Chapter VI

Set Connection and Numerical Functions

Concepts :

-> Extension to functions

-> Connected operators

-> Numerical Geodesy

-> Leveling and self-duality

Applications :

-> Extrema analysis

-> Contour preservation

-> Strong filters

-> Segmentation

Passage to Numerical Functions

Three passages from binary to grey tone images must be viewed.

- ***Geodesy***

  It is the simplest one. Dilation and erosion being increasing, it suffices to define numerical operations from binary ones, applied level by level.

- ***Applications***

  They are not the same as the binary case. Priority is now given to the processing of the ***extrema*** and to ***contours preservation***.

- ***Connections***

  This task is more difficult. We can either :

  - generalise the concept of a connection to lattices, and find connections which are adapted to numerical functions,

  - or use functions to induce ***set connections*** on their supports. This simpler (but less powerful) approach will be adopted here.

Lattice of Numerical Functions (*reminder*)

- In order to ovoid the distinction between continuous and digital cases, the axes $\overline{\mathbb{R}}$, $\overline{\mathbb{Z}}$, or any of their close subsets are all denoted by the generic symbol T. Set T is a totally ordered lattice of extrema 0 and m.

- When E is an arbitrary set, the functions f : E $\rightarrow$ T form in turn a totally distributive **lattice**, denoted by T^E, for the **product ordering**:

$$f \leq g \quad \text{iff} \quad f(x) \leq g(x) \quad \text{for all } x \in E ,$$

In this lattice, $\vee$ and $\wedge$, called numerical, are defined by:

$$(\vee f_i)(x) = \vee f_i(x) \qquad (\wedge f_i)(x) = \wedge f_i(x) .$$

- Moreover, in T^E the **impulse** functions :

$$k_{x,t}(y) = t \quad \text{if} \quad x = y \quad ; \quad k_{x,t}(y) = 0 \quad \text{if} \quad x \neq y \quad ,$$

are **sup-generator**, *i.e.* each f may be written $f = \vee k_{x,t}$ for convenient x's and t's.

Lattice of the Partitions (*reminder*)

Reminder : A **partition** of space E is a mapping D: E $\rightarrow \mathcal{P}$(E) such that

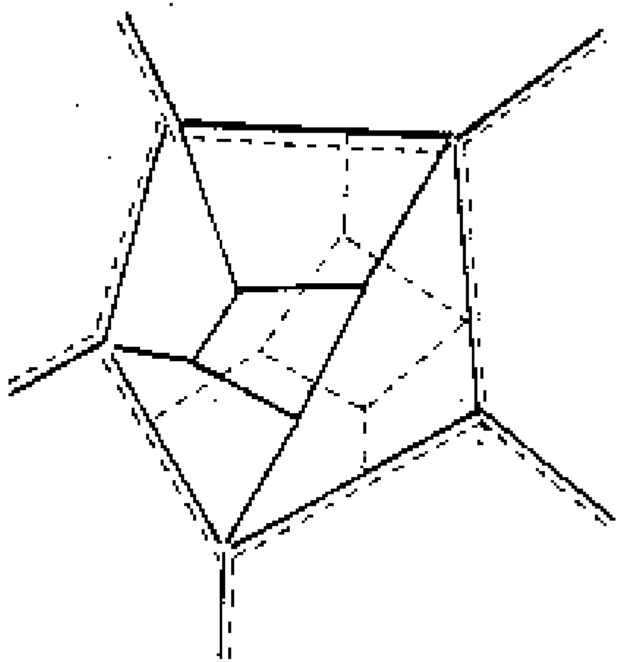
(i) $\forall$ **x** $\in$ **E ,** **x** $\in$ **D(x)**

(ii) $\forall$ **(x, y)** $\in$ **E,**

either **D(x) = D(y)**

or **D(x)** $\cap$ **D(y) =** $\varnothing$

The partitions of E form a **lattice** $\mathcal{D}$ for the ordering according to which D $\leq$ D' when each class of D is included in a class of D'. The largest element of $\mathcal{D}$ is E itself, and the smallest one is the pulverizing of E into all its points.



The sup of the two types of cells is the pentagon where their boundaries coincide. The inf, simpler, is obtained by intersecting the cells.

Set Connections induced by Functions

Goal : Let $\mathcal{C}$ be a connection on $\mathcal{P}(E)$ and $f : E \rightarrow T$. We look for a regional criterion σ on f such that :

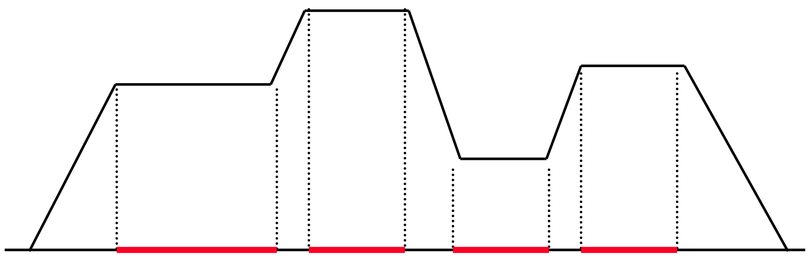
(i) $\forall\ x \in E$, $f(x)$ **fulfils** σ ;

(ii) $\forall\ A, B \in \mathcal{C}$, *with* $A \cap B \neq \emptyset$, **if f fulfils** σ **on A and B, then f fulfils** σ **on** $A \cup B$.

Result : Hence criterion σ generates a subclass $\mathcal{C}_\sigma$ of $\mathcal{C}$ which is a second connection on $\mathcal{P}(E)$.

In particular, $\mathcal{C}_\sigma$ ***partitions*** set E in maximal classes satisfying criterion σ.

Example : The zones where f is constant.

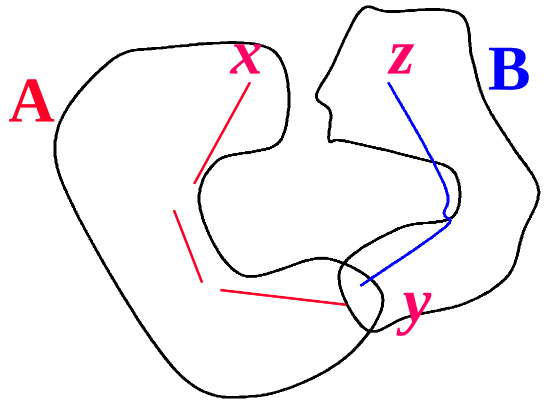


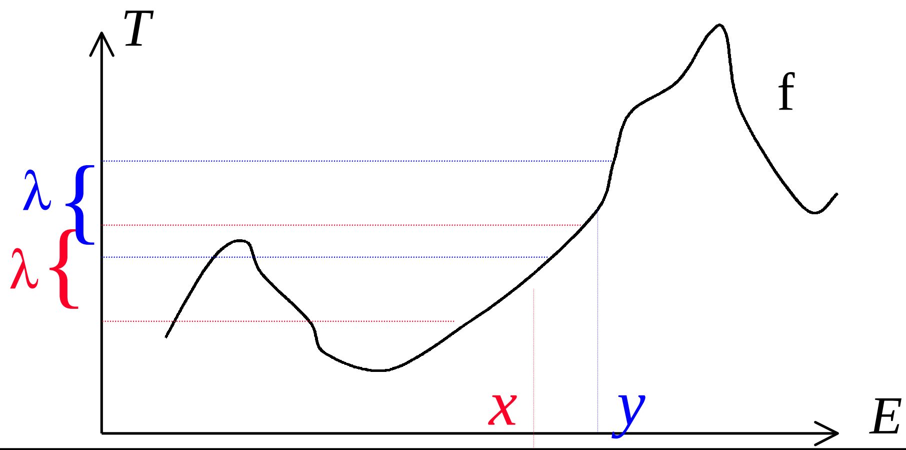
The connected components of $\mathcal{P}(R^1)$ according $\mathcal{C}_\sigma$ are either

- the red segments;

- or the points, elsewhere

"Smooth" induced Connection





Smooth connection : E = $\mathbb{R}^n$, provided with the ***arcwise*** connection, and function f : E → T is fixed. The class $\mathcal{C} \in \mathcal{P}(\mathbb{R}^n)$ composed of

i) the singletons plus the empty set ;

ii) all connected open sets $Y \in \mathcal{P}(\mathbb{R}^n)$ such that f is k-Lipschitz along all paths included in Y,

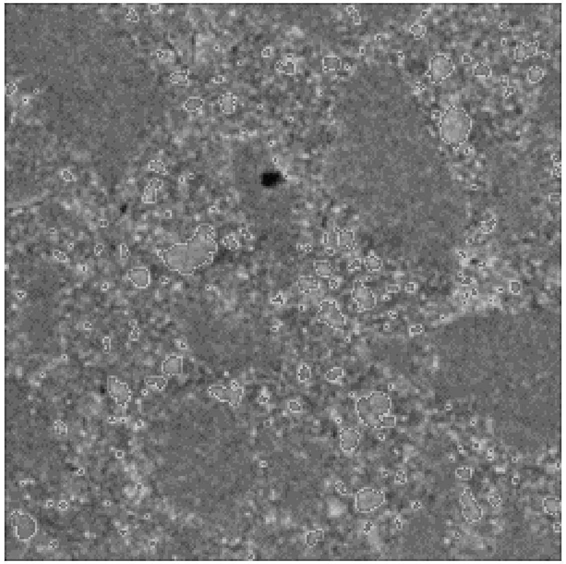
forms a second connection on $\mathcal{P}(\mathbb{R}^n)$, called "***smooth connection***".

Implementation : H(x) stands the unit disc of $\mathbb{Z}^2$ at point x. The partition which is associated with $\mathcal{C}$ has for non point classes the connected components of set

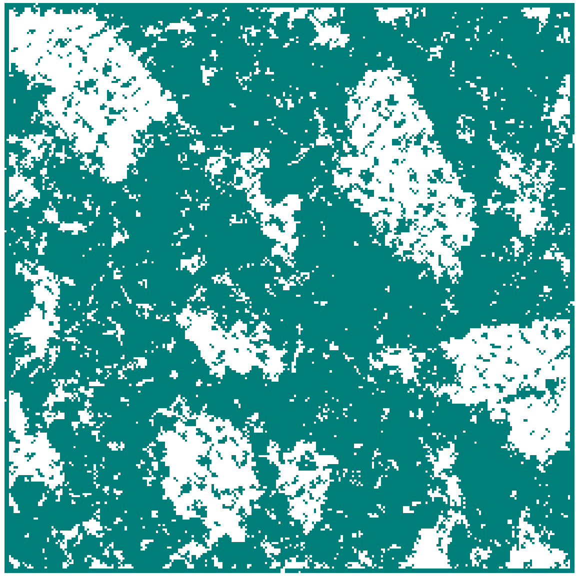
$$\mathbf{X = \{ x \in E \ ; \ sup\{ |f(x) - f(y)|, y \in H(x)\} \leq k \ \}}$$

An Example of Smooth Connection

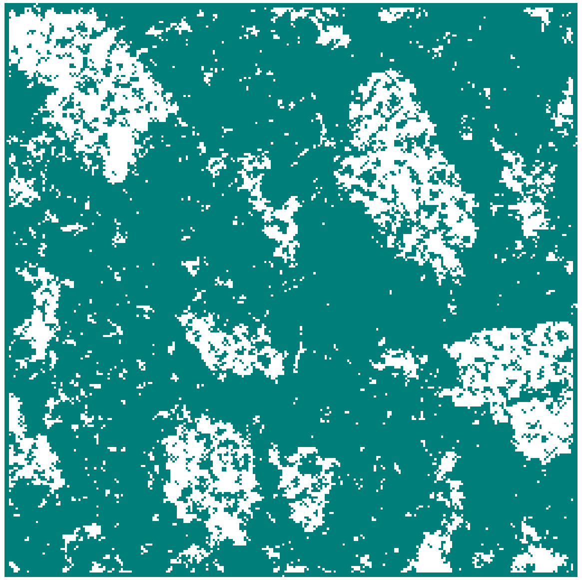
Comment *: the two phases of the micrograph cannot be distinguished by means of thresholds. The smooth connectivity classifies them according to their roughnesses*



a) Initial image: rock electron micrograph



b) smooth connection of slope 7



c) smooth connection of slope 6

(- in dark, the point connected components
- in white, each particle is the base of a cylinder)

"Jump" induced Connection

Jump connection : E = $\mathbb{R}^n$, provided with the ***arcwise*** connection, and function f : E $\to$ T is fixed. The class $C \in \mathcal{P}(\mathbb{R}^n)$ composed of

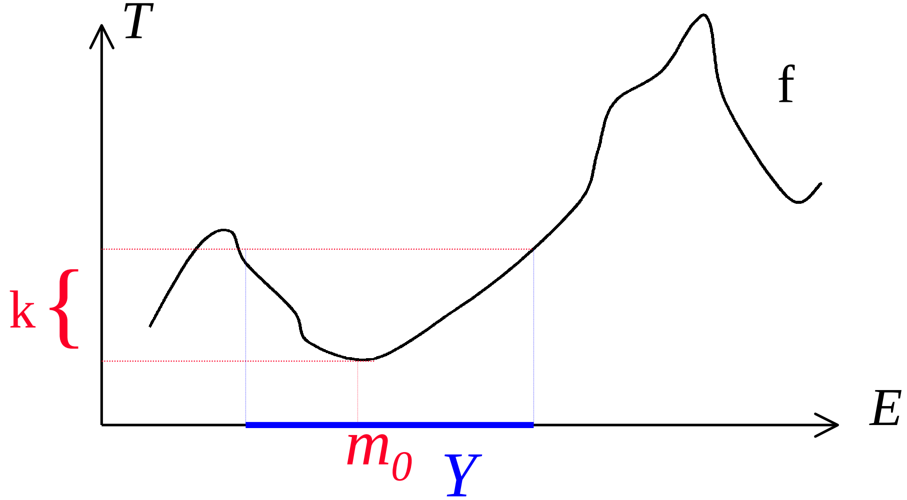
i) the singletons plus the empty set ;

ii) all connected sets around each minimum, and where the value of f is less than k above the minimum ;

forms a second connection on $\mathcal{P}(\mathbb{R}^n)$, called "***jump connection from minima***".

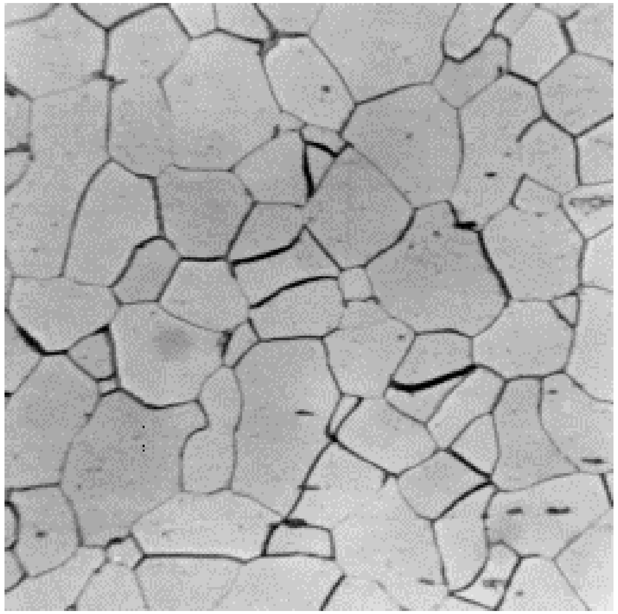
Similarly, one can start from the ***maxima***,

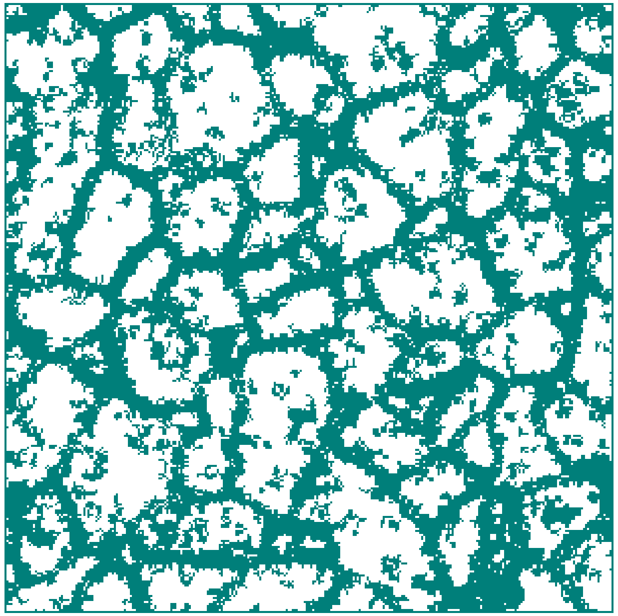
or take the intersection of both connections

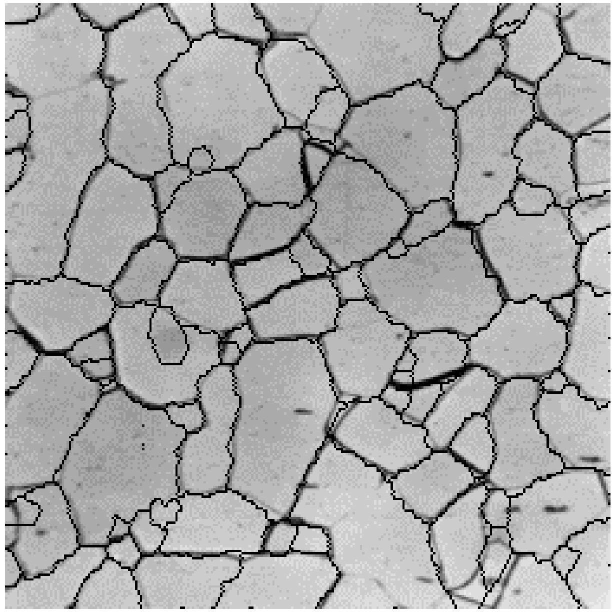


A connected component in the jump connection of range k from the minima .

An Example of symmetrical Jump Connection







a) Initial image: polished section of alumine grains

b) Jump connection of size 12 :

- in dark, the point connected components ;

- in white, the other ones

c) Skiz of the set of the dark points of image b)

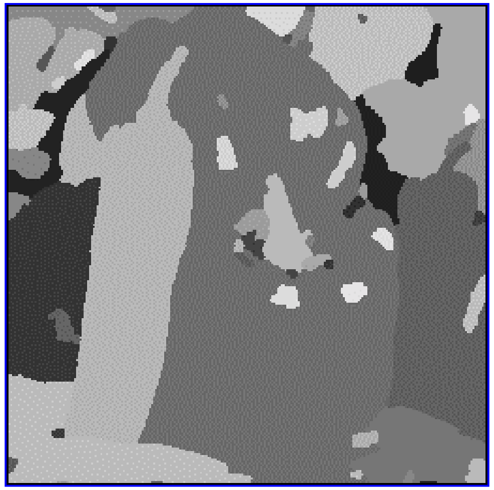
Connected Operators

Definition :

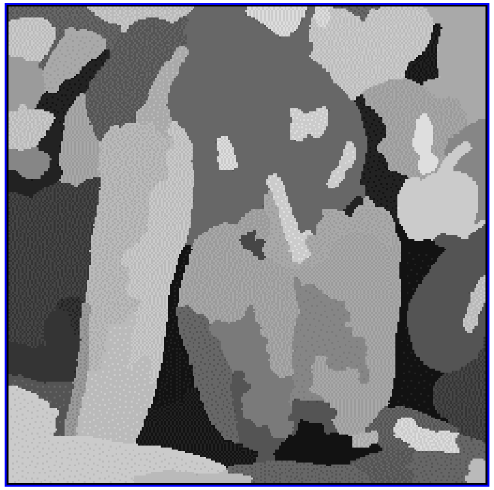
- A function operator $\psi : T^E \rightarrow T^E$ is said to be **connected** (for criterion σ) when the partition of E by $\psi(f)$ is larger than that of E by f .

a)

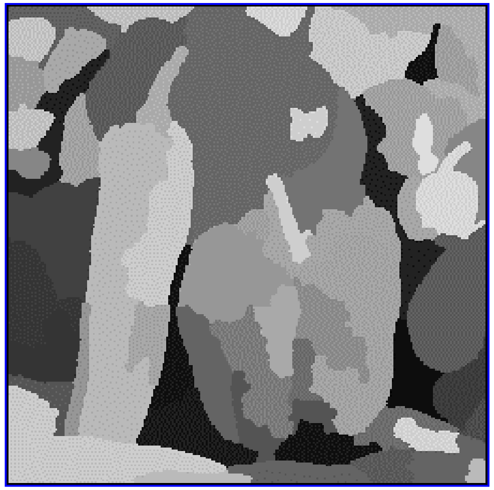
b)



c)



d)



Three mosaic images, due to C. Vachier, obtained by merging the watershed of the gradient of ***a):***

b) by dynamics ; ***c)*** by areas ; ***d)*** by volumes

Flat and Increasing Connected Operators

- From now on, we focus exclusively

  i) on the criterion σ of **flat zones** ;

  ii) and on those operators $\psi : T^E \rightarrow T^E$ that are **flat** and **increasing**.

Basic Properties :

- Every **binary** increasing connected (resp. and increasing) operator induces on T^E, via the cross sections, a ***unique*** increasing connected (resp. and increasing) operator (*H. Heijmans*) ;
- In particular, the geodesic implementations extend to the numerical case ;
- The properties of the set case, to be strong filters, to constitute semi-groups, etc.. are transmitted to the connected operators induced on T^E.

Note that a mapping may be anti-extensive on T^E, but extensive on the lattice $\mathcal{D}$ of the partitions (*e.g.*reconstruction openings).

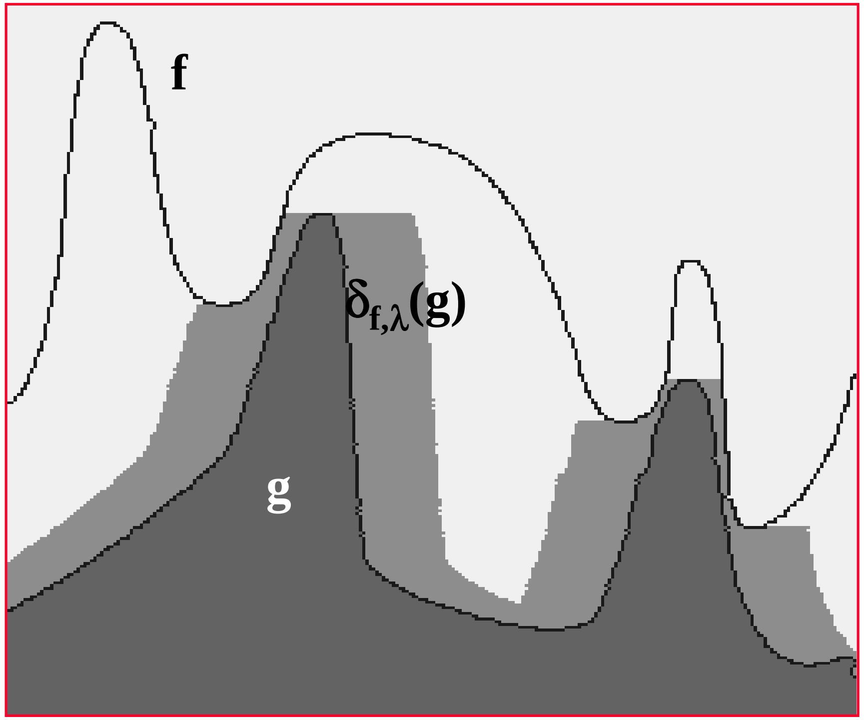
Numerical Geodesic Dilations (I)

- Let f and g be two numerical functions from $\mathbb{R}^d$ into T, with $g \leq f$.

The binary geodesic dilation of size λ of each cross section of g inside that of f at the same level induces on g a dilation $\delta_{f,\lambda}(g)$ (*S.Beucher*) .

- Equivalently, (*L.Vincent*) the sub-graph of $\delta_{f,\lambda}(g)$ is the set of those points of the sub-graph of f which are linked to that of g by
  - a non descending path
  - of length $\leq \lambda$.

numerical geodesic dilation of g with respect to f



Numerical geodesic Dilations (II)

- The digital version starts from the unit geodesic dilation:

$$\delta_f(g) = \inf(g \oplus B, f)$$

which is iterated n times to give that of size n

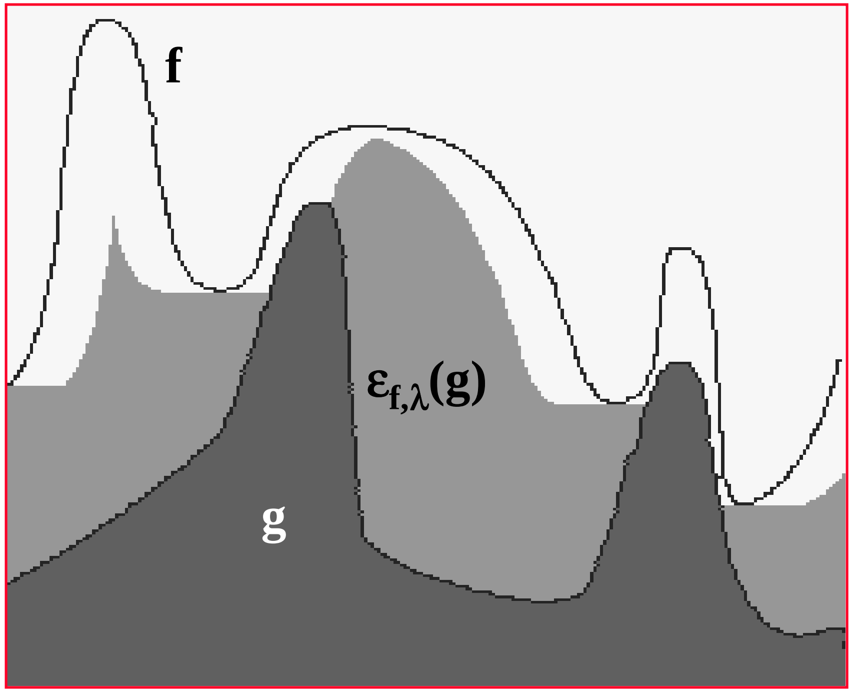
$$\delta_{f,n}(g) = \delta_f^{(n)}(g) = \delta_f(\delta_f \;....\; (\delta_f(g))).$$

- The Euclidean and digital erosions derive from the corresponding dilations by the following duality

$$\varepsilon_f(g) = m - \delta_f(m - g),$$

which is **different** from the binary duality.

numerical geodesic erosion of f with respect to g :



Numerical Reconstruction

- The reconstruction opening of f from g is the supremum of the geodesic dilations of g inside f, this sup being considered as a function of f:

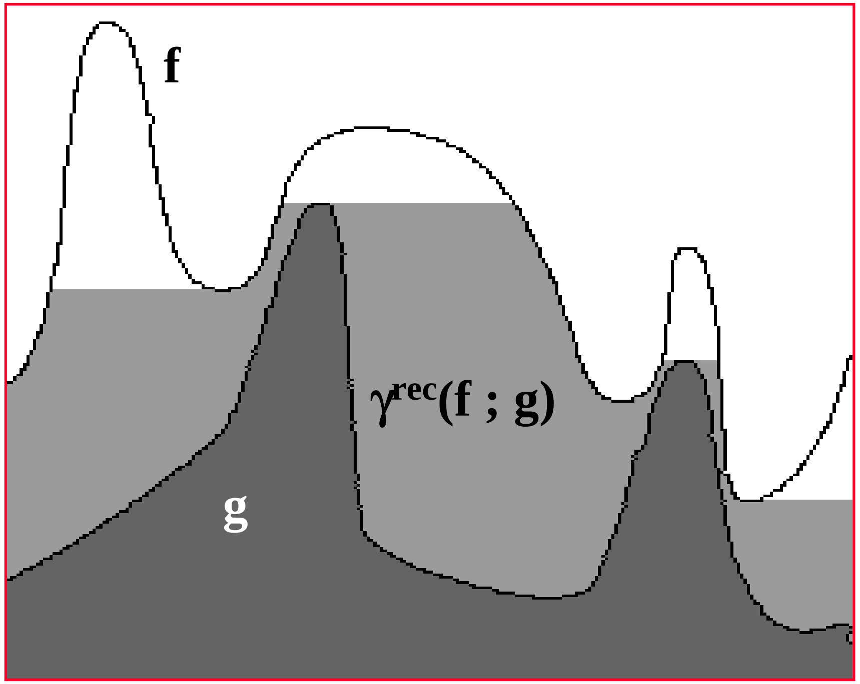
$$\gamma^{rec}(f\ ;\ g) = \vee\{\ \delta_{f,\lambda}(g)\ ,\ \lambda>0\ \}$$

The dual closing for the negative is

$$\varphi^{rec}(f\ ;\ g) = m - \gamma^{rec}(m- f\ ;\ m- g)$$

- The three major applications are :
  - **swamping**, *or reconstruction of a function by imposing markers for the maxima;*
  - *reconstruction from an erosion*
  - *contrast opening ,which extracts and filters the maxima.*

Numerical Reconstruction of g inside f :



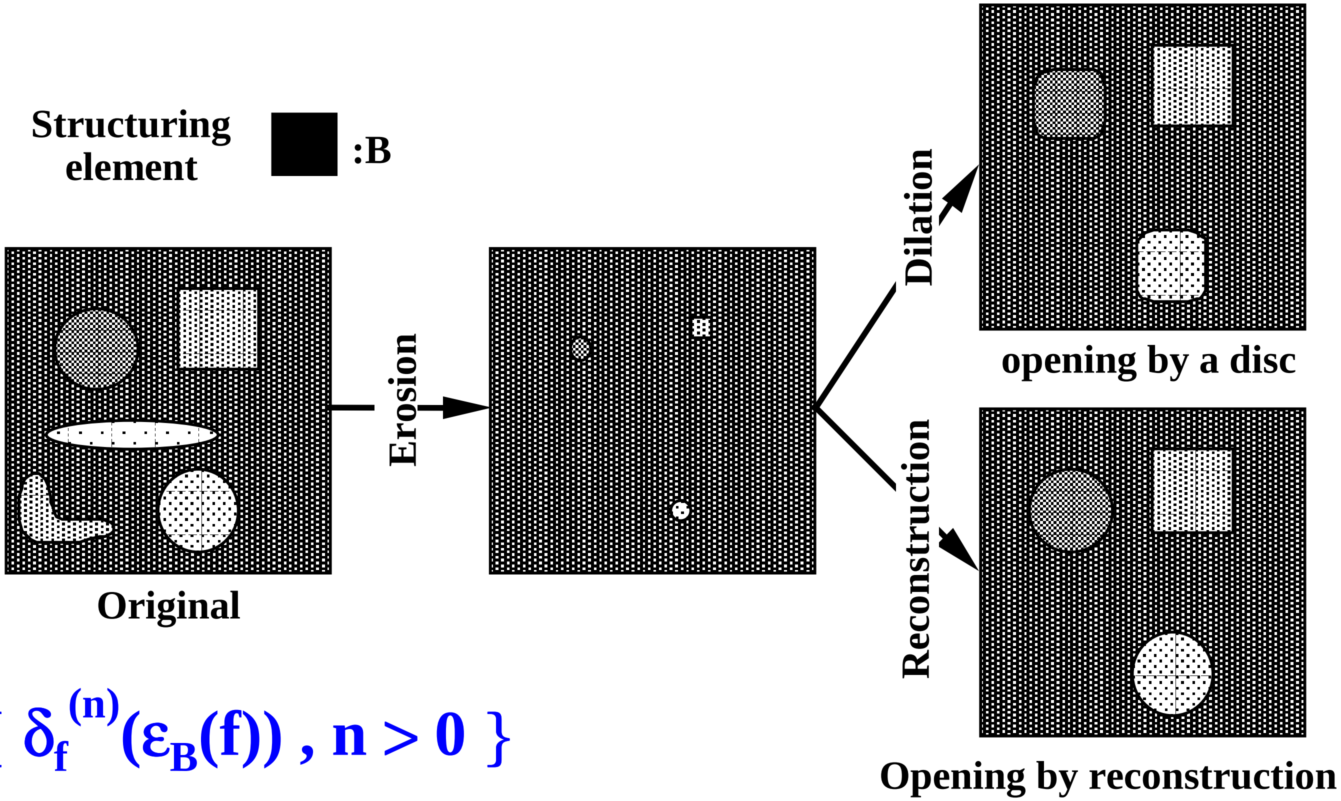
Reconstruction Opening by Erosion

Goal : contour preservation

Whereas the adjunction opening modifies contours, this transform is aimed to efficiently and precisely reconstruct the contours of the objects which have not been totally removed by the filtering process.

Algorithm

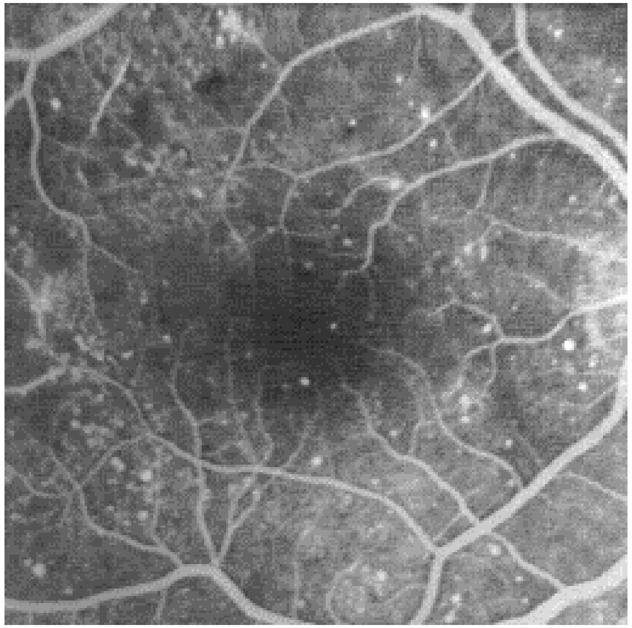
- the mask is the original signal ,
- the marker is an eroded of the mask.



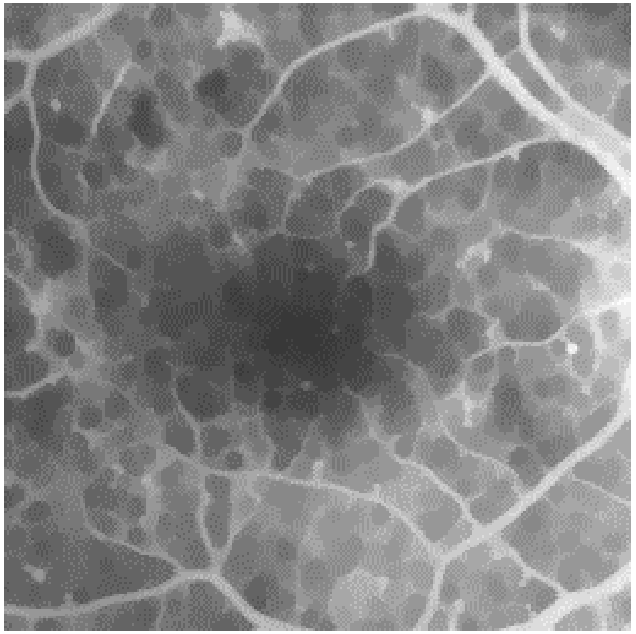
$$\gamma^{rec}(f ; \varepsilon_B(f)) = \vee\{ \delta_f^{(n)}(\varepsilon_B(f)) , n > 0 \}$$

Application to Retina Examination

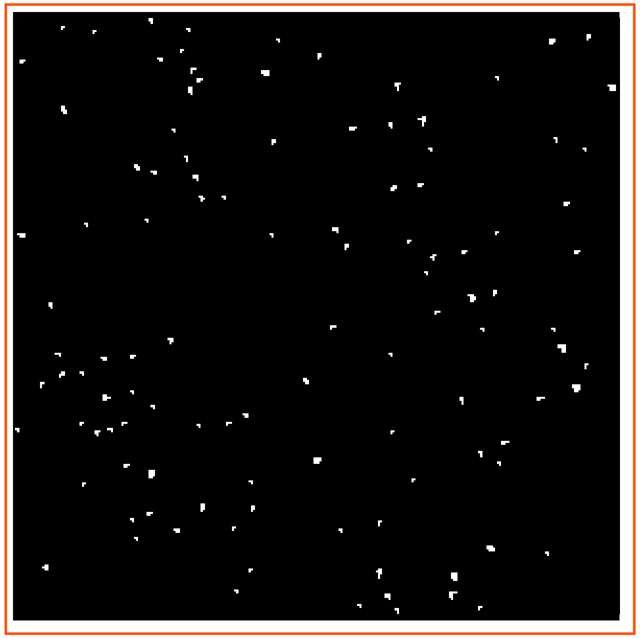
Comment: ***The aim is to extract and to localise aneurisms. Reconstruction operators ensure that one can remove*** **exclusively** ***the small and isolated peaks*** *(case study due to F. Zana and J.C.Klein).*



a) Initial image



b) closing by dilatation-reconstruction followed by opening by érosion- reconstruction



c) difference a) minus b) followed by a threshold

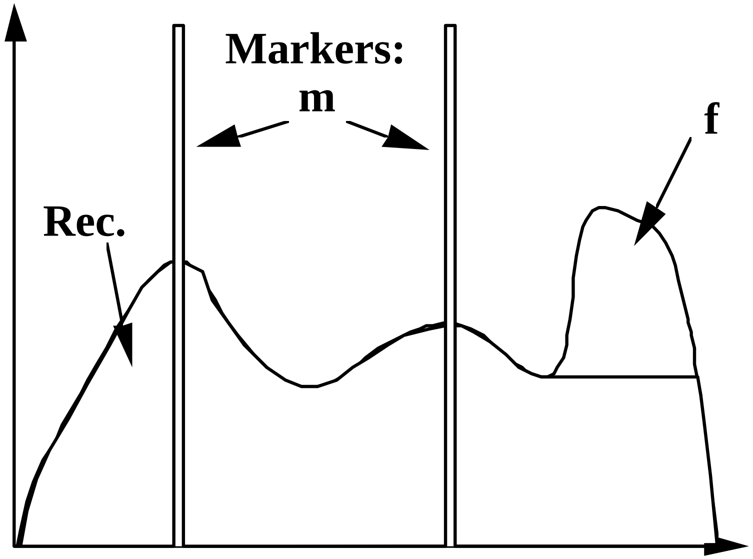
Reconstruction of a Function from Markers

Goal

To remove the useless maxima (or minima) of a function.

Algorithm

- The "marker" is a bi-valued (0,m) function identifying the peaks of interest.
- The reconstruction process result is the largest function ≤ f and admitting maxima at the marked points only. It is called the **swamping** of f by opening (*S.Beucher, F.Meyer*),



Swamping of f by markers m
(by opening)

An Example of Swamping : Contrast Opening

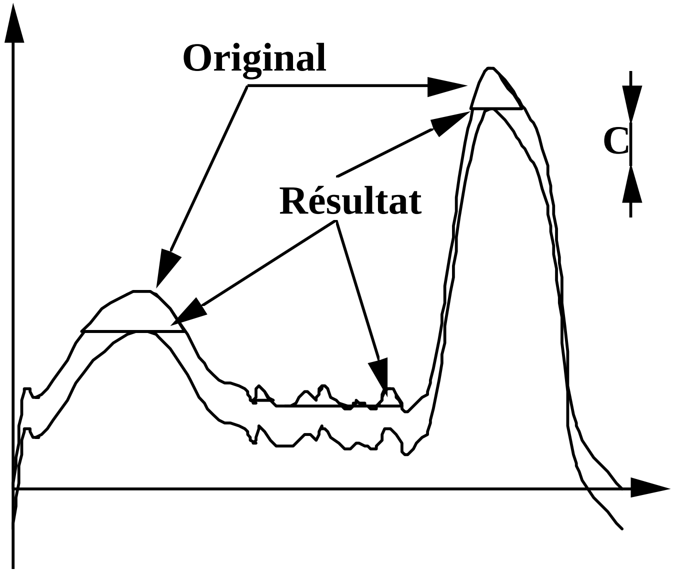
Goal

Both morphological and reconstruction openings reduce the functions according to size criteria which work on their cross sections. In opening by **dynamics**, the criterion holds on grey tones contrast (*M.Grimaud*).

Algorithm

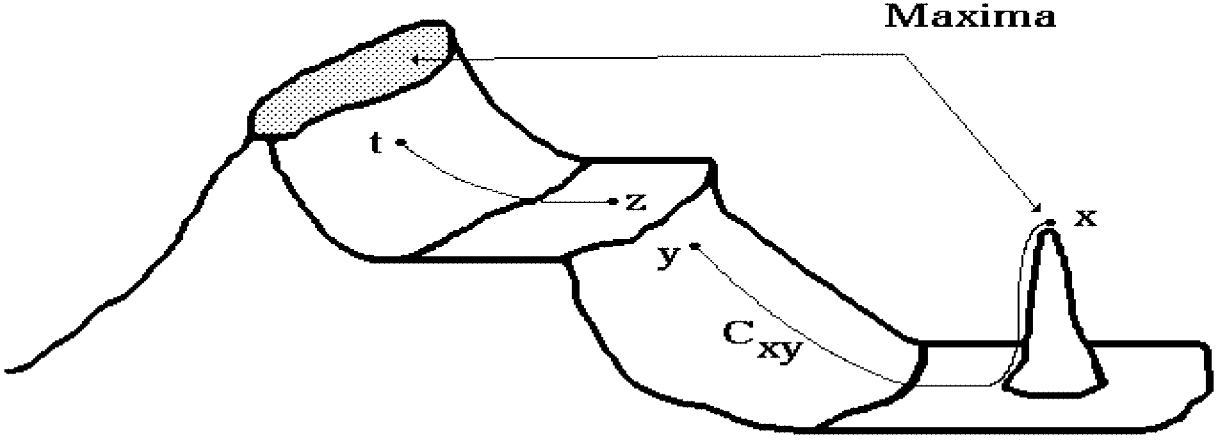
- Shift down the initial function f by constant c;
- Rebuilt f from function f - c, *i.e.*

$$\gamma^{rec}(f\ ;\ f\text{-}c) = \vee\{\ \delta_f^{(n)}(f\text{-}c)\ ,\ n > 0\ \}$$



The associated top-hat extracts all peaks of dynamics ≥ c

Application to Maxima Detection



- The **maxima** of a numerical function on a space E are the connected components of E where f is constant and surrounded by lower values.
- Therefore they are given by the residues of contrast opening of shift c = 1
- More generally, the residuals associated with a shift c extract the maxima surrounded by a descending zone deeper than c. They are called **Extended Maxima** *(S. Beucher).*

Strong Filters by Reconstruction

Here are a few nice properties of the filters by reconstruction

- ***Proposition**(J.Serra)* **:** Let γ^{rec} be a reconstruction opening on T^E that does not create pores and φ^{rec} be the dual of such an opening (not necessarily γ^{rec}). Then :

  $\nu = \varphi^{rec}\gamma^{rec}$ **and** $\mu = \gamma^{rec}\varphi^{rec}$ are **strong filters.**

  [Any pore of X, firstly blocked by φ^{rec}, and then recovered by γ^{rec}, is a pore of $X \cap \gamma^{rec}\varphi^{rec}(X)$, hence $\mu(I \cap \mu) = \mu$.]

  In particular, $I \wedge \gamma^{rec}\varphi^{rec}$ is an **opening** (appreciated for its top-hat when the notion is extended to numerical functions, see IV -9).

- ***Proposition** (J.Crespo, J.Serra)* **:** Let $\{\gamma_i^{rec}\}$ and $\{\varphi_i^{rec}\}$ denote a granulometry and a (not necessarily dual) anti-granulometry, then

- the corresponding alternating sequential filters N_i and M_i are **strong** ; and

- both operators $\Psi_n = \wedge\{\varphi_i\gamma_i, 1 \leq i \leq n\}$ and $\Theta_n = \vee\{\gamma_i\varphi_i, 1 \leq i \leq n\}$ are **strong filters**.

Semi-groups of Filters by Reconstruction

- ***Proposition** (Ph. Salembier, J.Serra)*: Let γ^{rec} be a reconstruction opening on E and φ be a closing that does not create particles. Then :

$$\varphi\gamma^{rec} \leq \gamma^{rec}\varphi \quad (\Leftrightarrow \gamma^{rec}\varphi\gamma^{rec} = \varphi\gamma^{rec} \Leftrightarrow \varphi\gamma^{rec}\varphi = \gamma^{rec}\varphi)$$

[$\varphi\gamma^{rec}$ is invariant under γ^{rec} since φ, can only enlarge the particles of $\gamma^{rec}(X)$]

- ***Proposition** (Ph. Salembier, J.Serra)*: Let $\{\gamma_i^{rec}\}$ be a granulometry and $\{\varphi_i\}$ be an anti-granulometry of the above types. Then:

a) for all i, both products $\nu_i = \varphi_i\gamma_i^{rec}$ and $\mu_i = \gamma_i^{rec}\varphi_i$ satisfy the relations

$$j \geq i \quad \Rightarrow \quad \nu_i\nu_j = \nu_j \quad \text{and} \quad \mu_i\mu_j = \mu_j$$

[we always have $j \geq i \Rightarrow \mu_j \leq \mu_i\mu_j$; in addition, here

$$\gamma_j^{rec}\varphi_j = \gamma_i^{rec}\gamma_j^{rec}\varphi_j = \gamma_i^{rec}\varphi_j\gamma_j^{rec}\varphi_j \geq \gamma_i^{rec}\varphi_i\gamma_j^{rec}\varphi_j \,]$$

b) Therefore, the associated A.S.F. N_i et M_i form a **semi group**

$$N_j N_i = N_i N_j = N_{sup(i,j)} \quad ; \quad M_j M_i = M_i M_j = M_{sup(i,j)}$$

An Example of a Pyramid of Connected A.S.F.'s

Flat zones connectivity, (i.e. $\varphi = 0$).
Each contour is preserved or suppressed,
but never deformed : the initial partition
increases under the successive filterings,
which are strong and form a semi-group.

Initial Image

ASF of size 1

ASF of size 4

ASF of size 8

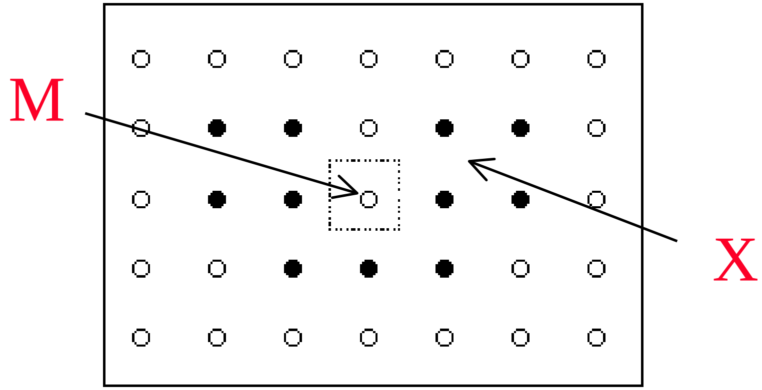
(hexagonal structuring elements)

Adjacency

- Marker based opening allows to design a *self-dual* operator, called leveling, and due to *F.Meyer*. We will introduce first the notion of adjacency :

- ***Adjacency:*** Let $\mathcal{C}$ be a connection on $\mathcal{P}$(E). Sets X,Y∈$\mathcal{P}$(E) are said to be adjacent when X ∪ Y is connected, whereas X and Y are disjoint.

Note that for the digital connection by a 2x2 square opening, the point marker M of the figure is adjacent to no grain of set X, but to X itself.



- ***Adjacency Prevention:*** Connection $\mathcal{C}$ is ***adjacent preventing*** when, for any element M∈$\mathcal{P}$(E) and any family $\{B_i ; i \in I\}$ in $\mathcal{C}$, to say that M is adjacent to none of the B_i is equivalent to saying that M is not adjacent to $\cup B_i$.

Levelling

- Given marker M, consider $A \in \mathcal{P}(E)$. Let

  ~ $\gamma_M(A)$ be the union of the grains of A that hit M or that are adjacent to it ;

  ~ $\varphi_M(A)$ be the union of A and of its pores that are included in M and non adjacent to M^c

- ***Definition** (F. Meyer) :* **leveling λ** is defined as the ***activity supremum***

$$\lambda = \gamma_M \curlyvee \varphi_M$$

i.e. $\lambda(A) \cap A = \gamma_M \cap A$, and $\lambda(A) \cap A^c = \varphi_M \cap A^c$.

Leveling λ acts as opening γ_M inside A , and as closing φ_M inside A^c .

- ***Self-duality**:* The mapping $(A,M) \rightarrow \lambda(A,M)$ from $\mathcal{P}(E) \times \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ is self-dual. If M itself depends on A, *i.e.* if $M = \mu(A)$, then λ, as a function of A only, is self-dual iff μ is already self-dual.

- The ***extension to functions*** (via their cross-sections) will be denoted by

$$(f, g) \rightarrow \Lambda\,(f, g)$$

Properties of Levelling

Here are a few nice properties of levelling :

- ***Proposition*** *(F.Meyer)*: The levelling (A,M)→ λ(A,M) is an increasing mapping from $\mathcal{P}(E) \times \mathcal{P}(E) \rightarrow \mathcal{P}(E)$; it admits the equivalent expression:

$$\lambda = \gamma_M \cup (\complement \cap \varphi_M)$$

- ***Proposition*** *(G.Matheron):* The two mappings

  $A \rightarrow \lambda_M(A)$, given M, and $M \rightarrow \lambda_A(M)$, given A,

  are *idempotent* (hence are *connected filters* on $\mathcal{P}(E)$).

- ***Proposition*** *(J. Serra)*: Levelling $A \rightarrow \lambda_M(A)$ is a strong filter, and is equal to the commutative product of its two primitives

$$\lambda = \gamma_M \circ \varphi_M = \varphi_M \circ \gamma_M$$

  iff connection *C* is *adjacency preventing*. Then, λ satisfies the stability relation : $\gamma_x(I \cup \lambda) = \gamma_x \cup \gamma_x \lambda$, *i.e.* preserves the *sense of variation* at the grains/pores junctions .

An Example

Initial image : *« Joueur de fifre », by E. MANET*

Markers : *hexagonal alternated filters, (non self-dual)*

Initial image, 83.776 pp
flat zones : 34.835

Marker $\varphi_1 \gamma_2$
flat zones : 53.813

Marker $\gamma_1 \varphi_2$
flat zones : 53.858

Duality for Functions

- If 0 and *m* stand for the two extreme bounds of the gray axis T, then the set complement operation is replaced by its function analogue f$\rightarrow$ *m* - f and we have for levelling Λ

$$m - \Lambda (m - f , m - g) = \Lambda (f, g) \qquad (1)$$

which means that f, g $\rightarrow$ Λ(f, g) is ***always*** a self dual mapping.

- In addition, if g derives from f by a self-dual operation, *i.e.* g = g(f) with

$$m - g (m - f) = g (f) \qquad (2)$$

(*convolution, median element*), then levelling f $\rightarrow$ Λ(f, g(f)) is self-dual.

- Observe that rel. (2) is distinct from that of invariance under complement

$$g (m - f) = g (f)$$

which is satisfied by the module of the gradient, or by the extended extrema, for example, and which does not imply self-duality for f $\rightarrow$ Λ(f).

An Example of Duality

Marker: ***extrema with a dynamics ≥ h (invariance under complement).***

Initial image
flat zones : 34.835

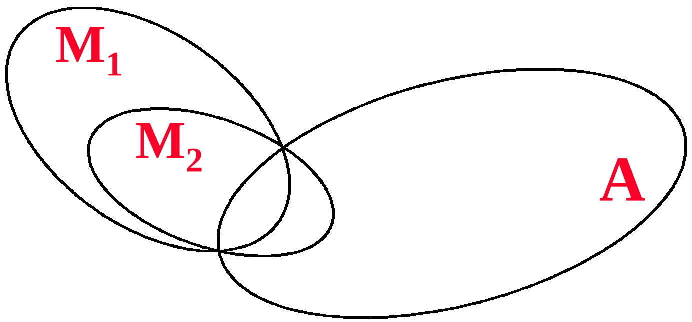
h = 80
flat zones : 57.445

Levelling as function of the marker

We now fix set A and study the mapping $M \rightarrow \lambda_A(M)$ as marker M varies. Set A generates on $\mathcal{P}(E)$ the A-activity ordering $\preceq_A$ by the relations

$$\mathbf{M_1 \preceq_A M_2}$$

i.e. if M_1 meets A or is adjacent to A,
then M_2 meets A or is adjacent to A
and if M_2 meets A^c or is adjacent to A^c,
then M_1 meets A^c or is adjacent to A^c.



***Proposition (J. Serra)*:** If $M_1 \preceq_A M_2$, then we have

$$\lambda_{M1}\, \lambda_{M2}\, (A) = \lambda_{M2}\, \lambda_{M1}\, (A) = \lambda_{M2}(A)$$

In practice, this granulometric pyramid allows to grade markers activities .

An Example of Pyramid

Marker: ***Initial image, where the h-extrema are given value zero (self-dual marker)***

Initial image
flat zones : 34.835

Levelling for h = 50
flat zones : 58.158

Levelling for h = 80
flat zones : 59.178

An Example of Noise Reduction

Marker: ***Gaussian convolution of size 5 of the noisy image***

a) Initial image, plus 10.000 noise points

b) Gaussian convolution of a)

c) Levelling of a) by b) flat zones : 46.900

References

***On Numerical Geodesy* :**

- The essentials of numerical geodesy were created by S.Beucher and F.Meyer during the eighties {BEU90},{MEY90}. Efficient developments are found in L.Vincent {VIN93}, P.Soille {SOI91}, C.Vachier {VAC95}. Ch. Ronse and J. Serra studied the links between symmetry, geodesy and connection {RON99}.

***On Connected Operators* :**

- In {MEY90} and in {SAL92}, reconstruction is used as a tool to modify the homotopy of a function, for multi-resolution purposes. The contrast opening is defined in {GRI92}. A systematic investigation of semi-groups and pyramids, by Ph.Salembier and J.Serra, is given in {SER93a} and used for sequences compression and filtering in {MGT96}, {SAL96}, {PAR94}, {CAS97}, and {DEC97}. Nice properties of $\vee$ and $\wedge$ were found by J.Crespo and Al {CRE95}.
- The notion of a levelling is due to F.Meyer {MEY98}, see also G.Matheron {MAT97}, and J.Serra {SER98b}. The larger class of the “grains operators” has been introduced and studied by H. Heijmans {HEI97}.