Hausdorff distances and interpolations

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Abstract

The space $\mathcal{K}'(E)$ of the non empty compact sets of a metric space E is itself metric for the so called Hausdorff distance. It is shown in this paper that Hausdorff metric admits at least two different geodesic, with provide interpolators between pairs of sets of $\mathcal{K}'(E)$. When E is affine, the space quotient of $\mathcal{K}'(E)$ under translation admits in turns reduced geodesics, that generate better set interpolators. Finally, by introducing a Hausdorff distance by erosion, one can provide a symmetric form to interpolators, but are no longer geodesics.

1 Introduction

This paper belongs to a series of three texts (ref. [1][2][3]) by S. Beucher, F. Meyer, and J. Serra respectively, originally in French and written in 1994. They have resulted in a patent [4], consequently we could not publish them for a certain time. The paper by Meyer was the subject of a communication in ISMM'96 in Atlanta [5], and the one by Beucher is presented in ISMM'98 [6]. These three related works, which develop three facets of a same idea, transcribe in a written form a number of fruitful discussions at the CMM. This paper is more upstream than the two other ones, and, unlike [5], uses the term "geodesic" in its mathematical sense, i.e. "segment in a metric space".

Before getting to the heart of the matter, we will remind briefly the classical Hausdorff distance, and introduce a variant of it (by erosions).

Let E be a metric space, of distance d, and let $\mathcal{K}'$ be the class of the non empty compact sets of E. Put

$$d(x,Y) = \inf\{d(x,y), y \in Y\} \qquad x \in E\,;\, Y \in \mathcal{K}'$$

and introduce the mapping $\mathcal{K}' \times \mathcal{K}' \to \mathbb{R}_+$

$$\rho(X,Y) = \max\left\{\sup_{x\in X} d(x,Y)\ ;\ \sup_{y\in Y} d(y,X)\right\} \tag{1}$$

Mapping ρ turns out to be a distance, called "Hausdorff distance", which holds on $\mathcal{K}'$ and no longer on E. Rel. (1) can be equivalently written by means of the dilations by the balls of space E. If $B_\lambda(x)$ stands for the compact ball of centre x and of radius λ, the dilation of $X \in \mathcal{K}'$ by λ has an expression

$$\delta_\lambda(X) = \cup \{B_\lambda(x)\ ,\ x \in X\} = \cup \{y : B_\lambda(y) \cap X \neq \emptyset\}\,. \tag{2}$$

Then, rel.(1) takes the following form, for all $X, Y \in \mathcal{K}'$:

$$\rho(X,Y) = \inf\{\lambda : X \subseteq \delta_\lambda(Y)\ ;\ Y \subseteq \delta_\lambda(X)\}\,. \tag{3}$$

This last expression, more geometrical than rel. (1) shows in particular a semi-sensitivity to noise: a small lacuna in set X_i or Y_i slightly changes ρ, whereas an isolated point far away from X and added to it modifies ρ drastically. Hausdorff distance satisfies a few algebraic properties, associated (of course) with union and dilation. Moreover it extends easily to grey tone and colour functions by means of their subgraphs [7][8]. Therefore all derived geodesics and interpolators apply also to functions (see S. Beucher's paper [6] in the present issue).

Consider now the subclass $\mathcal{A}$ of $\mathcal{K}'(E)$, of the regular compact sets, i.e.

$$X \in \mathcal{K}' \qquad \text{and} \qquad \overline{\overset{\circ}{X}} = X\ .$$

Then the non negative number

$$\sigma(X,Y) = \inf\{\lambda : \varepsilon_\lambda(X) \subseteq Y \quad ; \quad \varepsilon_\lambda(Y) \subseteq X\}$$

where ε_λ is the erosion adjoint to dilation δ_λ, is a distance on $\mathcal{A} \times \mathcal{A}$. Indeed, if $\sigma(X,Y) = 0$, then

$$Y \supseteq \underset{\lambda>0}{\cup} \varepsilon_\lambda(X) = \overset{\circ}{X} \quad \Rightarrow \quad Y \supseteq \overline{\overset{\circ}{X}} \supseteq X \qquad X, Y \in \mathcal{A}$$

and similarly $X \supseteq Y$, hence they are equal (the other axioms are proved as for distance ρ). We shall call σ the *Hausdorff distance by erosions.*

2 First Hausdorff geodesic

If it exists, a geodesic between X and Y, for the Hausdorff distance by dilations, will be a shortest segment from X to Y in space $\mathcal{K}'$, i.e. a family $\{Z_\alpha, 0 \leq \alpha \leq 1\}$ of non empty compact sets that interpolate X and Y. For $\alpha = 0$ we should obtain $Z_\alpha = X$, for $\alpha = 1, Z_\alpha = Y$, and as α increases, Z_α should progressively leave X and go to Y.

But does such a geodesic exist ? In many metric spaces, there are no geodesics, in some other ones, several ones, or even an infinity. What about our current case ? To answer the question, we need a small lemma.

Lemma 1 *Given S and X in $\mathcal{K}'$, if $\delta_\rho(S) \supseteq X$ for a $\rho \geq 0$, then $\delta_\rho(\delta_\rho(X) \cap S) \supseteq X$*

Proof. By hypothesis, every point $x \in X$ belongs to $\delta_\rho(S)$, i.e. there exists a point $s \in S$ such that $x \in \delta_\rho(s)$ or, equivalently, such that $s \in \delta_\rho(x)$, hence $s \in \cup\{\delta_\rho(x), x \in X\} \cap S$. Therefore every point $x \in X$ is covered by the ρ-dilate of an element of $\delta_\rho(X) \cap S$. Q.E.D.

The lemma allows to construct a first class of geodesics, as follows:

Theorem 1 *First geodesics: let E be a metric space of distance d, of compact balls δ_λ, and whose $\mathcal{K}'(E)$ denote the class of the non empty compacts sets. Then every pair X, Y in $\mathcal{K}'(E)$ from ρ apart, (for Hausdorff distance by dilations) admits the following geodesic:*

$$\left\{Z_\alpha = \delta_{\alpha\rho}(X) \cap \delta_{(1-\alpha)\rho}(Y) \quad , \quad \alpha \in [0,1] \quad \right\} \tag{4}$$

Proof : Fix α and put $S = \delta_{(1-\alpha)\rho}(Y)$. Since $\delta_{\alpha\rho}\left[\delta_{(1-\alpha)\rho}\right] = \delta_\rho$, we draw from rel. (3) that $\delta_{\alpha\rho}(S) = \delta_\rho(Y) \supseteq X$. By applying lemma 1 we have

$$\delta_{\alpha\rho}(Z_\alpha) = \delta_{\alpha\rho}\left(\delta_{\alpha\rho}(X) \cap \delta_{(1-\alpha)\rho}(Y)\right) \supseteq X \quad .$$

On the other hand, the definition of Z_α implies $\delta_{\alpha\rho}(X) \supseteq Z_\alpha$, hence

$$\rho(X, Z_\alpha) \leq \alpha\rho$$

One proves similarly that $\rho(Y, Z_\alpha) \leq (1-\alpha)\rho$. By combining these two results with the triangular inequality, we finally obtain

$$\rho \geq \rho(X, Z_\alpha) + \rho(Y, Z_\alpha) \geq \rho(X, Y) = \rho$$

which achieves the proof. Q.E.D.

Discussion: What is the practical value of algorithm (4)? When we apply it to two disjoint sets X and Y, it often swells the intermediate Z_α's. Typically, the midway set between the two *fine* and *horizontal* segments of the x-axis is a *thick* and *vertical* lens, which inflates more and more as the segments depart from one another.

At least, this first geodesic will have highlighted where the problems do lie. Indeed, the discussion suggests we focus

i/ on other possible geodesics, with finer interpolators,

ii/ on an approach which should treat separately the differences between the relative *positions* of X and Y, and their differences in *shape*.

iii/ on an approach which should not be restricted to extensive dilations, which always "swell", for example by involving dilations and erosions in a symmetrical manner.

The problems are set, it remains to examine how they can be solved.

3 Second Hausdorff geodesic

3.1 Convex case

The geodesic Z_α of rel. (4) does not involve dilations of X and Y by each other, but of both of them by the δ_λ's. However, if we focus on the sub-class $\mathcal{C}' \subseteq \mathcal{K}'$ of those elements of $\mathcal{K}'$ that are convex, we can easily exhibit another geodesic, based on cross dilations. In the following, αX stands for the transform of set X under the similitude of ratio α.

Proposition 1 *Let X and Y be two non empty compact convex sets in $\mathbb{R}^n$, at Hausdorff distance ρ from each other. Then the interpolator*

$$Z'_\alpha = (1-\alpha)\, X \oplus \alpha Y \tag{5}$$

is at distance $\alpha\rho$ from X and $(1-\alpha)\,\rho$ from Y, i.e. the family $\{Z'_\alpha\ ,\ \alpha \in [0,1]\}$ is a geodesic in space $\mathcal{C}'$.

Proof : For every set $X \in \mathcal{K}'$ and for any $\alpha \in [0,1]$ we have

$$(1-\alpha)\, X \oplus \alpha X \supseteq X \quad , \tag{6}$$

the equality being obtained iff X is convex [9]. Therefore, for all $X, Y \in \mathcal{K}'$ of distance ρ apart, we can write

$$Z'_\alpha \oplus \alpha\rho B = (1-\alpha)\, X \oplus \alpha\,(Y \oplus \rho B) \supseteq (1-\alpha)\, X \oplus \alpha X \supseteq X\,. \tag{7}$$

where B is the closed unit ball centered at the origin.

If, in addition, sets X and Y are convex, we have also

$$X \oplus \alpha\rho B = (1-\alpha)\, X \oplus \alpha\,(X \oplus \rho B) \supseteq (1-\alpha)\, X \oplus \alpha Y \supseteq Z'_\alpha\,. \tag{8}$$

Hence, $\rho\,(Z'_\alpha, X) \leq \alpha\rho$, and similarly $\rho\,(Z'_\alpha, Y) \leq (1-\alpha)\rho$. thus we have

$$\rho\,(Z'_\alpha, X) + \rho\,(Z'_\alpha, Y) \leq \rho = \rho\,(X, Y)$$

as well as the inverse inequality (triangular inequality for distances), which achieves the proof. Q.E.D.

In general, the second geodesic $\{Z'_\alpha, \alpha \in [0,1]\}$ does not reduce to the first one, since rel. (8), and its homolog version for Y, imply, when compared with rel. (4) that

$$Z_\alpha\,(X, Y) \supseteq Z'_\alpha\,(X, Y) \qquad \alpha \in [0,1] \quad , \quad X, Y \in \mathcal{C}' \tag{9}$$

For example, in the case of the two above segments, interpolator $Z'_{0,5}$ is itself a segment of same length as X and Y, and placed between them. We are far away from the thick lens $Z'_{0,5}$!

Another very positive feature lies in the fact that Z'_α commutes under translation on X or on Y. Indeed, the translate of X by vector h is nothing but the dilate $X \oplus \{h\}$ of X by $\{h\}$ (i.e. h considered as an element of $\mathcal{P}(\mathbb{R}^n)$). Therefore, the equality

$$Z'_\alpha (X \oplus \{h\}, Y) = (1-\alpha)(X \oplus \{h\}) \oplus \alpha Y = Z'_\alpha (X, Y) \oplus (1-\alpha)\{h\} \quad (10)$$

shows that in Z'_α the effect of the *shapes* of X and Y (which determines Z'_α up to a translation) is separated from the effect of their relative *positions* (which induce only translations on Z'_α).

Geodesics $\left\{Z'_\alpha\right\}$ admits an extension to the whole space $\mathcal{K}'$, given by the following theorem

Theorem 2 *Second geodesics: every pair X, Y in $\mathcal{K}'(E)$ from ρ apart for Hausdorff distance for dilations admits the following geodesic*

$$\left\{Z''_\alpha = \delta_{\alpha\rho}(X) \cap \delta_{(1-\alpha)\rho}(Y) \cap (1-\alpha) X \oplus \alpha Y \quad , \quad \alpha \in [0,1]\right\}$$

Proof : We will prove first that

$$x \in X \quad \Rightarrow \quad \delta_{\alpha\rho}(x) \cap \delta_{(1-\alpha)\rho}(Y) \cap Z'_\alpha \neq \emptyset$$

Given $x \in X$, there exists a $y \in \delta_\rho(Y)$ such that $x \in \delta_\rho(Y)$, i.e. that $U(\rho) = \delta_{\alpha\rho}(x) \cap \delta_{(1-\alpha)\rho}(Y) \neq \emptyset$. Therefore, one can find a value $0 < \rho_0 \leq \rho$ with $U(\rho_0)$ reduced to one point, s say. Point s belongs to $U(\rho_0)$ and to the vector $\overrightarrow{xy}$ of unit $\overrightarrow{u}$. More precisely, we have

$$\overrightarrow{s} = \overrightarrow{x} + \alpha\rho b \overrightarrow{u} = \overrightarrow{y} - (1-\alpha)\rho b \overrightarrow{u}$$

hence $\overrightarrow{s} = (1-\alpha)\overrightarrow{x} + \alpha\overrightarrow{y}$, i.e. $s \in Z'_\alpha$ and finally $s \in \delta_{\alpha\rho}(x) \cap \delta_{(1-\alpha)\rho}(Y) \cap Z'_\alpha \subseteq Z''_\alpha$. Since $x \in \delta_{\alpha\rho}(s)$,then $\delta_{\alpha\rho}(Z''_\alpha)$ covers point x, hence $\delta_{\alpha\rho}(Z''_\alpha) \supseteq X$. The brother inclusion $\delta_{\alpha\rho}(X) \supseteq Z''_\alpha$ derives from the definition of Z''_α, which achieves the proof. Q.E.D.

Fig. 1 shows an example of such a geodesic. The parasite swelling effect is partly mastered. However independence under translation is lost (it was true only for convex sets) and the implementations of the Z'_α require dilations of two arbitrary sets by each other. Perhaps could we try and follow a new lead ?

4 Reduced Hausdorff distances

Since the shape of the geodesic $\{Z_\alpha\}$ depends on the respective locations of sets X and Y, we could try and place them in the "most favourable" locations, by shifting and rotating Y (for example). The idea of studying the variation of $\rho(X, Y)$ as a function of the translates of Y was recently applied to template matching problems [9]. Here we will develop it to exhibit new metrics, and, from them, new geodesics. We restrict the displacements to translations (the rotations admit the same theoretical approach, but their implementation is less easy).

Figure 1:

In the following, E stands for a compact (but possibly large) region in $\mathbb{R}^n$ or $\mathbb{Z}^n$, equipped with a unit ball B. Denote by X_a (resp. X_b) the translate of set X (resp. Y) by vector a (resp. b), and put

$$\rho_1(X,Y) = \inf\{\rho(X_a, X_b)\ ,\ a, b \in E\} \tag{11}$$

Introduce the quotient space $\mathcal{K}_1$ of $\mathcal{K}'$ for the equivalence under translation, i.e. $X \equiv X_a\ , a \in E$. Operator ρ_1 maps $\mathcal{K}_1 \times \mathcal{K}_1$ into $\mathbb{R}_+$ since it does not change as X and Y are translated, and it is a symmetrical function of its two arguments. Clearly, given $X, Y \in \mathcal{K}'$, the mapping $(a,b) \to \rho\ (X_a, Y_b)$ from $E \times E$ into $\mathbb{R}_+$ is continuous, therefore the compacity of E implies that there exists a pair $(a,b) \in E \times E$ with

$$\rho_1(X,Y) = \rho(X_a, Y_b) \tag{12}$$

In particular, if $\rho_1(X,Y) = 0$ then $X_a = Y_b$, i.e. $X = Y$ and, of course, $X = Y$ implies $\rho_1(X,Y) = 0$.

Finally, operator ρ_1 satisfies the triangular inequality. To prove it, consider the quantity $\rho_1(X,Z) + \rho_1(Z,Y)\ ,\ X, Y, Z \in \mathcal{K}'$. Since ρ_1 is translation invariant, we can always replace Z by its translate Z_0 centred at the origin. If X_a and X_b denote the locations that minimize $\rho(X, Z_0)$ and $\rho(Y, Z_0)$ respectively, we can write

$$\rho_1(X,Z) + \rho_1(Z,Y) = \rho(X_a, Z_0) + \rho(Z_0, Y_b) \geq \rho(X_a, X_b) \geq \rho_1(X,Y)$$

Hence, ρ_1 is a distance. Moreover, its metrics admits geodesics of the first type. To show it, take a pair (X, Y_b) that minimizes ρ. Given an $\alpha \in [0,1]$, consider the first type interpolator Z_α between X and Y_b (rel. (4)). Z_α turns out to be in optimal position not only w.r. to ρ, but also w.r. to metric ρ_1, i.e. $\rho(X, Z_\alpha) = \rho_1(X, Z_\alpha)$. Indeed, we have

$$\rho = \rho_1(X,Y) = \rho(X, Y_b) \leq \rho(X, Z_\alpha) + \rho(Z_\alpha, Y_b)$$

but $\rho_1(X, Z_\alpha) \leq \alpha\rho$ and also $\rho_1(Z_\alpha, Y_b) \leq (1-\alpha)\rho$, hence

$$\rho \leq \rho_1(X, Z_\alpha) + \rho_1(Z_\alpha, Y_b) \leq \alpha\rho + (1-\alpha)\rho = \rho \; .$$

The equality results, which shows that Z_α is in optimal position for metric ρ_1. In conclusion, we can state

Theorem 3 *Let $\mathcal{K}_1$ denote the quotient space, under translation, of $\mathcal{K}'(\mathbb{R}^n)$ or $\mathcal{K}'(\mathbb{Z}^n)$. Then, the mapping $\mathcal{K}_1 \times \mathcal{K}_1 \to \mathbb{R}_+$ by rel. (11) defines a distance. Given X and Y in $\mathcal{K}'$ there exists at least one geodesic in the sense of ρ_1, which coincides with the geodesic $\{Z_\alpha\}$ in the sense of ρ, whose extremities are one of the pairs, (X_a, X_b) say, that minimizes $\rho(X, Y)$.*

From the reduction $\rho \to \rho_1$, it is always possible to go back easily. Having fixed set X and determined a translation b of Y such that $\rho(X, Y_b) = \rho_1(X, Y)$, it suffices then to take interpolator

$$Z_1(\alpha) = [(X \oplus \alpha\rho B) \cap (Y_b \oplus (1-\alpha)\rho B)]_{-\alpha b}$$

which generates in $\mathcal{K}'$ a segment from X to Y, as spans [0,1]. The proposed approach is valid in $\mathbb{R}^n$ as well as in $\mathbb{Z}^n$. In the latter case, the balls are digital polyhedra. An exact computation of the optimal vector b might be time consuming. In practice, one may content oneself with vector b^*, of which each coordinate $b_i^*, i \in [1, n]$ minimizes the 1-D Hausdorff distance between the projections of X_i of X and Y_i of Y on the corresponding axis No i. If x_i and $x_i^{'}$ (resp. y_i and $y_i^{'}$) stand for the extreme points of X_i (resp. Y_i), component b_i is

$$b_i^* = \left[\left(x_i + x_i^{'}\right) - \left(y_i + y_i^{'}\right)\right]/2$$

5 Interpolations for nested sets

This section is closer to the two related works [5] and [6], and establishes a common basis. Nice developments based on distance function have also been proposed by J.R. Casas [11] and also by P. Moreau and Ch. Ronse [12]. From now on, we particularize the pair (X, Y) under study, to be *ordered*, by taking $X \subseteq Y$. This dissymmetry suggests to play with both Hausdorff distances ρ (for dilations from X) and σ (for erosions from Y).

A point m at a distance $\leq \lambda$ from X and $\geq \lambda$ from Y^c belongs to set $(X \oplus \lambda B) \cap (Y \ominus \lambda B)$, hence to set

$$M = \cup \{(X \oplus \lambda B) \cap (Y \ominus \lambda B) \;\; , \quad \lambda \geq 0\} \; . \tag{13}$$

Conversely, every point $m \in M$ belongs to one of the terms of the union, so there exists a $\lambda \geq 0$ with $d(m, X) \leq \lambda$ and $d(m, Y^c) \geq \lambda$. In this sense interpolator M is midway between X and Y, and we shall call it the *median* of X and Y. It is easy to notice that the boundary of M is nothing but the skeleton by zone of influence, or skiz, between X and Y^c. This implies the immediate following consequences:

Proposition 2 *The median set $M(X,Y)$ is compact and comprised between X and Y. Its boundary is the locus of those points of $\mathbb{R}^n$ which are equidistant from X and Y^c, and these distances are smaller or equal to*

$$\mu = \inf\{\lambda : \lambda \geq 0\ ,\ X \oplus \lambda B \supseteq Y \ominus \lambda B\} \tag{14}$$

the equality being reached for at least one point of ∂M.

(Direct consequence of the SKIZ properties).Here is now an instructive property which shows how both Hausdorff distances are involved in the median $M(X,Y)$.

Theorem 4 *Given $X, Y \in \mathcal{K}'(\mathbb{R}^n)$, the median element $M(X,Y)$ is at Hausdorff dilation distance μ from X and $X \bullet \mu B$, and at Hausdorff erosion distance μ from Y and $Y \circ \mu B$, where μ is defined by rel. (14).*

Proof : Let ρ be the Hausdorff distance by dilation between sets M and X. We draw from prop. 6 that $X \subseteq M \subseteq X \oplus \mu B$ hence $\mu \leq \rho$. Now for all ρ strictly smaller than μ, the set $X \oplus \mu B$ cannot contain the non empty set $(X \oplus \mu B) \cap (Y \ominus \mu B)$, which is a subset of the boundary $\partial(X \oplus \mu B)$ (prop. 6 again). Hence $\mu = \rho$.

Consider now the Hausdorff distance by dilation ρ' between the closing $X \bullet \mu B$ and M. We have $X \bullet \mu B \oplus \mu B = X \oplus \mu B \supseteq M$, and also $M \oplus \mu B \supseteq X \oplus \mu B = X \bullet \mu B \oplus \mu B$, hence $\rho' \leq \mu$. Just as previously if ρ' is strictly smaller than μ, then $X \bullet \mu B \oplus \rho' B = X \oplus \rho' B$ cannot contain set M, hence $\rho' = \mu$.

The corresponding results for Hausdorff distance by erosions derive by duality. Q.E.D.

Note that in all these results, the distances between sets X and Y (by dilation or by erosion) do not intervene. Indeed they are not associated with μ, but rather with the sum of the successive median sets between X and Y. For example, if M_1 is the median between M and Y, of parameter μ, M_2 the median between M_1 and Y, etc., then $\mu + \sum \mu_i = \rho$, where ρ is the Hausdorff dilation distance between X and Y (similar result for the erosions by going from Y to X). This allows a series of progressive interpolations from X to Y [6], which are distinct from those, [5][10], which are obtained by replacing M by

$$M_\alpha = \bigcup_\lambda \{(X \oplus \alpha\lambda B) \cap (Y \ominus (1-\alpha)\lambda B)\ ,\ \alpha \in [0,1]\}\ .$$

6 Conclusion

There are various ways to define interpolators between two sets X and Y of a metric space E. If X and Y are supposed to be compact and non empty, Hausdorff distance provides one possible approach, because, as it has been proved in this paper, this distance admits geodesics, i.e. series of "best" interpolators going from X to Y.In fact, several geodesics co-exist, which are not equivalent. Therefore, some additional constraints allow more specific, hence better, interpolators. It is in particular the case wheni) X, Y are convex ;ii) E is equipped with a translation, and interpolations are introduced up to a translation ;iii) when $W \subseteq Y$.Up to now, the last case has been the most studied [1] [2] [11].

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