Connection, Image Segmentation and Filtering

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Abstract

The notion of a connection is a non metric generalisation of connectivity proposed by Serra. Its axiomatics lies on the idea that the union of connected components that intersect is still connected.

Such an approach allows a precise definition of image (or sequence) segmentation. It yields also powerful filters (levelings), and provides more classical ones with new properties (openings)

After an overview of set connection, its application to segmentation is developped, and illustrated by examples. The last sections are devoted to connected filters.

Key words : segmentation, connection, Lipschitz functions, jump and smooth connections, partitions, morphologial filters, leveling.

1 The connectivity concepts

1.1 Classical connectivity and image analysis

In mathematics, the concept of connectivity is formalized in the framework of topological spaces and is introduced in two different ways. First, a set is said to be connected when one cannot partition it into two non empty closed (or open) sets. This definition makes precise the intuitive idea that $[0,1]\cup[2,3]$ consists of two pieces, while $[0,1]$ consists of only one. But this first approach, extremely general, does not derive any advantage from the possible regularity of some spaces, such as the Euclidean ones. In such cases, the notion of *arcwise connectivity* turns out to be more convenient. According to it, a set A is connected when, for every $a,b\in A$, there exists a continuous mapping ψ from $[0,1]$ into A such that $\psi(0)=a$ and $\psi(1)=b$. Arcwise connectivity is more restrictive than the general one ; however, in $\mathbb{R}^d$, any open set which is connected in the general sense is also arcwise connected.

A basic result governs the meaning of connectivity ; namely, the union of connected sets whose intersection is not empty is still connected :

$$\{A_i \text{ connected}\} \text{ and } \{\cap A_i \neq \emptyset\} \Rightarrow \{\cup A_i \text{ connected}\} \tag{1}$$

In discrete geometry, the digital connectivities transpose the arcwise corresponding notion of the Euclidean case, by introducing some elementary arcs between neighboring pixels. This results in the classical 4- and 8-square connectivities, as well as the hexagonal one, or the cuboctahedric one in 3-D space. Is such a metric approach to connectivity adapted to image analysis ? We can argue that

a/ a suitable approach should apply to sets as as to functions;

b/ in discrete motion analysis, the trajectories of fast moving objects often appear as dotted tubes, and arcwise connections are unable to handle such situations;

c/ more deeply, one can wonder what is actually needed in image processing. As a matter of fact, when we examine the requirements for connectivity, we observe that the basic operation they involve consists, given a set A and a point $x \in A$, in extracting the particle of A at point x. For such a goal, an arcwise approach is obviously sufficent. But is it necessary?

1.2 The notion of a connection

These criticisms led J. Serra and G. Matheron to propose a new approach, in 1988 [15][7] where they take not rel.(1) as a consequence, but as a starting point. However, their definition is rather general and stated as follows.

Definition 1 *Connection: Let E be an arbitrary space. We call connected class or* connection $\mathcal{C}$ *any family in $\mathcal{P}(E)$ such that*

(i) $\emptyset \in \mathcal{C}$ and for all $x \in E$, $\{x\} \in \mathcal{C}$

(ii) for each family $\{C_i\}$ in $\mathcal{C}$, $\cap\, C_i \neq \emptyset$ implies $\cup C_i \in \mathcal{C}$.

As we can see, the topological background has been deliberately thrown out. The classical notions (e.g. connectivity based on digital or Euclidean arcs) are indeed particular cases, but the emphasis is put on another aspect, that answers the above criticism c/ in the following manner ([15], Ch. 2) :

Theorem 1 *The datum of a connection $\mathcal{C}$ on $\mathcal{P}(E)$ is equivalent to the family $\{\gamma_x\,, x \in E\}$ of openings such that*

(iii) for all $x \in E$, we have $\gamma_x(x) = \{x\}$

(iv) for all $A \subseteq E$, $x, y \in E$, $\gamma_x(A)$ and $\gamma_y(A)$ are equal or disjoint

(v) for all $A \subseteq E$, and all $x \in E$, we have $x \notin A \Rightarrow \gamma_x(A) = \emptyset$.

An alternative (and equivalent) axiomatics has been proposed by Ch. Ronse [12] ; it contains, as a particular case, another one by R.M. Haralick and L.G. Shapiro [4] ; however, both approaches are still set-oriented. The extension from sets to the general framework of complete lattices and in particular to numerical functions is due to J. Serra [16]. Historically speaking, the number of applications or of theoretical developments which was suggested (and permitted) by this theorem is considerable: It has opened a new way to an object-oriented approach for segmentation, compression and understanding of still and moving images (see a bibliogrphy in [17]).

When it is defined as follows, the notion of a partition turns out to be closely related to that of a connection

Definition 2 *Partition: Let E be an arbitrary set. A partition $\mathcal{D}$ of E is a mapping $x \to D(x)$ from E into $\mathcal{P}(E)$ such that*
(i) for all $x \in E : x \in D(x)$
(ii) for all $x, y \in E : D(x) = D(y)$ or $D_a(x) \cap D_a(y) = \emptyset$
$D(x)$ is called the class of the partition of origin x.

the first condition tells that classes $D(x)$ occupy the whole space E, the second one that two distinct classes have no common point.The next proposition shows that any connection connexion *partitions* all sets A, even when A is space E itself.

Proposition 2 *The openings $\{\gamma_x\}$ of connection $\mathcal{C}$ partition any set $A \subseteq E$ into the smallest possible number of components belonging to class $\mathcal{C}$, and this decomposition is increasing in that if $A \subseteq B$, then any connected component of A is included in a connected component of B.*

1.3 Connections on $P(E)$

Several instructive examples of connections on $\mathcal{P}(E)$ can be found in [5], in [12] and in [16]. Here we just recall a few of them, which are of interest for the present study.

i/ All arcwise connectivities on digital spaces are connections in the sense of definition 1;

ii/ In [15] ch.2, we start from a first connection $\mathcal{C}$ and consider an extensive dilation $\delta : \mathcal{P}(E) \to \mathcal{P}(E)$ that preserves $\mathcal{C}$ (*i.e.* $\delta(\mathcal{C}) \subseteq \mathcal{C}$). Then the inverse image $\mathcal{C}' = \delta^{-1}(\mathcal{C})$ of $\mathcal{C}$ under δ defines a new connection on $\mathcal{P}(E)$, which is richer. The $\mathcal{C}$ -components of $\delta(A)$, $A \in \mathcal{P}(E)$, are exactly the images $\delta(Y_i')$ of the $\mathcal{C}'$-components of A. If γ_x stands for the connected opening associated with connection $\mathcal{C}$ and ν_x for that associated with $\mathcal{C}'$, we have

$$\nu_x(A) = \gamma_x \delta(A) \cap A \quad \text{when} \quad x \in A \quad ; \quad \nu_x(A) = \emptyset \quad \text{when not} \qquad (2)$$

(similar technique applies also when δ stands for a closing, but without the statement on the connected components, and without Eq.2 [17])

In practice, the openings ν_x characterize the *clusters* of objects from a given distance d apart. Fig.1 illustrates this point by "reconnecting" dotted lines trajectories. But *a contrario*, such an approach can also provide a means to extract the objects which are isolated.They will be defined by the fact that for them $\nu_x(A) = \gamma_x(A)$, an equality which yields easy implementation [17].

iii/ Consider a *fixed* partition D and a point $x \in E$. The operation that associates

$$\gamma_x(A) = D(x) \cap A \quad \text{when} \quad x \in A \quad ; \quad \gamma_x(A) = \emptyset \quad \text{when not}$$

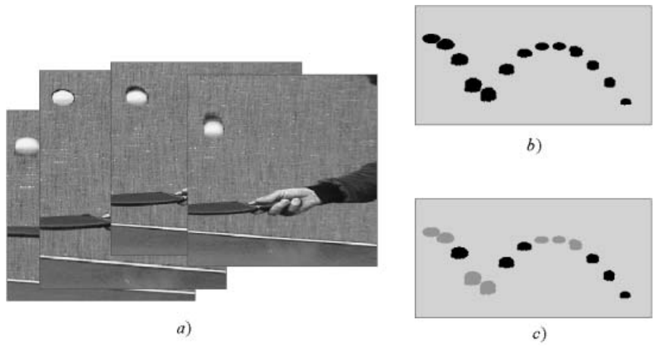


Figure 1: a) Sequence of images, b) space-time display of the ball, under a dilatation based connection, the three grey clusters are seen as three particles (they correspond to slow motions).

with any $A \subseteq E$,is clearly an opening. Moreover, as x varies, the $\gamma_x(A)$ and $\gamma_y(A)$ are identical or disjoint since they correspond to partition classes. Proposition 2 shows then that class

$$\mathcal{C} = \{\gamma_x(A), x \in E, A \in \mathcal{P}(E)\}$$

is a connection. Note that class $\mathcal{C}$ breaks the usual particles and regroups those which belong to a same class $D(x)$. When E is equipped with a previous connection $\mathcal{C}'$, such as the usual arcwise one, the the elements of $\mathcal{C} \bigwedge \mathcal{C}'$ are the connected components, in the sense of $\mathcal{C}$' of the intersections $A \cap D(x)$.

2 Connection and Segmentation

In image processing, an image or a sequence is said to be segmented when the area where it is defined has been partitioned into homogeneous zones in accordance with a given criterion. For instance, if this image is represented by a digital function $f : E \to \overline{R}$ where E is a set equipped with a connection, f *is segmented into flat and connected zones* when a partition D of E is created, such that for any $x \in E$, the class $D(x)$ is the largest connected component of E including point x and on which function f is constant and equal to $f(x)$.

All criteria do not lend themselves to such nice partitionings. Suppose, for instance, that we wish to partition E into various zones, connected or not, where function f is Lipschitz with parameter k equals 1. Three disjoint zones A, B, and C may very well be found, such that the criterion is satisfied on $A \cup B$ and on $A \cup C$, but not on $B \cup C$. In this case, there is no *largest* zone containing the points of A and where the criterion be satisfied. The criterion does not yield a segmentation.

In other words, the partitions refered to with a segmentation concept are *maximal* ones, i.e. they lead to largest classes. Besides, we can always construct a smaller partition, namely the one that reduces space $\mathcal{P}(E)$ on all its singletons $\{x\}$.

These remarks make us give a more formal expression, therefore more precise, to the two concepts of criterion and segmentation.

Definition 3 *Criterion: Let $f \in \mathcal{F}$ be a function from a set E into a complete lattice T. A criterion $\sigma : \mathcal{F} \otimes \mathcal{P}(E) \to [0,1]$ is a binary function, that decreases on $\mathcal{P}(E)$, with $\sigma[f(\{x\})] = 1$ for any $x \in E$. Denoting by $f(A)$ the set $\{f(x), x \in A\}$, we have:*
$\sigma[f(A)] = 1$ *(criterion satisfied on A)*
$\sigma[f(A)] = 0$ *(criterion refuted on A)*

The condition $\sigma[f(\{x\})] = 1$ guarantees that whatever criterion is considered, there is always at least one way to partition E into zones (the singletons) that satisfy it. Often, there is much more than one, and even in the case of the image segmentation, there is a larger partition. The decreasing condition simply means that if the criterion is satisfied on A, it is *a fortiori* satisfied on any $B \subseteq A$.

For instance, the criterion which is satisfied if A is reduced to one point or if

$$x \in A, \quad t_0 \leq f(x) \leq t_1$$

with t_0 and t_1 fixed, defines the *threshold criterion.*

Likewise, when space E is metric with distance d, the implication

$$x, y \in A \Rightarrow |f(x) - f(y)| \leq kd(x, y),$$

yields the *k-Lipschitz criterion.*

Definition 4 *Segmentation: Given a function f and a criterion σ, let $\{D_i\}$ be the non empty family of the partitions of E into homogeneous zones of f according to σ. Criterion σ is said to* segment *the functions when, for any function $f \in \mathcal{F}$, family $\{D_i\}$ is closed under supremum. Then the supremum partition $\curlyvee D_i$ defines the* segmentation of f according to σ.

This maximal partition is missing in the above k-Lipschitz criterion.

2.1 Connective criteria

What are the conditions that a criterion σ must satisfy to be a segmentation tool? The need for maximal partitions orients us towards a connection based approach, via the concept of a *connective criterion.*

Definition 5 *A criterion $\sigma : \mathcal{F} \otimes \mathcal{P}(E) \to [0,1]$ is* connective *when, for any family $\{A_i\}$ into $\mathcal{P}(E)$, we have*

$$\cap A_i \neq \emptyset \quad et \ \wedge \sigma[f(A_i)] = 1 \Rightarrow \sigma[f(\cup A_i)] = 1 \tag{3}$$

In other words, when a connective criterion σ is satisfied by a function f on a family $\{A_i\}$ of regions of the space, and if all these regions have one common point, then it is also satisfied on the union $\cup A_i$.In addition, we have

Proposition 3 *The infimum of any family $\{\sigma_j, j \in J\}$ of connective criteria is itself a connective criterion.*

Indeed, the connective criteria have the structure of a complete lattice. The following theorem, which ridges the gap between connection and segmentation is the corner stone of the theory (see the proof in [18])

Theorem 4 *Let $\mathcal{F}$ be the family of functions $f : E \to T$, where T is a complete lattice. A criterion σ segments the functions $f \in \mathcal{F}$ if and only if it is connective.*

Remark that, although space E was not a priori equipped with any connection at the beginning, the connective criterion supplies one to it. Now, if E is already equipped with some connection $\mathcal{C}'$, the intersection $\mathcal{C} \cap \mathcal{C}'$, which is a connection, generates the maximal partition whose classes satisfy both constraints. For instance, the criterion "function f is constant on A" leads to the partition of E by threshold of f. If, in addition, we demand that each class A be $\mathcal{C}'$-connected, then we find the segmentation of f into flat and connected zones, as previously described.

Remark also that the theorem does not impose any condition to lattice T. In the applications below T is totally ordered, but this is not an obligation, and the theorem applies to multi-spectral images or to any other type of lattice as well.

3 Examples of Segmentations by Connections

The five segmentations by connections that we now describe differ in various respects: smooth connection involves Euclidean dilations, quasi-flat zones connection requires geodesic ones, and watershed is no longer based on increasing operators. Note also that neither the Lipschitz connection nor the jump one involve paths.

3.1 Smooth connection

Space E is now a metric one, of distance d, and lattice T is the extended line. A function f is a local Lipschitz on set A when, for all $x, y \in A$ we have

$$d(x,y) \leq a \Rightarrow |f(x) - f(y)| \leq kd(x,y) \tag{4}$$

The connective criterion (4) induces the so-called *Lipschitz connection of range a and slope k.* In the Euclidian case, this relation (4) means that on the classes of the segmentation f is equals to both is erosion and its dilation by the "pencil" $H(k, a)$ (cylinder, covered with a cone) of slope k and radius a

$$x \in A \Leftrightarrow f(x) = (f \ominus H)(x) = (f \oplus H)(x). \tag{5}$$

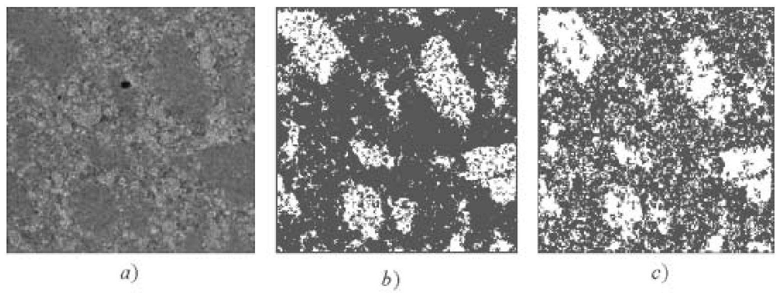


Figure 2: a) rock electronical micrography; b) and c) smooth connections with parameters 7 and 6 of image a)

This last relation provides also a digital algorithm to perform the segmentation. Remark that the expression of the criterion on A requires to know f on the dilate of A by the ball of radius a (and non only on A itself). As the range a decreases, with a fixed slope k, the maximal partitions $D_{a,k}(f)$ increase. This suggests to focus on partition supremum $D_k = \Upsilon\{\ D_{a,k}\ ,\ a > 0\}$. It results in the so-called *smooth connection,* which still corresponds to a connective criterion. In each class A of D_k, function f is ω-continuous along all paths included in the interior $\overset{\circ}{A}$ of A. Therefore, the *digital* smooth connection has a unit size. To implement it, it suffices to erode functions f and $-f$ by the cone $H(k,a)$ whose base is the unit square or hexagon, and whose height is k, the origin being placed at the top, and then to take the intersection of the two sets where f equals its erosion and its dilation respectively.

The smooth connection turns out to be a good segmentation tool to separate the smooth zones from the more granular ones with a similar grey level, as it often appears in electronical microscopy [17]. Figure 2 depicts two segmentations of a concrete micrography, carried out thanks to smooth connections.

3.2 Quasi- flat zones

Instead of demanding that f be ω-continuous along *all* paths included in A, we can also only require the ω-continuity for *at least one* path. This more comprehensive new criterion is still connective, and leads to the connection according to the "*quasi-flat*" zones, due to F. Meyer [10]. This time, the digital implementation involves geodesic reconstructions.

3.3 Watershed lines

Finally, let us mention one of the the oldest pathwise connection, namely the watershed coontours. The criterion "all A points are flooded from the same minimum" being connective, the watersheds partition the definition area E into arcs connected catchement bassins, plus into a set of point connected components.

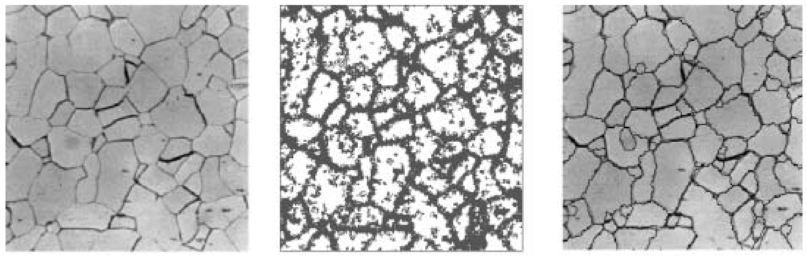


Figure 3: (a) original micrograph of alumina, (b) jump connection from the maxima , with h=15, (c) derived SKIZ.

The latter form the watershed contours.

3.4 Jump connection

The previous connection is noticeably improved when the variations origins are set on the minima or maxima of function f. Suppose that a connection $\mathcal{C}$ has been defined on E. Fix the range k, and let m be a minimum of function f. Consider the connected component $A(m)$ that contains the abcissa of minimum m, and such that

$$A(m) = \{z : z \in E, \quad 0 < f(z) - m \leq k\}.$$

Let $D_{k,m}$ the partition composed of $A(m)$ plus point classes on $E\backslash A(m)$.Take the supremum D_k of all partitions $D_{k,m}$ associated with all minima of function f. Then iterate the process on the set $E\backslash \cup A(m)$, i.e. extract the zones above each minimum m such that $k < f(z) - m \leq 2k$; iterate again withe $2k < f(z) - m \leq 3k$, etc... The segmentation according to this criterion, obviously connective, leads to the *jump connection.* The alternative process, from the maxima, is built by duality, and both may be combined in a symmetrical way, where the ascent stops when it crosses a descent, and where the zones straddling an ascent and a descent are equally sub-divided [16] [17]. The example presented in Fig.3 illustrates the use of such a transformation. Fig.3a depicts the optical micrograph of a polished section of alumina grains. The partition of the space under jump connection is depicted in Fig.3b, whereas Fig.3c shows the superposition of the skeleton by influence zones of the set Fig.3b on the original image.

In practice the jump connection turns out to be one of the best techniques to segment images, thanks to the quality of the segmentations it creates (few

point zones, visually significant classes), and to its fast computation. Note that as the slope k varies geometrically ($k = 1, 2, 4, ..$) the partitions that segment f increase. Also the plot of the areas of the non-point classes versus range k is very informative. Fig.4 illustrates this point.A connection may be

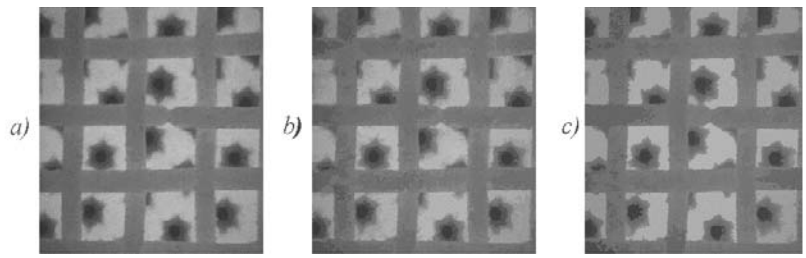


Figure 4: a) Initial image : gas burner; b) jump of range 12 : 783 zones; c) jump of range 24 : 63 zones.

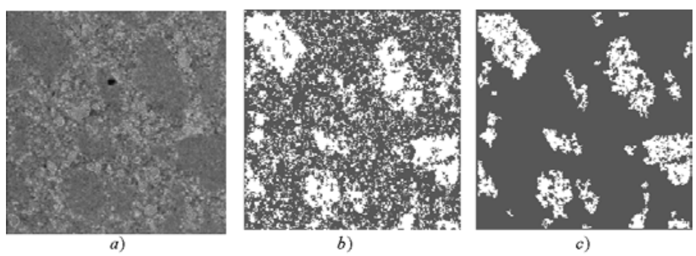


Figure 5: a) electron micrograph of a rock ; b) jump connection of range 12 ; c) intersection of jump connection (range 12) and smooth connection (range 6).

used to strengthen another one. Thus, the smooth connections of Fig.2 are improved when the infimum is considered with a jump connection with the suitable parameter, as depicted in Fig.5.

4 Set Connected Filters

The way the connective algorithms work, by region clustering, suggests to use connections for filtering images, i.e. for simplifying them, removing some ppossible noise, etc, while preserving the visual quality of the contours. We shall reach this goal by building *connected morphological filters*, and in particular *openings and closings by reconstruction*, and then *levelings*.

For now on E is an arbitrary set, and $\mathcal{P}(E)$ is supposed to be equipped with connection $\mathcal{C}$. For every set $A \in \mathcal{P}(E)$, the two families of the connected components of A (the "grains") and of A^c (the "pores") partition space E. Then, an operation $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ is said to be *connected* when the partition associated with $\psi(A)$ is coarser that that of A [13]. Clearly, taking the complement of a set, or removing some grains, or filling pores generate connected operators. The major class of mappings we have in view is that of the (connected or not) morphological filters. Let us briefly recall it

- A mapping ψ is said to be a *morphological filter* on $\mathcal{P}(E)$ when it is increasing and idempotent:

$$\begin{array}{ll} A, B \subseteq E, \;\; A \subseteq B \Rightarrow \psi(A) \subseteq \psi(B) & \qquad \textit{increasingness} \\ \psi(\psi(A)) = \psi(A) & \qquad \textit{idempotence} \end{array}$$

- In particular, a filter that is extensive (resp. anti-extensive) is called a *closing* (resp. an *opening*) :

$$\begin{array}{llllll} \gamma & \text{an opening} & : & \gamma = \text{a filter and} & \gamma(A) \subseteq A & , \quad A \subseteq E \\ \varphi & \text{a closing} & : & \varphi = \text{a filter and} & \varphi(A) \supseteq A & , \quad A \subseteq E \end{array}$$

4.1 Set opening by reconstruction and some derivatives

A comprehensive class of connected filters derives from the classical *opening by reconstruction*. Its definition appears in [15], ch.7.8. Significant studies which use this notion may be found in literature, such as [13] (connected operators),[2](stable operators), [5](grain operators).

An opening by reconstruction is obtained by starting from an increasing binary criterion τ (e.g. "*the area of A is $\geq$ 10*"), to which one associates the trivial opening

$$\begin{array}{ll} \gamma^{\tau}(A) = A & \text{when } A \text{ satisfies the criterion} \\ \gamma^{\tau}(A) = \emptyset & \qquad\qquad \text{when not} \end{array}$$

The corresponding opening by reconstruction γ is then generated by applying the criterion to all grains of A, independently of one another, and by taking the union of the results :

$$\gamma(A) = \cup\{\gamma^{\tau}\gamma_x(A), \quad x \in E\}$$

The closing by reconstruction φ (for the same criterion) is the dual of γ for the complement, *i.e.* if $\complement$ stands for the complement operator, then

$$\varphi = \complement\gamma\complement\,.$$

For example, in R^2, if we take for criterion τ, " *have an area* ≥ 10", then $\gamma(A)$ is given by the union of grains of A whose areas are ≥ 10, and $\varphi(A)$ is the union of A and all its pores whose areas are ≤ 10. Similarly, if criterion τ is expressed by "hit a fixed marker M", then $\gamma(A)$ is the union of the grains that hit A, wheras $\varphi(A)$ is composed of A and of all pores that miss M .

4.2 Set Levelings

Levelings have been introduced by F. Meyer, in [10], as gray tone connected operators on digital spaces, for the usual digital arcwise connections based on neighbor pixels in square or hexagonal grids. In [9], G. Matheron proposes a generalization to an arbitrary space (hence, without *a priori* connection). Here, connection arrives as a final result, and is generated by an extensive dilation.

Now in both cases, levelings turn out to be *flat* operators, *i.e.* that treat each grey level independently of the others. This circumstance suggests to try and generalize F. Meyer's approach by focusing on *set* levelings, but re-interpreted in the framework of an arbitrary connection $\mathcal{C}$. J. Serra entered this way of thinking [17], which allowed him to obtain theorem 11

The central notion of adjacency [17], which governs the structure of the levelings below, is defined as follows

Definition 6 *Let $\mathcal{C}$ be a connection on $\mathcal{P}(E)$, and let $X, Y \in \mathcal{C}$. Sets X and Y are said to be adjacent when $X \cup Y$ is connected, whereas X and Y are disjoint.*

Two sets A and M of $\mathcal{P}(E)$ are said to be adjacent when they are disjoints, and when one can find one connected component in A a,d another in M whose union is connected. When two sets interset each other or are adjacent, we say that they *touch each other*, which is denoted by $A \parallel M$ ("A touches M").

Consider then the opening by reconstruction obtained by all grains of A that touch a given marker M

$$\gamma_M(A) = \cup \{\gamma_x(A)\, ,\, x \in E\, ,\, \gamma_x(A) \parallel M\}$$

The dual closing $\varphi_M(A)$ is the complement of the unions of those pores of A that touch M. In the following, it is the closing $\varphi_{M^C}(A)$ wich is used, i.e. the union of A and of its pores that miss M^c. We have the following theorem [17]

Theorem 5 *Given a marker $M \subseteq E$ the two operations γ_M and φ_{N^c} commute. If we call* leveling *their product λ_M, we have*

$$\lambda = \gamma_M\ \varphi_{N^c} = \varphi_{N^c}\gamma_M\ .$$

Several properties derive from this key result. Firstly, the leveling λ_M is a morphological filter (increasing and idempotent) on $\mathcal{P}(E)$. Moreover its exceptional robustness is expressed by the relationship

$$A, B \subseteq E,\ \lambda_M(A) \cap A \subseteq \lambda_M(B) \subseteq \lambda_M(A) \cup A \quad \Rightarrow \quad \lambda_M(A) = \lambda_M(B)$$

Small perturbations do not modify the "stong" filter λ_M.

Make now variable the parameter M, and consider the two operands mapping $(A, M) \to \lambda(A, M)$ from $\mathcal{P}(E) \times \mathcal{P}(E)$ into $\mathcal{P}(E)$. The leveling λ turns out to be an increasing and *self-dual* operation, i. e.

$$\lambda(A^c, M^c) = [\lambda(A, M)]^c$$

Finally, one can make fixed the set A under study, and leave variable marker M,and denote by λ_A the resulting mapping from $\mathcal{P}(E)$ into itself . IThe relevant formalism to go further is that of the activity ordering for sets (and no longer for set mappings)[9]. As a matter of fact, any fixed set A generates an ordering denoted by $\preceq_A$, from the two relationships

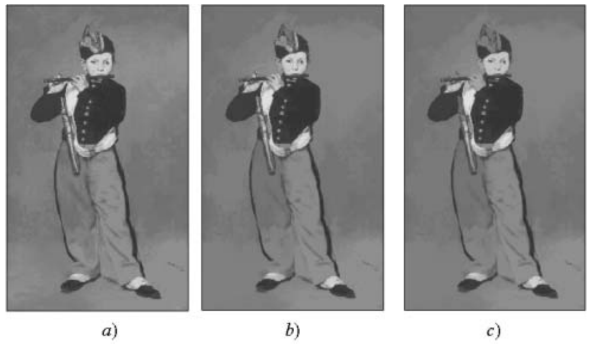


Figure 6: a) Fifre player, b) (resp.c) leveling having for marker the extrema of dynamics >50 (resp. >80).

$$M_1, M_2 \subseteq E \qquad \begin{array}{c} M_1 \cap A \supseteq M_2 \cap A \\ \\ M_1 \cap A^c \subseteq M_2 \cap A^c \end{array} \qquad \Leftrightarrow \quad M_1 \preceq_A M_2$$

With respect to this activity ordering, le levelings λ_A forme a *granulometrical semi-group of openings* [17], i.e.

$$M_1 \preceq_A M_2 \Rightarrow \lambda_{\lambda A(M_1)}\,(M_2) = \lambda_{\lambda(M_2)}\,(M1) = \lambda_A\,(M_2)$$

This last granulometric type pyramid is specially usefull in practice, for it allows to grade the activity effects of markers: it means that we can directly implement a highly active marker, or, equivalently, reach it by intermediary steps. An example is given in fig.6.

5 Extension to Functions

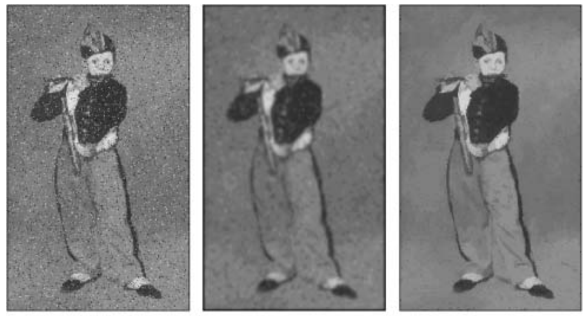
The extension to numerical functions of the previous set results is straightforward, and has been succesfully used in number of studies such as [6][14][19][11] Denote by T a totally ordered lattice such that $[0,1]$; $[0,\infty]$; $\overline{\mathbf{R}}$ or $\overline{\mathbf{Z}}$o, and by T^E lthe lattice of all numerical functions $f : E \longrightarrow T$. An increasing operator Ψ on T^E is said to be *flat* if there exists an increasing set operator ψ such that

$$X\,[\Psi(f), t] = \psi\,[X(f,t)] \tag{6}$$

where $X(f,t)$ stands for the thresholding of function f at level t, i.e. :

$$X(f,t) = \{x : x \in E, \quad f(x) \geq t\} \tag{7}$$

In the discrete cases of digital imagery, relation (6) is sufficient to characterize the function operator Ψ associated with an increasing set operator ψ. In other words, as soon as a set operation ψ is increasing, it suffices to replace, in



a) *b)* *c)*

Figure 7: a) initial image plus Poisson noise; b) convolution of a) by a disc of radius 5; c) leveling of a) by marker b) (the noise is removed, but the contours are those of the initial image).

its algorithm, the set oriented $\cup$ and $\cap$ by *sup* and inf respectively to obtain the gray yone version. The rule applies for set levelings, since they are increasing. Their numerical versions still satisfy the three basic properties cstated above, namely proposition 3, the self duality, and the granulometric semi-group.

In practice, the role of the marker is crucial. In fig.5, the marker is obtained by replacing f by zero out of the maxima and minima of f with a dynamics $\geqslant k$, and by leaving f unchanged on these extrema. The maxima of f with a dynamics $\geqslant k$ are obtained from the opening by reconstruction $\gamma_{\text{rec}}(f)$ of f by marker $f-k$, where k is a positive constant. Then these maxima are located at those points x where $f(x)-\gamma_{\text{rec}}(f)(x)=k$ (the similar minima are obtained by duality). The corresponding levelings are shown in fig.5a and 5b, for markers g_{30} and g_{60}, of dynamics 50 and 80 respectively (over 256 gray levels).

These two markers are self-dual by construction, and satisfy the condition of activity increasingness of theorem 16. Their progressive leveling action appears clearly when confronting fig.5a and 5b. Notice the relatively correct preservation of some fine details such as buttons, eyes, eyebrows, fingers, etc.. These details are preserved because of their high dynamics.

In figure 6, the leveling is used for noise reduction, from a marker obtained by Gaussian moving average of size 5, namely fig.6b, of the initial noisy image fig.6a. It results in fig.6c where the noise reduction of fig.6b is preserved, but where the initial sharpness of the edges is recovered.

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