point x is arbitrary in E the pointwise convergence of $\delta(f_n)$ results, hence the continuity of δ. Q.E.D.

Corollary 12 *The class generated by finite sup, inf and composition product of dilations and erosions whose structuring elements admit a modulus of continuity is composed of* continuous *increasing operators. When all the moduli of the structuring elements are anti-extensive, then these increasing operators each map L_φ into itself.*

[Easy proof].

Despite the assumption of finiteness (which could be overcome by supplementary hypotheses of compactness for the K's), this corollary ensures the continuity for a comprehensive number of operators in Mathematical Morphology, and among others for the morphological filters (openings, closings, their products and the alternating sequential filters). It shows, a contrario, that semi-continuity arises from *rapid variations* of the structuring elements, but not from the substitution $\bigvee \rightarrow \bigwedge$.

We conclude this section by brief comments about linear operators on the L_φ. Concerning convolution, one easy prove the following

Proposition 13 *Let $g(dh)$ be a measure such that $\int_E |g(dh)| \leq 1$. Then the convolution by g maps each L_φ into itself and is continuous.*

Consequently, all the half residuals of the operations (i.e. the difference between a function and its transform) described by corollary 12 map each L_φ into itself and are continuous (e.g. the top hat mappings). An approach with variable kernels $g(dh)$ could be developed in a way similar to what we did for dilations. It should lead to similar results.

4 Random φ-continuous functions

4.1 Definition

We are now in a position to provide a random status to the φ-continuous classes, which will conclude this paper. Given a modulus of continuity φ, the lattice L_φ is compact as a closed subset of the (compact) set F of the upper semi-continuous functions from E into $\overline{\mathbb{R}}$.The events (1) that generate the σ-algebra σ on F admit a similar meaning in L_φ, and the compactness of L_φ ensures that there do exist probabilities on the σ-algebra of the measurable space (L_φ, σ). Moreover, we draw from proposition 11 that the dilations (and the erosions) involved in theorem 7, as well as their finite sup, inf, and compositions preserve φ-continuous random functions, with possible changes of moduli φ.

The random functions which will be obtained from (L_φ,σ) will result in relatively regular realizations. For example, a Lipschitz Boolean function will accept sharp valleys, but without strict verticalities.

How does the φ-continuity affect the characteristic functional Eq.(5)? To answer this question, we need to establish first the following lemma :

Lemma 14 *: A function $f : E \to \overline{\mathbb{R}}$, E a metric space, is $\varphi-$continuous if and only if for all $x \in E, t \in \mathbb{R}$ and $h > 0$, we have*

$$f(x) < t \qquad \text{and} \qquad y \in B_{\theta(h)} \Longrightarrow \qquad f(y) < t + h \tag{8}$$

where function θ designates the largest inverse of modulus φ, i.e.

$$\theta(h) = \sup \{d : \varphi(d) \leq h\} \qquad h, d \in \mathbb{R}_+$$

and where $B_{\theta(h)}(x)$ is the closed ball centered at x and of radius $\theta(h)$.

Proof. Suppose f to be φ-continuous, and $f(x) < t$ for some $x \in E$ and some $t \in \mathbb{R}$. Fix the value h. If point $y \in B_{\theta(h)}(x)$, then $d(x, y) = \theta(h)$,i.e. $\varphi[d(x, y)] \leq h$. Since f is φ-continuous, we have

$$f(y) \leq f(x) + \varphi[d(x, y)] < t + h$$

Conversely, suppose that implication (8) is true for all $x \in E, t \in \mathbb{R}$ and $h > 0$. If there is no pair (x,t) such that $f(x) < t$, then $f = +\infty$, hence is φ-continuous. If not, $f(x)$ admits finite upper bounds t and rel.(8) implies, for any point y at distance $\theta(h)$ from x apart, that

$$f(y) < h + t \qquad \Longrightarrow \qquad f(y) \leq h + \bigwedge \{t, t > f(x)\} = h + f(x)$$

and finally $|f(y) - f(x)| \leq h = \varphi[d(x,y)]$ .Now point y may, in turn, play the role of starting point for an arbitrary point z, since $f(y) < \infty$, and this achieves the proof. ■

In more geometrical terms, the lemma says that when a point (x,t) is strictly above the subgraph of the $\varphi-$continuous function f, in the product space $E \times \overline{\mathbb{R}}$, then the whole cone of summit (x,t), of generator $\varphi(d)$ and oriented upwards, is strictly above the subgraph of f.

In terms of Random Sets, the property depicted by Lemma 14 has a meaning of a *condition* : when we know that compact set K misses the subgraph of f, then all the sections of the cone generated from K, miss it too. This result can be stated as follows :

Proposition 15 *Let f be a random u.s.c. function from a metric space E into $\overline{\mathbb{R}}$. Function f is almost surely $\varphi-$continuous if and only if there exists a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ continuous at the origin, and such that we have for all $K \in \mathcal{K}, t \in \mathbb{R}$ and $h \geq 0$:*

$$\Pr\left\{\sup\left\{f(x), x \in \delta_{\theta(h)}(K)\right\} < t + h / \sup\left\{f(x), x \in K\right\} < t\right\} = 1 \tag{9}$$

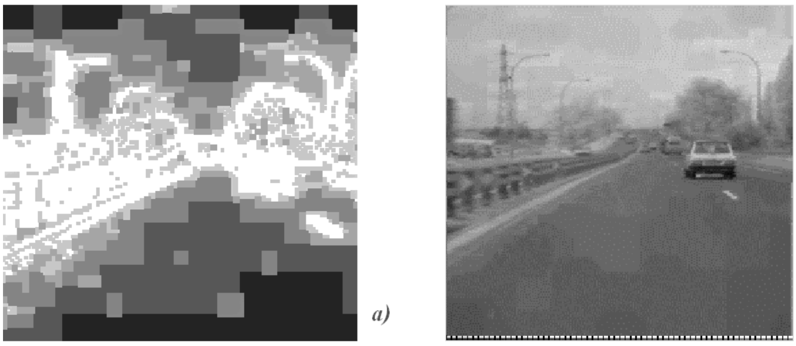
where $\theta(h) = \sup\{d : \varphi(d) \leq h\}$.

Proof. Observe that implication (8) extends to compact sets K, since the dilate of set $\delta_{\theta(h)}(K)$ of set K by the closed ball $B_{\theta(h)}$ is the union of the balls $B_{\theta(h)}(x)$, as x spans K. Hence we have

$$\sup\{f(x), x \in K\} < t \qquad \Longrightarrow \qquad \sup\{f(x), x \in \delta_{\theta(h)}(K)\} < t + h \tag{10}$$

Therefore the event of the left member of Eq. (9) is almost sure, which yields Eq. (9). Conversely, the datum of Eq. (9) means almost surely implication (10), hence the a.s. φ-continuity of f. ■

The random functions which will be obtained from (L_φ, σ) will result in relatively regular realizations. For example, a Lipschitz Boolean function will accept sharp valleys, but without strict verticalities.



a) b)

Figure 2: *a) Sampling zones derived from the inverse modulus r ; b) Sampled image obtained from pattern a).*

4.2 Application to sampling

In image processing, the sampling problems are often set in terms of subsampling. We will treat here the following one:

Starting from the datum of a whole image f, what is the minimal number of values of f to be kept in order to estimate it everywhere with a given accuracy, and where must we implant the sampling points?

Consider f as a realization of a φ-continuous random function, and introduce a *local version* of the modulus φ by associating, with each point x, the maximum variation of f over the closed ball $B_x(r)$ of radius r and centered at point x:

$$\varphi_x(r) = E\left[\max\left\{(\delta_h f - f)(x)\ ; (f - \varepsilon_h f)(x)\right\}\right] \tag{11}$$

where δ_h and ε_h are the dilation and the erosion by ball B_h.

Second, consider the larger inverse $r_x(\varphi)$ of $\varphi_x(r)$. The value $r_x(\varphi)$ is the size of the maximum ball centered at x such that the variation, in the sense of Eq.(11) is $\leq \varphi$. Set accuracy φ to a fixed value, φ_0 say; hence $r_x(\varphi_0) = r(x)$ becomes a numerical function of x only. The sampling protocol is then achieved by a downstream approach which allocate a sample density inverse to function r.

For a numerical illustration, start from a digital image of $2^i \times 2^i$ pixels (Fig. 2), with $i = 8$. The largest possible grid $G(i)$ has a spacing $2^i = 256$, and four points at the four corners of the image. The gray scale ranges over 256 levels, and the accuracy φ_0 is fixed to be equal to 10 levels. The cross section

$$X(i) = \left\{x : r(x) \leq 2^i\right\}$$

of r, corresponds to the flatest zones of the image under study. So they are sampled with the largest grid, i.e. reduced to the set

$$Y(i) = X(i) \cap G(i)$$

The points of $Y(i)$ admit a certain zone of influence $k(i)$, such that the dilate $\delta_{k(i)}[Y(i)]$ indicates the portion of the space "known" from sampling $Y(i)$. Iterate, by putting

$$\begin{array}{rcl} X(i-1) & = & \left\{x : h(x) \leq 2^{i-1}\right\} \quad \backslash \quad \delta_{k(i)}[Y(i)] \\ Y(i-1) & = & X(i-1) \cap G(i-1) \qquad\qquad . \end{array}$$

Function $k(i)$ is calculated to be $\leq 2^i$ and to make contiguous the zones of influence, as i varies. For square grids, for example, one can take:

$$k(i) + k(i-1) = 2^i - 1.$$

These conditions lead to a pixel reduction by four in the example of fig. 2. In terms of data compression, such a result is acceptable, but not outstanding. However, by extending the samples in their respective zones of influence, one generates the new image f^*shown in Fig.2b, so that for all treatments ψ designed by corollary 12 (anti-extensive case), we still have

$$E\left[\max\left|(\psi f)(x) - (\psi f^*)(x)\right|\right] \leq \varphi_0$$

which is not a trivial result.

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