Composition products by over-filters

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1 Introduction

The short remarks that follow are just a comment to the Sergio Faria's talk last month, who gave an overview of his current work on morphological optimisations. The representation theorems of mathematical morphology decompose any operator θ (resp. increasing operator, resp.opening) into a supremum of hit-or-miss mappings (resp. of erosions, resp. of adjunction openings). Such decompositions suggest a use of hit-or-miss operators, erosions and adjunction openings as universal bases for image processing. In this way of thinking, every problem of image processing should be solved by means of a training test of images/solutions, plus an adequate union of erosions (or hit-or-miss, or adjunction openings), whose structuring elements should derive from some optimisation.

Unfortunately, the implicit analogy with the vector space structure that underly this approach does not resist to a more accurate examination. The unions involved in such representations are not countable, so that when we pass to digital versions, combinatorial explosions occur inevitably. For example, the operator "*suppress all particles of areas smaller than 1000*" yields billions of adjunction openings, since it requires the family of all structuring elements whose area is equal to 1000. Then, a technique to get around this trouble consists in restricting the possible stucturing elements to those that are inluded in a window of a given size, in E (for sets) or in $E \otimes T$ (for numerical functions). But by so doing one restricts also the class of possible operators, hence their efficency. In particular, by reducing the size of the window to value one, as it is generally made, one implicitely decides to ignore every feature larger than the noise. In such cases, even if Matheron or Banon-Barrera theorems are still called upon, they are no longer used. The approaches are simply based on that a union of erosion is increasing, a union of adjunction openings is an opening, and a union of hit-or-miss operations is anything...

However, the principle of optimising a decomposition of operations remains a good idea, and if for computational reasons the window has to be kept down, one can think to composition products between the general operators θ and more specific ones, such as the connected over-filters ψ. The latter, which depend on few parameters, should compensate the spacial narowness of θ, while introducing nice features. In the following we try and develop such an approach, after having recalled the two representation theorems, by erosions and by hit-or-miss, of the translation invariant case.

2 Binary representation theorems

The importance of erosions and dilations, for sets and functions, is best illustrated by the following theorem which is due to Matheron([2], p.218). Define the kernel $\mathcal{V}(\psi)$ of a translation invariant operator ψ on $\mathcal{P}(R^n)$ by

$$\mathcal{V}(\psi) = \{A \in \mathcal{P}(R^n) \mid O \in \psi(A)\}.$$

Where O designates the origin of the coordinates.

Theorem 1 *(G.Matheron) Let ψ be an increasing translation-invariant operator on $\mathcal{P}(R^n)$. Then ψ can be decomposed as a union of Minkowski substractions, and, likewise as an intersection of Minkowski additions. More precisely*

$$\psi\,(X) = \psi(X) = \bigcup\{X \ominus B, B \in \mathcal{P}(R^n), B \in \mathcal{V}(\psi)\}. \tag{1}$$

Banon and Barrera [1] have extended Matheron's result for operators which are not necessarily increasing and shown that any translation invariant mapping $\theta : \mathcal{P}(R^n) \to \mathcal{P}(R^n)$ is a supremum of *hit-or-miss mappings.* This transformation, which turns out to be historically the first morphological operator [3], is defined as

$$X \circledast (A, B) = \cup\,\{h \in E \mid A_h \subseteq X \text{ and } B_h \subseteq X^c\} = (X \ominus A) \cap (X^c \ominus B)\,.$$

where $A, B \subseteq R^n$ are structuring elements. The hit-or-miss operator is equivalently be written in the following way

$$X \circledcirc (A, B) = \cup\,\{h \in R^n \mid A_h \subseteq X \subseteq B_h\}\,,$$

which avoids the use of a complement.Obviously, there exists the following relation

$$X \circledcirc (A, B) = X \circledast (A, B^c)\,.$$

For $A, B \subseteq R^n$ we define the "interval" $[A, B]$ as

$$[A, B] = \{X \in \mathcal{P}(R^n) \mid A \subseteq X \subseteq B\}$$

and the bi-kernel $\mathcal{W}\,(\psi)$ of operator ψ as

$$\mathcal{W}\,(\psi) = \{(A, B) \in \mathcal{P}(R^n) \times \mathcal{P}(R^n) \mid [A, B] \subseteq \mathcal{V}\,(\psi)\}\,.$$

Theorem 2 *(Banon-Barrera) Let ψ be an arbitrary translation-invariant operator on $\mathcal{P}(R^n)$ of bi-kernel $\mathcal{W}\,(\psi)$. Then we have*

$$\psi\,(X) = \cup\,\{X \circledcirc (A, B) \mid (A, B) \in \mathcal{W}\,(\psi)\} \tag{2}$$

3 Representation theorems for numerical functions

How can we extend Banon-Barrera theorem to functions? First of all, the points of the space have to be replaced by the pulse functions:

$$\begin{cases} i_{x,t}(y) = t & if \quad y = x \\ i_{x,t}(y) = 0 & if \quad x \neq y \end{cases}$$

Secondly, we have to avoid a hit-or-miss formalism that involves a complement, since this set operator does not extend to numerical functions.This can obtained by using the $\circledcirc$ form of the hit-or-miss mapping, namely by introducing the following operator, for $g, g' \leq \mathcal{F}(R^n)$

$$f \circledcirc (g, g') = \vee \{i_{h,t} \in \mathcal{F}(R^n) \mid g_h \leq f + t \leq g'_h\}.$$

associated with each $x \in R^n$ and $t \in \overline{R}$.

Finally, generalizing the previous set oriented case, define for each pair $g, g' \in \mathcal{F}(R^n)$

- the "interval" $[g, g']$ as
$$[g, g'] = \{f \in \mathcal{F}(R^n) \mid g \leq f \leq g'\}$$
- the kernel of an operator $\theta : \mathcal{F}(R^n) \to \mathcal{F}(R^n)$ as
$$\mathcal{V}(\theta) = \{g \in \mathcal{F}(R^n) \mid O \leq \theta(0)\}$$
- and the bi-kernel $\mathcal{W}(\theta)$ of the operator ψ as
$$\mathcal{W}(\theta) = \{(g, g') \in \mathcal{F}(R^n) \times \mathcal{F}(R^n) \mid [g, g'] \subseteq \mathcal{V}(\theta)\}.$$

Theorem 3 *Let θ be an arbitrary translation-invariant operator on $\mathcal{F}(R^n)$. Then*

$$\theta(f) = \vee \{f \circledcirc (g, g') \mid (g, g') \in \mathcal{W}(\theta)\} \tag{3}$$

Proof. *"$\geq$ part" : Let $i_{h,t} \in f \circledcirc (g, g')$ for some$(g, g') \in \mathcal{W}(\theta)$. Then $g_h \leq f + t \leq g'_h$, hence $f_{-h} - t \in [g, g'] \subseteq \mathcal{V}(\theta)$. Therefore $O \in \theta(f_{-h} - t)$, and by translation-invariance of θ, $i_{h,t} \in \theta(f)$.*

"$\leq$ part": Let $i_{h,t} \in \theta(f)$, that is, $O \in \theta(f_{-h} - t)$. Then $[f_{-h} - t, f_{-h} - t] \in \mathcal{W}(\theta)$. In addition, it is obvious that $i_{h,t} \in f \circledcirc (f_{-h} - t, f_{-h} - t)$ and thus $h \in \vee \{f \circledcirc (g, g') \mid (g, g') \in \mathcal{W}(\theta)\}$. ■

Remark that the theorem treats the set oriented case via the supporting functions. Note also that if operator θ is increasing and $g \in \mathcal{V}(\theta)$, then $(g, +\infty) \in \mathcal{W}(\theta)$ and since $X \circledcirc (A, B)$ is decreasing with respect to g', we have $f \circledcirc (g, +\infty) = f \ominus g$, and formula (3) reduces to $\theta(f) = \bigcup\{f \ominus g \mid g \in \mathcal{V}(\theta)\}$. So Banon-Barrera's theorem indeed generalizes Matheron's one.

Both theorems admit a dual form with respect to complement. Let us show for exemple the dual statement of Matheron's thorem in the set case, by introducing the operator θ^* that derives from θ by duality, i.e.

$$\theta^*(X) = [\theta(X^c)]^c \text{ or as well } \theta(X) = [\theta^*(X^c)]^c$$

Then, relation (1) is equivalent to

$$\begin{aligned} \theta(X) &= \cup \{(X^c \ominus B) \mid B \in \mathcal{V}(\theta^*)\}^c = \cap \{(X^c \ominus B)^c \mid B \in \mathcal{V}(\theta^*)\}^c \\ \theta(X) &= \cap \left\{X \oplus \check{B} \mid \check{B} \in \mathcal{V}(\theta^*)\right\}. \end{aligned}$$

This last form gives a dual statement for Matheron's theorem, according to which any increasing and translation invariant mapping is decomposable into an infimum of Mankowski additions.

4 Composition products

Firstly, we recall two notions :

1/ an operator ψ is connected, for connection $\mathcal{C}$, when

$$\forall y \in E : C_y\psi(A) = [\cup C_i(A), i \in I] \cup [\cup C_j(A), j \in J] \tag{4}$$

2/ a mapping θ does not create connected components when

$$C_x\theta(A) \neq \emptyset \Longrightarrow \exists\, y \in A : C_x\theta(A) = C_y\theta(A) \tag{5}$$

where $C_x(*)$ stands for the connected component at point $x \in E$.

From now on, symbol ψdesignates a connected over-filter (i.e.ψ is connected, increasing and $\psi\psi \geq \psi$) and θ a mapping that does not creates connected components

Proposition 4 *Let ψ be a connected over-filter on $\mathcal{P}(E)$ and let θ be an operator on $\mathcal{P}(E)$ that does not create connected components.*

There we have

$$\psi\theta \geq \psi \qquad \Rightarrow \qquad \psi\theta\psi \geq \theta\psi. \tag{6}$$

Proof. *Given $A \in \mathcal{P}(E)$, suppose that $\psi\theta \geq \psi$ and let $x \in \theta\psi(A)$. Since $x \in C_x\theta\psi(A)$ then, according to relation (5), there exists $y \in \psi(A)$ such that $C_x\theta\psi(A) = C_y\theta\psi(A)$. Now, the over-potence of ψ and the inequality $\psi\theta \geq \psi$ imply that*

$$y \in \psi(A) \subseteq \psi\psi(A) \subseteq \psi\theta\psi(A).$$

Therefore $C_y\psi\theta\psi(A)$ is non empty, and since ψ is connected, relation (4) implies

$$y \in C_y\theta\psi(A) \subseteq C_y\psi\theta\psi(A),$$

i.e. finally $x \in C_x\theta\psi(A) = C_y\theta\psi(A) \subseteq \psi\theta\psi(A)$. ■

The requirement $\psi\theta \geq \psi$ may be fulfilled in several ways. Here are three possible inputs

- $\theta \geq \psi$
- θ is extensive , i.e. $\theta \geq I$
- $\theta = \psi \vee \theta_0$, where θ_0 and ψ do not create connected components

The condition (5) of non creation of connected components is not very demanding. Start from an arbitrary mapping $\theta_0\mathcal{P}(E) \to \mathcal{P}(E)$ and suppress the connected components of $\theta_0(A)$ that miss A; the result defines the following mapping θ :

$$\theta(A) = \theta_0(A) \diagdown \cup \{C_x[\theta_0(A)] \mid x \in E, C_x(\theta_0(A)) \cap A = \emptyset\}$$

which, by construction, satisfies condition (5).

When $\psi = \gamma$ is a connected opening, the anti-extensivity of γ yields $\theta\gamma \geq \gamma\theta\gamma$, and implication (6) becomes

$$\gamma\theta \geq \gamma \qquad \Longrightarrow \qquad \theta\gamma = \gamma\theta\gamma. \tag{7}$$

If, in addition, operator θ is increasing, then we have

$$\gamma\theta \geq \gamma \qquad \Longrightarrow \qquad \theta\gamma = \gamma\theta\gamma \leq \gamma\theta.$$

5 Idempotent upperbounds

The most useful case occurs when $\theta \geq \psi$. Firstly we have

$$\theta\psi \geq \psi\psi \geq \psi,$$

which provides a lower bound for $\theta\psi$. Secondly, the composition product $\theta\psi$ inherits of the over-potence property of ψ, and induces an idempotent operator. More precisely, we can state the following

Proposition 5 *Let $\{\psi_i \ , i \in I\}$ be a family of connected over-filters, and let $\{\theta_i, i \in I\}$ be a family of operators that do not create connected components. If for each index i we have $\theta_i \geq \psi_i$, then all products $\theta_i\psi_i$ as well as the supremum $\vee\{\theta_i\psi_i, i \in I\ \}$ are over-potent.*

Proof. *The inequality $\theta_i \geq \psi_i$ and the over-potence of ψ_i imply that $\theta_i\psi_i\theta_i\psi_i \geq \psi_i\psi_i\theta_i\psi_i \geq \psi_i\theta_i\psi_i$ and relation (6) that $\theta_i\psi_i\theta_i\psi_i \geq \theta_i\psi_i$ so that the product $\theta_i\psi_i$ is over-potent. Similarly, for every $i_0 \in I$, the supremum $\vee\{\theta_i\psi_i, i \in I\ \}$ satisfies the inequalities*

$\{\vee\theta_i\psi_i\}\{\vee\theta_i\psi_i\} \geq \theta_{i_0}\psi_{i_0}\{\vee\theta_i\psi_i\} \geq \psi_{i_0}\psi_{i_0}\{\vee\theta_i\psi_i\}$
$\geq \psi_{i_0}\{\vee\theta_i\psi_i\} \geq \psi_{i_0}\{\vee\theta_i\psi_i\} \geq \psi_{i_0}\theta_{i_0}\psi_{i_0} \geq \theta_{i_0}\psi_{i_0}$
hence $\{\vee\theta_i\psi_i\}\{\vee\theta_i\psi_i\} \geq \{\vee\theta_i\psi_i\}$. ■

Given one product $\theta\psi$, we now consider the class $\mathbb{C}$ closed under supremum and self composition generated by $\theta\psi$, i.e. the intersection of all classes closed under $\vee$ and self composition that contain $\theta\psi$.We will study the structure of class $\mathbb{C}$ by adopting the same type of approach as Matheron did for generating filters,

Proposition 6 *Let be $\theta\psi$ be the composition product of two operators as defined in the previous proposition, and let $\mathbb{C}$ be the class closed under $\vee$ and self composition generated by $\theta\psi$. Then class $\mathbb{C}$ admits a larger element ξ which is idempotent, and satisfies the relationships*

$$\xi = \theta\psi\xi = \theta\xi = \psi\xi$$

Proof. *All the elements of $\mathbb{C}$ are over-potent. By closure under $\vee$, the supremum mapping*

$$\xi = \vee\{\eta, \eta \in \mathbb{C}\} \tag{8}$$

is iteslf an element of $\mathbb{C}$*, hence* $\xi\xi \geq \xi$*. But* $\mathbb{C}$ *is closed under self composition, so that* $\xi\xi$ *belongs to the class, and according to rel.(8)* $\xi\xi \leq \xi$*, which results in the idempotence of* ξ*. Moreover, by over-potence of* $\theta\psi$*, and since* ξ *is the maximum element of the class, we find* $\theta\psi\xi = \xi$ *.Then we have, on the one hand* $\psi\xi = \psi\theta\psi\xi \geq \theta\psi\xi = \xi$*, and on the other one* $\xi = \theta\psi\xi \geq \psi\psi\xi \geq \psi\xi$*, hence the equality* $\xi = \psi\xi$*, and also* $\xi = \theta\psi\xi = \theta\xi$*.* ■

In digital spaces, equation (8) reduces to the idempotent limit of $\theta\psi$ under iteration, i.e.

$$\xi = [\theta\psi]^n$$

for n large enough. In particular that when ψ is an opening, i.e.$\psi = \gamma$, then the condition $\gamma \leq \theta$ is equivalent to $\gamma \leq \gamma\theta$ and from rel.(7) we draw

$$\theta[\gamma\theta\gamma] = \theta^2\gamma.....[\theta\gamma]^n = [\theta]^n\gamma$$

6 Conclusion

This brief analysis aims to open the door to variants in hit-or-miss optimisations. Such variants have been constructed in order to compensate the narowness of the supremum θ of hit-or-misses by composing it with a connected filter ψ involving possible larger sizes. Optimisations can be reached by several ways from the starting points suggested above. One can use the double inequality

$$\psi \leq \theta\psi \leq \psi\theta\psi$$

and minimise the distances between $\psi(f)$ and $\psi\theta\psi(f)$ for a training set. Alternatively, one can try and obtain the idempotent limits before any optimisation, or even work directly with the product $\theta\psi$ as a unique entity to optimise. Finally, one can consider θ in a piecewise manner as made of several suprema θ_i of hit-or-misses, each being associated with an over-filter ψ_i, and then apply the optimisation to $\vee\theta_i\psi_i$.

References

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