Lattices of Numerical Functions

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1 Introduction

Up to now, when we where dealing with numerical functions, we have systematically considered the lattice $\overline{\mathbb{R}}^E$ of all possible functions from some space E into the totally ordered lattice $\overline{\mathbb{R}}$ (or $\overline{\mathbb{Z}}$, or into anamorphoses of $\overline{\mathbb{R}}$ or $\overline{\mathbb{Z}}$).This allowed us to work with a lattice where the $\vee$ and the $\wedge$ are the numerical ones at each point $x \in E$, and to postpone the discussion about the pertinence of such a lattice. However, such a framework is obviously too general. The numerical functions on $\mathbb{R}^n$, for example, carry number of exotic specimen that are irreductible to any digitization.

In this chapter, we propose two reductions that reflect two points of view on the physical word. Firstly we can base the approach on some disymmetry between foreground and background so that very narrow crestlines sharp peaks, and vertical cliffs be well represented. By so doing, we obtain the lattice of the upper semi continuous (in brief u.s. c.) functions that transposes to numerical functions the closed sets of the binary case. It permits to generalize Poisson points into Poisson pulses for example.

The second reduction puts the emphasis on the *digitability* of the functions and of the operations under study. This demands the continuity of both functions and operators. The goal is not trivial because the class of the continuous functions does not form a complete lattice. For example, the infimum of all functions $\{x^\alpha, 0 \leq x \leq 1, \ \alpha \in \mathbb{Z}_+\}$ is not a continuous function. We shall have to be more restrictive and focus on the classes of ω-continuous functions.

Which lattice suits the best for representing grey tone images? We just saw that the extreme generality of $\overline{\mathbb{R}}^E$ involves too many teratological functions, so that the whole class *must* be reduced as soon we want to structure it. For example, the requirements for a stochastic version require conditions compactness that cannot be fulfilled by lattice $\overline{\mathbb{R}}^E$, but which are satisfied by the lattice of the upper semi-continuous functions ([7], pp.468-469). More generally this lattice is convenient in all situations when the processing reduces to dilations

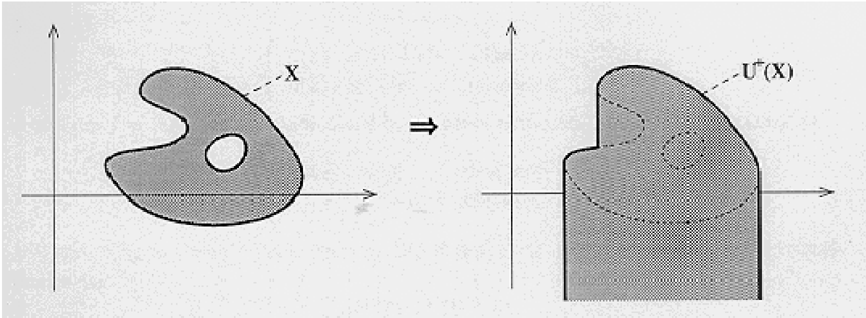


Figure 1: Umbra $U(X)$ of a set $X \subset \mathbb{R}\times\overline{\mathbb{R}}$

(e.g. testing random functions). On the other hand, it is not closed under substraction: the difference of two u.s.c. functions may be not continuous at all, and operators such that the top-hats mappings do not apply the lattice into itself. Finally, the lattices of ω-continuous functions are both regular enough and self-dual for the opposite (i.e. $f \in \mathcal{L}_\omega \Leftrightarrow -f \in \mathcal{L}_\omega$) so that they seems to be *the* general model to digital situations that refer to an underlying Euclidean space (video processing, microcopic images, tomography, etc..). But these lattices are incompatible with that of the semi-continuous functions: nobody is perfect !

2 Upper semi-continuous functions

2.1 Comments on functions and umbrae

Is it possible to identify the function lattice $\overline{\mathbb{R}}^E$, or some of its sub lattices, with a class of *sets* in the product space $E \otimes \overline{\mathbb{R}}$? This question will be expressed in a more precise way if we introduce the notion of an umbra.

Definition 1 *With every set X of the product space $E \otimes \overline{\mathbb{R}}$ there corresponds the set $U(X)$ of $E \otimes \overline{\mathbb{R}}$ defined by the relation to every function $f : E \longrightarrow \overline{\mathbb{R}}$ (and more generally to see Fig.1),*

$$U(X) = \{(x, z) \in E \otimes \overline{\mathbb{R}}, \quad z \leq z', (x, z') \in X\} \cup E_{-\infty}.$$

In particular, when set X is the graph of a function $f : E \longrightarrow \overline{\mathbb{R}}$ the associated umbra becomes

$$U(f) = \{(x, z) \in E \otimes \overline{\mathbb{R}}, f(x) \leq z\} \cup E_{-\infty}$$

Figure 1 illustrates this definitionAlthough we have

$$f \leq g \iff U(f) \subseteq U(g) \ ,$$

the correspondence "function - umbra" is not a bijection (an umbra and its topological closure can yield the same function, see exercise n° 8-1 below)).

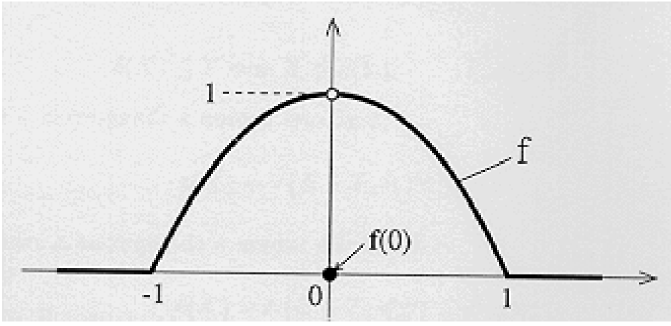


Figure 2: A function without a maximum (it is not u.s.c.).

2.2 Lattices of semi-continuous functions

Definition 2 *Given a topological space E, a numerical function $f : E \to \overline{\mathbb{R}}$ is* upper semi-continuous *(u.s.c.) when its umbra $U(f)$ is a closed set in $E \otimes \overline{\mathbb{R}}$.*

The use of semi-continuity appears as soon as extrema are involved in the analysis under study, at least in continuous cases. For example (Fig. 2), could we extract the maxima of the following function $f : \mathbb{R} \to \overline{\mathbb{R}}$:

$$f(x) = \begin{cases} 1 - x^2 & \text{when } 0 < |x| < 1, \\ 0 & \text{when } |x| \geq 1 \text{ or } x = 0 \end{cases} \qquad ?$$

Actually, the maximum of such a function, although it is bounded, *does not exist*. Conversely, as soon as we refer to the "maximum" of a function over a continuous space, we implicitely introduce the requirement that it is u.s.c. (or lower semi-continuous when looking for minima).

The class $\mathcal{F}_u$ of the upper semi-continuous functions $f : E \to \overline{\mathbb{R}}$ forms a complete lattice. In this lattice, the inf of a family $\{f_i\}$ is the function which admits, at each point $x \in \mathbb{R}^2$, the numerical inf of the $f_i(x)'$s, but the sup is the function whose umbra is the topological closure of the union of the umbrae of the f_i's. We have

$$\begin{aligned} \inf_i f_i &= \{f \in \mathcal{F}_u, U^+(f) = \bigcap_i U(f_i)\}, \\ \sup_i f_i &= \{f \in \mathcal{F}_u, U^+(f) = \overline{\bigcup_i U(f_i)}\}, \end{aligned}$$

a notation which shows that the lattice $\mathcal{F}_u$ and that of the closed umbrae can be *identified* . As a matter of fact, lattice $\mathcal{F}_u$ turns out to be the most direct transposition of the closed sets to numerical functions. The pulses, which belong to lattice $\mathcal{F}_u$, are sup-generating co-primes, but not strong ones. Lattice $\mathcal{F}_u$ is distributive, and infinite $\bigwedge$-distributive, but not infinite $\bigvee$-distributive. For the classical Choquet toplolgy on closed sets, lattice $\mathcal{F}_u$ is CCO.

In 1969, G. Matheron used the identificationbetween s.c.s. functions and umbrae to extend his random closed sets theory to the upper semi-continuous functions from an L.C.D space E into $\overline{\mathbb{R}}$ [3]. He proved that the class $\mathcal{F}_u$ is a *compact family* of the closed sets of $E \otimes \overline{\mathbb{R}}$. The topology on $\mathcal{F}_u$ is obtained as the restriction of Matheron topology, as dfined in chapter 2, to this specific class. Consequently, the open sets in $\mathcal{F}_u$ are generated by the parts of $\mathcal{F}_u$ whose elements f satisfy the two conditions:

$$\overline{X}_f(G) = \sup\{f(x), x \in G\} > b \quad \text{and} \quad \inf\left\{\overline{X}_f(G), G \supset K\right\} < a \qquad (1)$$

as G spans the open sets of E, and K its compact sets ($a, b \in \overline{\mathbb{R}}$). When both E and $\overline{\mathbb{R}}$ are equipped with a metric and that d stands for the product metric on $E \otimes \overline{\mathbb{R}}$, the topology defined by relations (1) results in the following criterion of convergence

Criterion 3 *A sequence $\{f_n\}$ converges towards f in $\mathcal{F}_u$ if and only if for all $x, t \in E \otimes \overline{\mathbb{R}}$, the sequence $d[(x,t), U(f_n)]$ converges in $\overline{\mathbb{R}}_+$.*

The expression of this criterion of convergence via the umbrae is geometrically more intuitive than the general formalism by means of neigborhoods and sub-sequences. We shall no nlonger develop this topology but indicate only three worthwhile results that must be quoted. They hold on cross sections, increasing mappings and Minkowski addition respectively.

- The upper semi continuous functions $f \in \mathcal{F}_u$ can be characterized by their cross sections

  $$X_t(f) = \{x : f(x) \geq t\} \qquad -\infty \leq t \leq +\infty$$

  which are closed and monotonically decreasing sets of E, i.e.

  $$t' < t \Longrightarrow X_{t'} \supseteq X_t \quad \text{and} \quad t' \uparrow t \Longrightarrow X_{t'} \downarrow X_t \ .$$

  Conversely ([7], p.426), any stack $\{X_t,\ -\infty \leq t \leq +\infty\}$ of closed sets satisfying these two conditions generates a unique u.s.c. function f with

  $$f(x) = \sup\{t : x \in X_t\} \qquad x \in E.$$

- An increasing mapping ψ on $\mathcal{F}_u$ is upper semi-continuous if and only if $f_n \downarrow f$ in $\mathcal{F}_u$ implies $\psi(f_n) \downarrow \psi(f)$.Here the adjective "upper semi-continuous" concerns of course the mapping under study. This property, that derives from prop.1-2-4 in [4], orients us towards *samplig techniques by covering.*

- If $g \in \mathcal{F}_u$ is a structuring function whose all cross sections are compact (except possibly at $-\infty$), then the Minkowski addition $f \to f \oplus g$ from $\mathcal{F}_u$ into itself is continuous, whereas the Minkowski substraction $f \to f \ominus g$ is upper semi-continuous only.

However, the limits of Lattice $\mathcal{F}_u$ are rapidly reached. Class $\mathcal{F}_u$ is not closed under difference, and does not allow us to model the residuals. Also, in lattice $\mathcal{F}_u$, the infinum $\bigwedge f_i$ is identical to numerical inf, whereas the supremum $\bigvee f_i$ is the *topological closure* of the numerical supremum; hence the symmetry between $\bigvee$ and $\bigwedge$ is lost. This is the reason why Minkowski addition, but not subtraction, is continuous. Now one cannot design an experiment able to bring to the fore such a distinction. In practice, one passes from a dilation to an erosion of function f by replacing it by $-f$, or by $m - f$, and negation is continuous. Is a continuity that no experiment will never distinguish from semi-continuity a worthwhile property of the model? Finally, $\mathcal{F}_u$ is not a vector space, and this is a pity, for linear techniques in image analysis are of barycentric type (e.g. convolution). But is it possible to construct a function lattice, sufficiently regular and which should accept some linear operations?

3 Gauges

Consider an arbitrary metric space E of distance d ($\mathbb{R}^n, \mathbb{Z}^n$, or conditional. versions of these spaces, planar graphs, etc.).With any numerical function f from space E into $\overline{\mathbb{R}}$ (or into $\overline{\mathbb{Z}}$) one can always associate a second function, ω say, from $\mathbb{R}_+$ into $\overline{\mathbb{R}_+}$ as follows:

$$\omega_f(h) = \sup\{|f(x) - f(y)| \quad y \in E,\ d(x,y) \leq h\}. \tag{2}$$

Clearly, equation (2) results in an increasing positive function. More precisely, it characterizes the *gauge of function f.* Remember [1] that a mapping $\omega : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ is said to be a *gauge*, if it is increasing and when we have

$$\omega(o) = 0 \qquad \text{and} \qquad \omega(u+v) \leq \omega(u) + \omega(v) \qquad \text{(sub-additivity)} \tag{3}$$

Indeed, let $|f(x) - f(y)|$ be an element involved in the supremum $\omega_f(u+v)$ of Eq (2), then $d(x,y) \leq u+v$ and there exists a closed ball $B_x(u)$ (resp. $B_y(v)$) of radius u and centre x (resp. radius v and centre y) such that $B_x(u) \cap B_y(v) \neq \emptyset$. Then for any point $z \in B_x(u) \cap B_y(v)$, we have that

$$|f(x) - f(z)| \leq \omega(u) \ \text{ and } \ |f(z) - f(y)| \leq \omega(v)$$

and by triangular inequality $|f(x) - f(y)| \leq \omega(u) + \omega(v)$. Then, by taking the supremum of the left member over x and y in E, under the condition $d(x,y) \leq u+v$, we obtain the sub-additivity inequality (3).

To prove the converse, we must start from a given gauge ω and exhibit a numerical function $b : E \to \overline{\mathbb{R}}$ that satisfies Eq.(2). Take for b the quantity

$$b\left(x\right)=\omega\left[d_{E}\left(0,x\right)\right]\qquad x\in E$$

where ω is the given gauge. We derive from the increasingness and the sub-additivity of ω that

$$\omega\left[d\left(o,y\right)\right]\leq\omega\left[d\left(o,x\right)+d\left(x,y\right)\right]\leq\omega\left[d\left(o,x\right)\right]+\omega\left[d\left(x,y\right)\right]$$

i.e.

$$b\left(x\right)-b\left(y\right)\leq\omega\left(d\left(x,y\right)\right)$$

as well as the similar inequality obtained by interchanging x and y, so that we can finally write

$$\left|b\left(x\right)-b\left(y\right)\right|\leq\omega\left(d\left(x,y\right)\right). \tag{4}$$

Therefore we can state the following

Theorem 4 *Given a metric space E and a numerical function $f : E \to \overline{\mathbb{R}}$, the quantity ω defined by the equation*

$$\omega_{f}(h)=\sup\{\left|f\left(x\right)-f(y)\right|,x,y\in E,d(x,y)\leq h\}.$$

is a gauge. Conversely, given a gauge ω, there exists always a numerical function $f : E \to \overline{\mathbb{R}}$ that satifies this equation.

Geometrically speaking, the sub-additivity condition (3) amounts to be a concavity reduced to the triplets (x, y, z) of coordinates that involve the origin (see exercise 8-2 below). Also, it is noticeable that Eq.(2) applies to both continuous or discrete spaces. Actually, when the gauge is continuous at the origin it serves to bridge the gap between the two cases (see section 6 of this chapter).

The gauge assiociated with function f allows us to express a nice property of finitness. Indeed the sub-additivity implies taht for all positive integers k we have $\omega(ku) \leq k\omega(u)$. Therefore if $\omega(u)$ is finite for some $u > 0$ and simultaneously $f(x) < \infty$ for some point x, then the finiteness of $f(x)$ extends to the whole space E

$$\{\exists u>0;\exists x\in E:\omega\left(u\right)<\infty\text{ and }f(x)<\infty\}\Leftrightarrow\{\forall h\in E,\ f(x+h)<\infty\}.$$

The gauge characterization involved in theorem (4) suggests to regroup the families of functions depending of a same gauge as follows

Definition 5 *Given a gauge ω, the set of functions $f : E \to \overline{\mathbb{R}}$ such that*

$$\omega_{f}\ \leq\omega$$

is called the ω-gauge class *and is denoted by $\mathcal{G}_\omega$.*

Clearly, the ω-gauge classes reduce as gauges ω decrease, and we have

$$\omega_1 \leq \omega_2 \Rightarrow \mathcal{G}_{\omega 1} \subseteq \mathcal{G}_{\omega 2}.$$

More generally they all satisfy the following three obvious properties

Proposition 6 *Consider the set of all ω-gauge families of functions $f : E \to \overline{\mathbb{R}}$. Then*

i) the constant functions are the only elements to be common to all ω-gauge classes;

ii) each class $\mathcal{G}_\omega$ is closed under vertical translation

$$a \in \mathbb{R} \text{ and } f \in \mathcal{G}_\omega \Rightarrow a + f \in \mathcal{G}_\omega$$

iii) each class $\mathcal{G}_\omega$ is symmetrical

$$f \in \mathcal{G}_\omega \Leftrightarrow -f \in \mathcal{G}_\omega$$

4 Gauges Based Lattices

4.1 Gauges and Numerical Lattices

Given a lattice $\mathcal{L}$, a subset $\mathcal{L}' \subset \mathcal{L}$ is said to be a *sub-lattice* of $\mathcal{L}$ when $\mathcal{L}'$ is closed under the supremum and the infinum of $\mathcal{L}$, and admits the same two extrema as $\mathcal{L}$. Then all concepts or mappings defined for $\mathcal{L}$, and which involve uniquely sup and inf, have a meaning over $\mathcal{L}'$. We will see now that such a property is satisfied by the ω-gauge families.

Theorem 7 *Given a metric space E and a gauge ω, the class $\mathcal{G}_\omega$ of the ω-gauge functions from E into $\overline{\mathbb{R}}$ is a sub-lattice of $\overline{\mathbb{R}}^E$.*

Proof. Let $\{f_i, i \in I\}$ be a family in $\mathcal{G}_\omega$; put $f = \bigvee f_i$. If $f(x) = +\infty$ for some $x \in E$, then $f = +\infty$, hence f belongs to $\mathcal{G}_\omega$. If, $f(x) < +\infty$, then $f(y)$, which is bounded by

$$\omega [d(x, y)] + f(x)$$

is finite, and we can write, for all $x, y \in E$:

$$f(y) - f(x) \leq \omega [d(x, y)] \quad \text{and as well} \quad f(x) - f(y) \leq \omega [d(x, y)] .$$

A similar result may also be obtained for $\bigwedge f_i$, which achieves the proof. ■

Theorem 7 is classically presented in the Lipschitz framework (see for ex. a proof in [1]). The property it states owes more to the compacity of $\overline{\mathbb{R}}$ than to its complete ordering, and this point is less known. As a counter example, take $0 < a < b < 1$ and the lattice $L = \{x : x \in \mathbb{R}, -\infty \leq x < a \quad \text{or} \quad b \leq x \leq 1\}$.

Consider the family of Lipschitz functions $f_i :]-\infty, a] \to L$ defined by $f_i(x) = x - \varepsilon_i$ with $\varepsilon_i \downarrow 0$. For $x = a$ and $y = a - \alpha$, $(\alpha > 0)$, we have $d(x, y) = \alpha$ but

$$\bigvee f_i(x) = b, \bigvee f_i(y) = a - x \quad \text{hence} \quad \left|\bigvee f_i(x) - \bigvee f_i(y)\right| = b - a + \alpha > \alpha!$$

Remark that no possible continuity of gauge ω intervenes in the proof of the theorem. This provides it with a large degree of generality that can be maintained in the next result, about the topology of the the gauge based lattices.

4.2 Topology for the Lattices $\mathcal{G}_\omega$

The criterion (**??**) will now allow us to derive the $\bigvee$ and $\bigwedge$ continuities for the G_ω lattices when the arrival space is $\overline{\mathbb{R}}$, and next when it is $\left(\overline{\mathbb{R}}\right)^n$, n a finite positive integer.

Theorem 8 *Let E be a metric space, ω be a gauge and $\mathcal{G}_\omega$ be the lattice of the ω-gauge functions from E into $\overline{\mathbb{R}}$. Then the unique topology that makes $\mathcal{G}_\omega$ CCO, with continuous $\bigvee$ and $\bigwedge$ is the topology of the pointwise convergence.*

Proof. Consider two distinct functions f and g of $\mathcal{G}_\omega$ with $f \not\leq g$. There exist at least one point $x \in E$ and a real member a with (for example) the strict inequalities

$$g(x) < a < f(x)$$

Introduce the two following elements f_0 and g_0 of $\mathcal{G}_\omega$:

$$f_0(y) = a - \omega\left[d(x, y)\right] \qquad g_0(y) = a + \omega\left[d(x, y)\right] \qquad \forall y \in E \tag{5}$$

Function f does not belong to the lower bounds of g_0, since $f(x) > a$, i.e. $f \notin M^{g_0}$. Similarly, we have $g \notin M_{f_0}$. Moreover, any function $s \in \mathcal{G}_\omega$ is either $\leq g_0$ (when $s(x) \leq a$) or $\geq f_0$ (when $s(x) \geq a$), so we can write $M^{g_0} \cup M_{f_0} = \mathcal{L}_\omega$. Therefore criterion **??** applies, and lattice $\mathcal{G}_\omega$ is CCO with continuous $\bigvee$ and $\bigwedge$ for a certain topology. One can find out this topology by means of a general characterization [5], but in the present case, it suffices to observe that $\mathcal{G}_\omega$ is a compact sub-lattice of the upper semi continuous functions, sub-lattice on which both topologies of Matheron and of the pointwise convergence coincide. Now, the $\bigvee$ is continuous for the first one, hence also in the pointwise sense. Similartly, $\mathcal{G}_\omega$ is a compact sublattice of the lower semi continuous function, hence $\bigwedge$ is continuous in the pointwise sense, which achieves the proof. $\blacksquare$

This result extends to ω-gauge functions $f : E \to \overline{\mathbb{R}}$ a theorem already established by G. Matheron in the Lipschitz case [theorem 6.5 in [5]]. The extension may be pursued further. First, space $\overline{\mathbb{R}}$ may be replaced by any compact segment $S \subset \overline{\mathbb{R}}$. Clearly, the ω-continuous functions from E into S form a compact quasi sub-lattice of G_ω ("quasi" because the extreme elements are not preserved). The proof may be reproduced integrally for them. Also, $\overline{\mathbb{Z}}$ may be substituted for $\overline{\mathbb{R}}$ and any subset of $\overline{\mathbb{Z}}$ for S. Second, the theorem extends to product lattices.

Corollary 9 *Theorem 8 remains true when $\overline{\mathbb{R}}$ is replaced by any product $\mathcal{T} = \Pi\{\mathcal{T}_j, j \in J\}$ of closed subsets $\mathcal{T}_j$ of $\overline{\mathbb{R}}$ or of $\overline{\mathbb{Z}}$.*

Proof. As previously, consider two distinct functions f and g of $\mathcal{T}$, i.e. $f = \{f_j,\ j \in J\}$ and $g = \{g_j,\ j \in J\}$. There exists at least one label $v \in J$ such that $f_v \neq g_v$, with $f_v(x) > g_v(x)$, strictly, for a point $x \in E$. Lattice $\mathcal{T}_v$ enters the framework of theorem 8, which determines two distinct functions f_{v_0} and g_{v_0} from equations 5. Let then f_0 be the function $E \to \mathcal{T}$ whose label v is equal to f_{v_0}, and whose all other components f_j, $j \in J$, $j \neq v$ coincide with the inf in the corresponding lattice $\mathcal{T}_j$. Similarly, define g_0 to be the function equal to g_{v_0} for $j = v$ and equal to the sup in $\mathcal{T}_j$ for all $j \neq v$. The criterion of proposition **??** is still satisfied for f_0 and g_0, which results in the corollary. ■

5 Numerical Lattices Characterization

The three properties of proposition (6) depend obviously on the Euclidean metric on the arrival space $\overline{\mathbb{R}}$, but are true for any distance on E, and their expressions do not involve this underlying distance on E. Conversely, if we take for E an arbitrary space that does not need to be metric, and if we assume only the three conditions of proposition (6) are axioms to be satisfied for a numerical lattice $\mathcal{T}$ of functions $f : E \to \overline{\mathbb{R}}$, what are the characteristics of $\mathcal{T}$? We will treat this problem by following an approach partly sketched in ([5], p.133). With each point x_0 of E associate the function

$$\rho_{x_0} = \vee\{f : f \in \mathcal{T},\quad f(x_0 \leq 0)\} \tag{6}$$

which belongs to $\mathcal{T}$. From axiom i), function $f = 0$ belongs also to $\mathcal{T}$, hence

$$\rho_{x_0}(x_0) = 0 \text{ and } \rho_{x_0}(y) \geq 0.$$

Then from axiom ii),we have for all $a \in \mathbb{R}$

$$\vee\{f : f \in \mathcal{T},\quad f(x_0 \leq a)\} = a + \rho_{x_0}.$$

Moreover, we have $\rho_x(y) = \rho_y(x) = \rho(x, y)$. Indeed, let f be a function that participates to the supremum ρ_x. Put $\varepsilon_1 = 0 - f(x)$ and $\varepsilon_2 = \rho_x(y) - f(y)$. By axiom ii), any function $f + \max(\varepsilon_1, \varepsilon_2) + \varepsilon$, with $\varepsilon > 0$,belongs to lattice $\mathcal{T}$, so that when $\varepsilon_1, \varepsilon_2$ and $\varepsilon \downarrow 0$, we obtain

$$0 = \wedge\{f(x)\ \ ;\ \ f \in \mathcal{T},\ f(y) \geq \rho_x(y)\}.$$

By multiplying the two members by -1, and applying axiom iii), we find $0 = \vee\{f(x)\ \ ; f \in \mathcal{T}, f(y) \geq -\rho_x(y)\}$, or equivalently (axiom ii))

$$\rho_x(y) = \vee\{f(x)\ ;\quad f \in \mathcal{T},\quad f(y) \geq 0\} = \rho_y(x)$$

Take now an arbitrary, but fixed, function $f \in \mathcal{T}$ and consider the quantity $f(x_0) + \rho_{x_0}$. It is equal to $f(x_0)$ at point x_0, and $\geq x_0$ everywhere else. Hence we have

$$f = \wedge\{f(x_0) + \rho_{x_0},\quad x_0 \in E\} = \vee\{f(x_0) - \rho_{x_0},\quad x_0 \in E\} \tag{7}$$

the second equation deriving from the first one by duality (axiom iii)).Therefore, lattice $\mathcal{T}$ is generated by the prime elements $a+\rho_{x_0}$, as well as by the co-prime elements $a-\rho_{x_0}$

$$\begin{aligned} P &= \{a+\rho_{x_0},\ a\in\mathbb{R},\ \ x_0\in E\} \\ Q &= \{a-\rho_{x_0},\ a\in\mathbb{R},\ \ x_0\in E\} \end{aligned}$$

It is easy to see that lattice $\mathcal{T}$ satisfies the conditions of criterion (**??**): when f and g are different, with $f\not\leq g$, there are at least one point $x_0\in E$ and a real member a with $g(x_0)<a<f(x_0)$. Then the two functions $f_0=a-\rho_{x_0}$ and $g_0=a+\rho_{x_0}$ play the same role as in Eq (5). Therefore the unique topology for which the $\vee$ and the $\wedge$ operations are continuous is that of the pointwise convergence. We can state the following

Theorem 10 *Let E be an arbitrary space. Every family of mappings $f:E\to\overline{\mathbb{R}}$ which is a numerical lattice,$\mathcal{T}$ say, and which satisfies the three conditions of proposition (6) is CCO with continuous $\vee$ and $\wedge$ for the topology of the pointwise convergence.*

This lattice is both sup-generated by the co-primes $\{a-\rho_{x_0},\ a\in\mathbb{R},\ \ x_0\in E\}$ and inf-generated by the primes $\{a+\rho_{x_0},\ a\in\mathbb{R},\ \ x_0\in E\}$, where $\rho_{x_0}=\vee\{f:f\in\mathcal{T},\ \ f(x_0\leq 0)\}$.

Equations (7) can be interpreted geometrically by considering the familly $\{-\rho_x,\ x\in E\}$ as a structuring function of associated dilation δ_ρ and erosion ε_ρ. Then the two equations (7) indicate that all functions $f\in\mathcal{T}$ are both dilated by δ_ρ and eroded by ε_ρ (hence open by $\varepsilon_\rho\ \delta_\rho$ and closed by $\delta_\rho\varepsilon_\rho$). In particular each ρ_x, which is an element of $\mathcal{T}$, is thus open and closed by itself. This point is illustrated by figure (3) in the translation invariant case.

When space E is metric, the gauge of lattice $\mathcal{T}$ admits an expression as suprema of $\rho(x,y)$, since we have

$$\omega(h)=\vee\{\rho(x,y),\ \ d(x,y)\leq h\}$$

In the euclidean space $\mathbb{R}^n$ in particular, if we assume lattice $\mathcal{T}$ to be closed under translation and rotation, then $\rho(x,y)=\rho(0,d(y-x))$ i.e. since ω is increasing, $\omega(h)=\rho(0,h)$. To summarize, we can state the following

Proposition 11 *Lattice $\mathcal{T}$ is the image of lattice $\overline{\mathbb{R}}^E$ under dilation δ_ρ by function ρ, as well as the image of under erosion ε_ρ by $-\rho$ i.e.*

$$\mathcal{T}=\delta_\rho(\overline{\mathbb{R}}^E)=\varepsilon_\rho(\overline{\mathbb{R}}^E)$$

In particular, when E is the Euclidean space $\mathbb{R}^n$, then lattice $\mathcal{T}$ is generated by all suprema of translates of the gauge $\omega(h)=\rho(0,h)$ over the space $\mathbb{R}^n\otimes\overline{\mathbb{R}}$, and, equivalently, by all infima of translates of $-\rho$ over the space $\mathbb{R}^n\otimes\overline{\mathbb{R}}$.

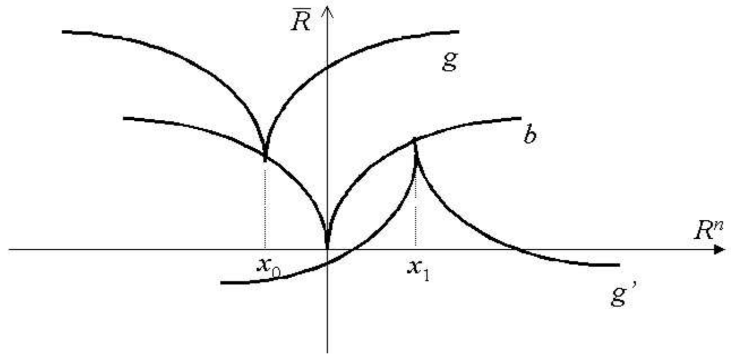


Figure 3: Function b is a 1-D gauge. The super-graph of b is generated by the union of the super-graphs of the translates of b,and the sub-graph of b by the union of the sub-graphs of its opposite translates.

Clearly, theses results are still valid in the digital case, mutatis mutandis, i.e. among others by choosing a translation invariant metric on $\mathbb{Z}^n$.

The above characterization theorem (10) is strong because it does not suppose any property to space E, but in compensation we have no mean to control the level of regularity of the lattices it deals with. Since space E is not a priori equipped with a metric, we cannot compare function ρ_x with a pre-existing gauge ω. Now, some lattices $\mathcal{T}$ may contain hightly irregular elements. Take for example the class of all functions $f : \mathbb{R}^1 \to \overline{\mathbb{R}}$ such that $0 \leq |f(x) - f(y)| \leq 1$. They satisfy the conditions of theorem (10), therefore the $\vee$ and the $\wedge$ are continuous operators. Nevetheless the function $f(x) = 0$ when x is rational and $f(x) = 1$ when not, belongs to the lattice, although it has no physical meaning (there is no experimental means to sample such a function). The same example shows also that translation is not a continouous operator in the lattices satisfying theorem (10). Take for example the function $f(x) = 0$ when $x \leq 1$ and $f(x) = 1$ when $x > 1$, and the sequence $f_n(x) = f(x + \frac{1}{n})$. As $n \to \infty$ we have $f_n(1) \to 1 \neq f(1) = 0$.

The case of this lattice example teaches us still more. Clearly it admits for gauge the function $\omega(h) = 0$ for $h = 0$ and $\omega(h) = 1$ for $h \neq 0$. Therefore neither the presence of a gauge, nor the existence of continuous $\vee$ and $\wedge$ prevent the lattice form higly irregular elements. We have to look for another idea, and we will reach the goal by imposing some continuity condition to the gauges.

6 Moduli of continuity

In fact we wish that, when space E is metric, all functions of the lattice $\mathcal{T}$ under study be correctly approached when we sample them. If a stands for the grid

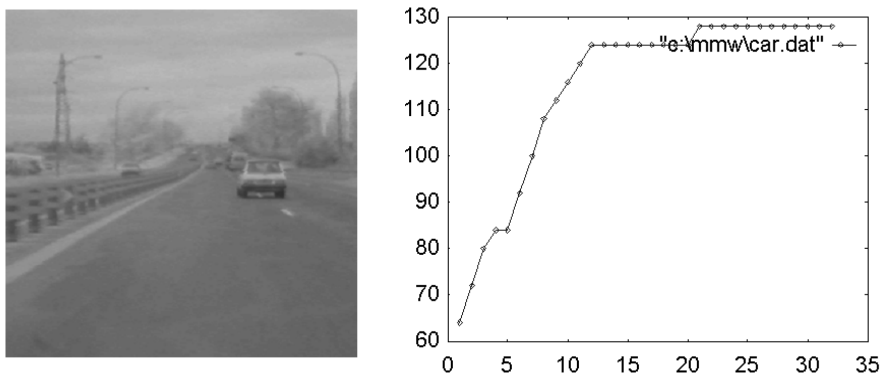


Figure 4: a) Video image ; b) Corresponding modulus ω, for the digital square metrics.

spacing, we want to be able to estimate all values $f(x), x \in E,\ f \in \mathcal{T}$ with an error that tends towards zero with spacing a. Formally speaking, this means exactly that the gauge ω_f of lattice $\mathcal{T}$ must be *continuous at the origin.*This leads to the following definition

Definition 12 *When the gauge* ω_f *of a function* f *is continuous at the origin then it is called a* modulus of continuity*. The family of all functions* $f : E \to \overline{\mathbb{R}}$*(or* $\overline{\mathbf{Z}}$*) whose moduli* ω_f *are smaller than a given gauge* ω *is said to be* equicontinuous of modulus ω*, or* ω*-continuous, and is denoted by the symbol* $\mathcal{L}_\omega$ *.*

Not only each function $f \in \mathcal{L}_\omega$ is uniformly continuous, but this uniform continuity is of the same type for all the $f \in \mathcal{L}_\omega$. Indeed, according to a classical result, lattice $\mathcal{L}_\omega$ is a compact sub-space of $\overline{\mathbb{R}}^E$ for the topology of the uniform distance (Ascoli 's theorem, p.99 in [1]). The modulus of continuity describes, in some sense, the degree of regularity of the function f under study. This approach can be compared with the use of the variogram, or the covariance, in the order two analysis. But, unlike the variogram, which provides the *quadratic mean* of the variation between two points, here we take into account the *maximum* absolute difference. One can see from Fig.4 how the range of ω reflects the sizes of the features in f, and its asymptote their dynamics.

The most usual prototypes of classes $\mathcal{L}_\omega$ are the *Lipschitz families,* and also the first ones historically speaking (H. Lebesgue introduced the moduli of continuity in 1910, but R. Lipschitz had proposed condition (8) half a century before). Each Lipschitz class is obtained by taking for ω a constant value $k > 0$, so that we have

$$|f(x) - f(y)| \leq k.d(x, y) \tag{8}$$

All above properties of the ω-gauge classes $\mathcal{G}_\omega$ are of course still valid for the ω-continuous classes $\mathcal{L}_\omega$, with in addition the following one

Proposition 13 *For each modulus ω, if function $f \in \mathcal{L}_\omega$ is finite at point $x \in E$, then it is finite everywhere.*

Proof. The sub-additivity of the modulus, added to its continuity at the origin, imply that modulus $\omega(h)$ is finite for $h < \infty$. Therefore when $f(x) < \infty$, then $f(y) < \infty$ for all points $y \in E$. ■

Moreover, for every lattice $\mathcal{L}_\omega$, and on any compact sub-set X of E, the pointwise convergence of a sequence f_n of functions towards function f is equivalent to the uniform convergence, and to Matheron topology convergence of criterion (3). Also, in the Euclidean cases, translation and rotation are continuous operation: things become really more regular.

7 Dilations and Increasing operators

We will now study the dilations that act on the lattice $\overline{\mathbb{R}}^E$ of the numerical functions from E into $\overline{\mathbb{R}}$. As we saw in chapter 1, the lattice $\overline{\mathbb{R}}^E$ is *sup-generated* by the pulse functions $u_{x,t},\ x \in E, t \in \overline{\mathbb{R}}$:

$$u_{z,t}(y) = t \quad \text{when} \quad y = z \quad \text{and} \quad u_{z,t} = -\infty \quad \text{otherwise,}$$

and every function $f \in \overline{\mathbb{R}}^E$ admits a decomposition

$$f = \bigvee \{u_{z,t} \quad , \quad z \in E \quad , \quad t \leq f(z)\}.$$

Let $\delta : \overline{\mathbb{R}}^E \to \overline{\mathbb{R}}^E$ be a dilation on $\overline{\mathbb{R}}^E$. The transforms $\delta(u_{z,t})$ of the pulses are in turn sup-generators in the space image $\delta\left(\overline{\mathbb{R}}^E\right)$ since

$$\delta f = \bigvee \{\delta(u_{z,t}), \quad z \in E, \quad t \leq f(z)\} \qquad f \in \overline{\mathbb{R}}^E$$

It is more convenient, here, to introduce the *reciprocal structuring function* $g_{y,t}$, namely

$$g_{y,t}(z) = \delta(u_{z,t}(y)) \qquad y, z \in E$$

In the following, we shall focus on the dilations whose structuring functions commute with translation on $\overline{\mathbb{R}}$, i.e. such that

$$g_{z,t} = g_{z,0} + t \qquad (\text{with } g_z = g_{z,0}).$$

Then the expression of the dilate at point $y \in E$ reduces to

$$(\delta f)(y) = \bigvee \{g_y(z) + f(z) \quad , \quad z \in E\} \tag{9}$$

All the dilations encountered in practice are particular cases of the last representation (9).

7.1 Dilations on $\mathcal{L}_\omega$

We now focus on the sub lattice $\mathcal{L}_\omega$ of the ω-continuous functions on $\overline{\mathbb{R}}$, and we wonder about the image $\delta(\mathcal{L}_\omega)$ of $\mathcal{L}_\omega$ under a dilation of the type Eq. (9). Pertinent results are obtained when the variation of the g_x's over space E is provided with a certain regularity, that we will formalize when space E is metric.

Proposition 14 *Let $\mathcal{G}$ be a family of numerical functions over a metric space E,*

i/ which admit a common finite upper bound

ii/ whose cross sections

$$X_t(g) = \{y : g(y) \geq t\} \qquad g \in \mathcal{G}$$

are compact, for all $t \in \overline{\mathbb{R}} \backslash \{-\infty\}$.

If g_ρ stands for the dilate of g by a circular cylinder of radius ρ and height $k\rho$, i.e.

$$g_\rho(z) = \sup\{g(y), \quad y \in B_\rho(z)\} \ + \ k\rho$$

Then the quantity

$$h(g, g') = \inf\left\{\rho : g \leq g'_\rho, \quad g' \leq g_\rho\right\} \qquad g, g' \in \mathcal{G}$$

is a Hausdorff type distance on $\mathcal{G}$.

[easy proof].

Consider now a structuring family $\{g_x, x \in E\}$, which is supposed to satisfy the two conditions of proposition 14, and whose variation over E is governed by a modulus of continuity ω', i.e.

$$h(g_x, g_y) \leq \omega'\left[d(x, y)\right] \qquad x, y \in E \tag{10}$$

we may state [8][9]:

Theorem 15 *Let E be a metric space, and $\delta : \overline{\mathbb{R}}^E \to \overline{\mathbb{R}}^E$ be a dilation on the lattice $\overline{\mathbb{R}}^E$, whose structuring functions $\{g_x, x \in E\}$ admit a modulus of continuity ω' (i.e. satisfy 10). Then δ maps the sub-lattice L_ω of the ω-continuous functions in the sub-lattice $L_{(\omega+k)\circ\omega'}$ of the $(\omega + k) \circ \omega'$-continuous functions.*

Proof. Let $f \in \mathcal{L}_\omega$. Put $h(g_x, g_y) = h$. At point y, we have:

$$(\delta f)(y) = \sup\{f(z) + g_y(z), z \in E\}.$$

But $g_y(z) \leq \sup\{g_x(u), u \in B_h(z)\} + \ kh$ (Hausdorff distance) and $f(z) \leq f(u) + \omega(d(z, u))$. Hence, we have

$$\begin{aligned} (\delta f)(y) \quad &\leq \quad \sup\{f(u) + g_x(u) + \omega(d(z, u)) \quad ; \quad z \in B_h(u), u \in E\} + \ kh \\ &\leq \quad \sup\{f(u) + g_x(u), +u \in E\} + \omega(h) + kh = (\delta f)(x) + \omega(h) + kh \end{aligned}$$

and the similar inequality, by interverting x and y. Finally:

$$|(\delta f)(y) - (\delta f)(x)| \leq \omega(h) + k(h) \leq (\omega + k) \circ \omega'(d(x, y))$$

■

7.2 Particular cases

1. Suppose E to be affine and take for g_x the translate by vector x of the structuring function g associated with the origin. Then $h(g_x\,, g_y) = d(x, y)$ and $(\omega + k) \circ \omega' = \omega$. The dilations that are translation invariant preserve all equicontinuous lattices $\mathcal{L}_\omega$.
2. Take for g_x a *flat* structuring function, of compact support K_x, i.e.

$$\begin{array}{lcccc} g_x\,(y) & = & 0 & \text{when} & y \in K_x \\ g_x\,(y) & = & -\infty & \text{when} & \text{not} \end{array}$$

Then the expression 9 of a dilation reduces to

$$(\delta f)\,(y) = \bigvee \{f\,(z)\,, z \in K_y\} \tag{11}$$

where the geometrical role of the (variable) structuring elements $\{K_y, y \in E\}$ appears clearly. The dilations of the type 11, which are said to be flat, exhibit a number of remarkable features. For a flat dilation δ of structuring elements $\{K_x, x \in E\}$, with

$$h\,(K_x, K_y) \leq \omega'\,[d\,(x, y)]\,,$$

where h is the set-oriented Hausdorff distance, the theorem proves that any ω-continuous function is transformed into a $\omega \circ \omega'$-continuous one. In particular, when $\omega' \leq$ Identity, δ maps every $\mathcal{L}_\omega$ into itself. This case occurs for example when E is affine and $K_x = K_0 + x, x \in E$ (translation invariance), or also when $K_x = \{K_0 + x\} \cap Z$ where Z is a rectangular window.

Theorem 15, which has been stated for dilations admits by duality a similar version for erosions, and of course extends to any inf of dilations which have the same modulus ω'. Another instructive feature concerns the structuring functions, for which no continuity is required. For example, the two conditions of proposition 14 may be satisfied by upper semi continuous functions.

7.3 Continuity and increasing operators

For the sake of pedagogy, we will treat the "flat" case only, which the most used in applications.

Proposition 16 *Let E be a metric space (distance d), $K : E \to \mathcal{K}(E)\backslash\emptyset$ be a structuring element such that*

$$h\,[K(x), K(y)] \leq \omega'\,[d(x, y)]$$

(h, Hausdorff distance) for some modulus ω', and let $\delta : E \to \overline{\mathbb{R}}$ be the dilation of structuring element K. Then, for each modulus ω, the mapping $\delta : L_{\omega \to} L_{\omega \circ \omega'}$ is continuous.

Proof. Given an arbitrary point $x \in E$, consider a family f_n in $\mathcal{L}_\omega$ with $f_n \to f$ for the pointwise convergence. We draw from theorem 8 that

$$\bigvee \{f_x(y), y \in K_n(x)\} \to \bigvee \{f(y), y \in K(x)\}.$$

Since point x is arbitrary in E the pointwise convergence of $\delta(f_n)$ results, hence the continuity of δ.

■

Corollary 17 *The class generated by finite sup, inf and composition product of dilations and erosions whose structuring elements admit a modulus of continuity is composed of* continuous *increasing operators. When all the moduli of the structuring elements are anti-extensive, then these increasing operators each map L_ω into itself.*

[Easy proof].

Despite the assumption of finiteness (which could be overcome by supplementary hypotheses of compactness for the K's), this corollary ensures the continuity for a comprehensive number of operators in Mathematical Morphology, and among others for the morphological filters (openings, closings, their products and the alternating sequential filters). It shows, a contrario, that semi-continuity arises from *rapid variations* of the structuring elements, but not from the substitution $\bigvee \to \bigwedge$.

We conclude this section by brief comments about linear operators on the $\mathcal{L}_\omega$. Concerning convolution, one easy proves the following

Proposition 18 *Let $g(dh)$ be a measure such that $\int_E |g(dh)| \leq 1$. Then the convolution by g maps each L_ω into itself and is continuous.*

Consequently, all the half residuals of the operations (i.e. the difference between a function and its transform) described by corollary 17 map each $\mathcal{L}_\omega$ into itself and are continuous (e.g. the top hat mappings). An approach with variable kernels $g(dh)$ could be developed in a way similar to what we did for dilations. It should lead to similar results.

8 Exercises

8.1 1) Threshold Mapping and Semi-continuous Functions

The *threshold mapping* defined as follows:

$$[\psi(f)](x) = \begin{cases} f(x) & \text{when } f(x) \geq 1, \\ -\infty & \text{otherwise.} \end{cases} \tag{12}$$

This operation is shown on Fig. 5.

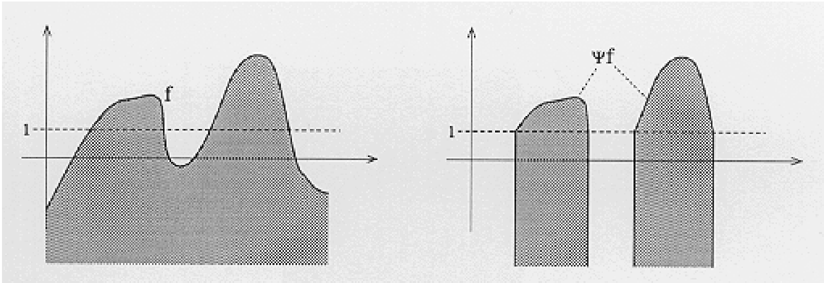


Figure 5: The threshold mapping ψ.

In set terms, the transformation ψ consists in intersecting the umbra $U(f)$ by the closed half space

$$E_1 = \{(x,y), x \in E, z \geq 1\},$$

and in taking the umbra of the result:

$$U(\psi(f)) = U[E_1 \cap U(f)] \cup E_{-\infty}. \tag{13}$$

Show by means of an example that the limit of the thresholds of a converging sequence of functions is not the threshold of the limit function.

[If functions and umbrae were equivalent, then the two algorithms (12) and (13) should give the same result. Let's try and apply the two algorithms to the sup of the following family (see Fig. 6):

$$\begin{cases} f_i(x) &=& 1 - 1/i & \text{when } |x| \leq 1, \\ f_i(x) &=& -\infty & \text{otherwise.} \end{cases}$$

If the sup f of this family is understood in the sense of the function lattice, it is equal to:

$$\begin{cases} f(x) &=& 1 & \text{when } |x| \leq 1, \\ f(x) &=& -\infty & \text{otherwise,} \end{cases}$$

and according to the rel.(12), $\psi f = f$. But if the sup is understood in the sense of the umbrae lattice, i.e.

$$U(f) = \bigcup_i U(f_i),$$

then from rel.(13), we derive $U[\psi(f)] = E_{-\infty}$, i.e. $\forall x \in E, \psi f(x) = -\infty$. In other words, in the Euclidean case, the function lattice and the set oriented lattice of umbrae are not equivalent at all. Nevertheless, in the discrete case of functions $f : \mathrm{Z}^n \longrightarrow \mathrm{Z}$, the two approaches coincide and one can transpose the way of reasoning from sets to functions.]

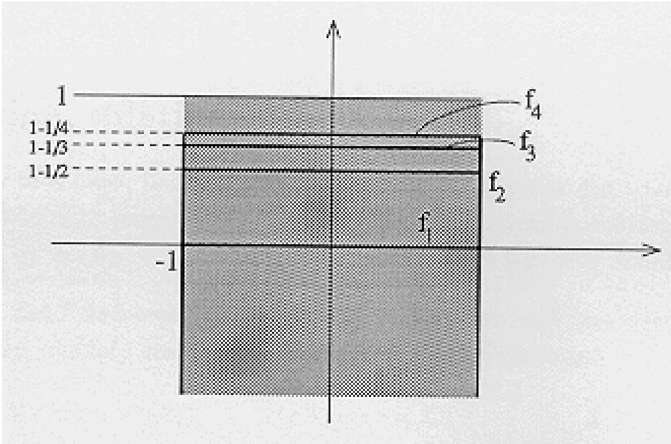


Figure 6: The family of functions (f_i).

8.2 2) Gauges and concave functions

The aim of this exercise is to determine a geometrical condition that a given function must fulfill to be a gauge. Firstly we remind the definition and a property concerning the concavity of a function.

Definition 19 *A finite numerical function, defined on an interval D of $\mathbb{R}$ is concave when its subgraph in the product space $\mathbb{R}^2$ is a convex set, i.e. when*

$$f(\alpha_1 x_1 + \alpha_2 x_2) \geq \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

for all $\alpha_{1,}\alpha_2 \geq 0$ such that $\alpha_{1+}\alpha_2 = 1$

To say that f is concave is equivalent to saying that the function ω

$$\omega(x,y) = \frac{f(y) - f(x)}{y - x} \qquad \text{(where } x \neq y\text{)} \tag{14}$$

is decreasing with respect to each of the variables.Geometrically speaking, rel.(14) amounts to saying that for any triplet $(x, y, z) \in D$ with $x < y < z$, the point $(y, f(y))$ of $\mathbb{R}^2$ lies above the segment of extremities $(x, f(u))$ and $(z, f(z))$, as depicted in figure 7a.

Every increasing and concave function ω such that $\omega(o) = 0$ is a gauge, since the convexity rel.(14) implies that

$$\omega(u + v) - \omega(o + v) \leq \omega(u) - \omega(o) \tag{15}$$

Conversely, the sub-additivity condition is less demanding and turns out to be a concavity reduced to the triplets (x, y, z) of coordinates that involve the

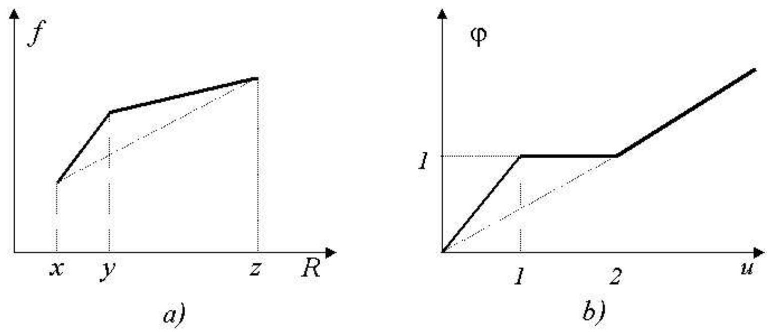


Figure 7: a)Concave function b)Example of a gauge that is not concave.

origin. For example the function

$$\begin{aligned}\omega(u) &= u \qquad 0 \leq u \leq 1\\ \omega(u) &= 1 \qquad 1 \leq u \leq 2\\ \omega(u) &= u/2 \qquad u \geq 2\end{aligned}$$

which is non concave (see fig.7b) satisfies rel.(15) and is continuous at the origin, hence is a modulus of continuity.

However, the restricted concavity of rel.(15) is already informative. It rejects for example all functions $\omega(u)$ where the behaviour near the origin is $u^{\alpha}+0(u^{\alpha})$, as soon as $\alpha < 1$.

8.2.1 Lipschitz class on an arbirary space

This exercise, due to G. Matheron ([5], p. 131), prolongates theorem (10). We still assume that the three properties of proposition (6) are satisfied for some lattice $\mathcal{T}$, and we suppose in addition that if $f(x) < \infty$, $f \in \mathcal{T}$, then for all points $y \in E$ we have $f(y) < \infty$.

1- show that $\rho_x(y)$, as defined by relation (6) is equal to $d(x,y) = d_1(x,y) \vee d_2(x,y)$ with

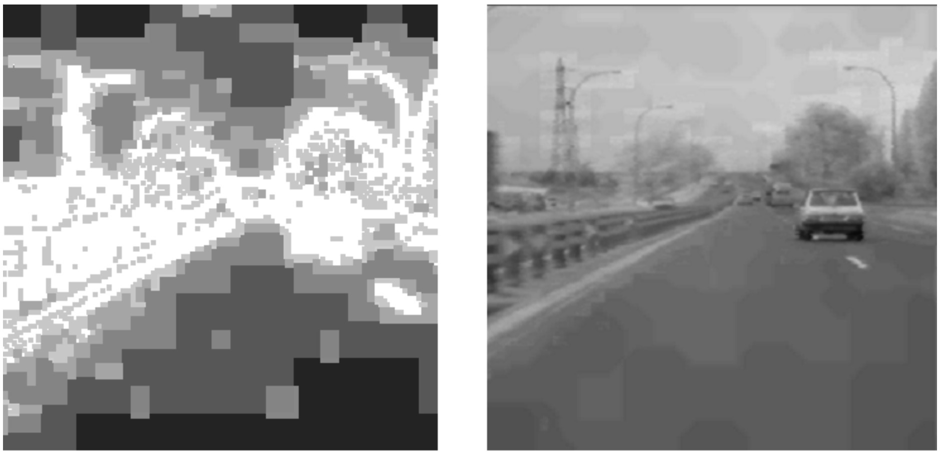
$$d_1(x,y) = \wedge\{a : a \in \mathbb{R}_+, \forall f \in \mathcal{T},\ f(x) \leq f(y) + a\}$$

2-Prove that d is a distance on E, and derive that lattice $\mathcal{T}$ is Lipschitz for distance d,i.e.

$$f \in \mathcal{T} \Rightarrow |\ f(x) - f(y)| \leq d(x,y)$$

8.3 Sampling and $\omega-$continuity

This exercise, due to J. Serra [9], deals with subsampling. *Starting from the datum of a digital image f, what is the minimal number of values of f to be*



a) b)

Figure 8: a) sampling zones derived from the inverse modulus r b) sampled image obtained from pattern a)

kept in order to estimate it everywhere with a given accuracy, and where must we implant the sampling points?

Consider f as a realization of a ω-continuous random function, and introduce a *local version* of the modulus ω by associating, with each point x, the maximum variation of f over the closed ball $B_x(r)$ of radius r and centered at point x:

$$\omega_x(r) = E\left[\max\left\{\left(\delta_h f - f\right)(x)\ ; \left(f - \varepsilon_h f\right)(x)\right\}\right] \qquad (16)$$

where δ_h and ε_h are the dilation and the erosion by ball B_h.

Second, consider the larger inverse $r_x(\omega)$ of $\omega_x(r)$. The value $r_x(\omega)$ is the size of the maximum ball centered at x such that the variation, in the sense of Eq. (16) is $\leq \omega$. Set accuracy ω to a fixed value, ω_0 say; hence $r_x(\omega_0) = r(x)$ becomes a numerical function of x only. The sampling protocol is then achieved by a downstream approach which allocate a sample density inverse to function r.

For a numerical illustration, start from a digital image of $2^i \times 2^i$ pixels (Fig. 2), with $i = 8$. The largest possible grid $G(i)$ has a spacing $2^i = 256$, and four points at the four corners of the image. The gray scale ranges over 256 levels, and the accuracy ω_0 is fixed to be equal to 10 levels. The cross section

$$X(i) = \left\{x : r(x) \leq 2^i\right\}$$

of r, corresponds to the flatest zones of the image under study. So they are sampled with the largest grid, i.e. reduced to the set

$$Y(i) = X(i) \cap G(i)$$

The points of $Y(i)$ admit a certain zone of influence $k(i)$, such that the dilate $\delta_{k(i)}[Y(i)]$ indicates the portion of the space "known" from sampling $Y(i)$. Iterate, by putting

$$\begin{array}{rcll} X(i-1) & = & \left\{x : h(x) \leq 2^{i-1}\right\} & \setminus \quad \delta_{k(i)}[Y(i)] \\ Y(i-1) & = & X(i-1) \cap G(i-1) & . \end{array}$$

Function $k(i)$ is calculated to be $\leq 2^i$ and to make contiguous the zones of influence, as i varies. For square grids, for example, one can take:

$$k(i) + k(i-1) = 2^i - 1.$$

These conditions lead to a pixel reduction by four in the example of fig. 2. In terms of data compression, such a result is acceptable, but not outstanding. However, by extending the samples in their respective zones of influence, one generates the new image f^*shown in Fig. 2b, so that for all treatments ψ designed by corollary 12 (anti-extensive case), we still have

$$E\left[\max\left|(\psi f)(x) - (\psi f^*)(x)\right|\right] \leq \omega_0$$

which is not a trivial result.

8.4 Robust Lattices

Independently of the CCO topology of $\overline{\mathbb{R}}$, introduce an ecart d_T on $\overline{\mathbb{R}}$, such that $x_i \to x$ in $\overline{\mathbb{R}}$ implies $d_T(x_i, x) \to 0$. This ecart will be said to be *robust* for $\overline{\mathbb{R}}$, when for all pair $\{a_i\}$ and $\{b_i\}$, $i \in I$ of elements of $\overline{\mathbb{R}}$, the two inequalities

$$\begin{array}{l} d_T \ \{\bigvee a_i, \bigvee b_i\} \leq \sup\{d_T(a_i, b_i)\} \\ d_T \ \{\bigwedge a_i, \bigwedge b_i\} \leq \sup\{d_T(a_i, b_i)\} \end{array}$$

are true [8].

1- Show that ecart d_T is robust if and only if we have

$$a \leq x \leq y \leq b \ \ in\ \overline{\mathbb{R}} \Rightarrow d_T(x, y) \leq d_T(a, b)$$

2- Show, by means of a counter example, that the CCO topology of $\overline{\mathbb{R}}$ is an unavoidable requirement for the previous result.

2-Prove that theorem 8 remains valid when ecart $\mid f(x) - f(y)|$is replaced by $d_T(x, y)$.

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