A Lattice Approach to Image Segmentation

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Summary

After a formal definition of segmentation as the largest partition of the space according to a criterion σ and a function f, the notion of a morphological connection is reminded. It is used as an input to a central theorem of the paper (Theorem 8), that *identifies* segmentation with the connections that are based on connective criteria. Just as connections, the segmentations can then be regrouped by suprema and infima. The generality of the theorem makes it valid for functions from any space to any other one. Two propositions make precise the AND and OR combinations of connective criteria.

The soundness of the approach is demonstrated by listing a series of segmentation techniques. One considers first the cases when the segmentation under study does not involve initial seeds. Various modes of regularity are discussed, which all derive from Lipschitz functions. A second category of examples involves the presence of seed around which the partition of the space is organized. An overall proposition shows that these examples are a matter for the central theorem. Watershed and jump connection based segmentations illustrate this type of situation. The third and last category of examples deals with cases when the segmentation occurs in an indirect space, such as an histogram, and is then projected back on the actual space under study.

The relationships between filtering and segmentation are then investigated. A theoretical chapter introduces and studies the two notions of a *pulse opening* and of a *connected operator*. The conditions under which a family of pulse openings can yield a connected filter are clarified. The ability of segmentations to generate pyramids, or hierarchies, is analyzed. A distinction is made between weak hierarchies where the partitions increase when going up in the pyramid, and the strong hierarchies where the various levels are structured as semi-groups, and particularly as granulometric semi-groups.

The last section is based on one example, and goes back over the controversy about "lattice" versus "functional" optimization. The problem is now tackled via a case of colour segmentation, where the saturation serves as a cursor between luminance and hue. The emphasis is put on the difficulty of grouping the various necessary optimizations into a single one.

Key words : segmentation, connection, connective criteria, $\omega-$continuity, Lipschitz, quasi-flat zones, jump connection, smooth connection, levelling, colour segmentation, seeds, watersheds, measurements, 3-D set processing, connected operators, variational methods.

1 Introduction

1.1 The audio-visual perception

Most of the audio-visual perceptions of our daily life, although apparently very simple, turn out to be of a fearsome complexity when one seeks to reproduce them on a computer. Reading a postcard, listening to an instrumental duet, determining by a glance that a key is in place, watching a TV show, etc.. are so much integrated to our standard behavior that we perform them without paying attention.

All these activities share the common feature of being part of our time space environment, but that is not enough to characterize them. For example, when a radio speaker uses an unknown language, his discourse sounds to us like a confuse and rather continuous noise, where we are totally unable to place cuts between the words. But as soon as we know the language the slightest bit, this inability regresses : a few significant words emerge from the magma and it may happen that this partial segmentation be sufficient for the overall understanding. Finally, when we know the language correctly, the reverse process arises and we are able to recover what the speakers says even when his speech production is disastrous or when the broadcast is fuzzy. Therefore, what various auditors extract from the same audio signal varies considerably according to their own capabilities, and these differences impact the *segmentation* of the discourse, i.e. the extraction of time segments during which the audio signal has a unitary meaning for us.

As any example, that of the speaker is particular, but the four main emphasized phenomena are quite general and appear in visual perception as well:

1. What has been segmented is a certain *perceptive space*. It was the time in the example, but it could be a visual field or an hybrid of both. But are they the only spaces ? When we mentally share the two voices of an instrumental duet, in what kind of space such a segmentation can be described ? Must we add some frequency dimension to the time axis, as in sonograms ? But does the harmonic decomposition of the signal $f(t)$ belong to the *arrival* space ? How to mix input and output ?

In vision, a similar situation also arises, perhaps less frequently, but in a ruder way. The process that sends back the segmented image into the initial space is well captured in the ambiguous configurations that the gestaltists appreciate so much. Everybody knows these cobblings of cubes may be seen from their diagonals in positive or negative relief. Objectively, the drawing remains unchanged, but we choose to see it in a certain way, which excludes the other. But in which space does this choice occur?

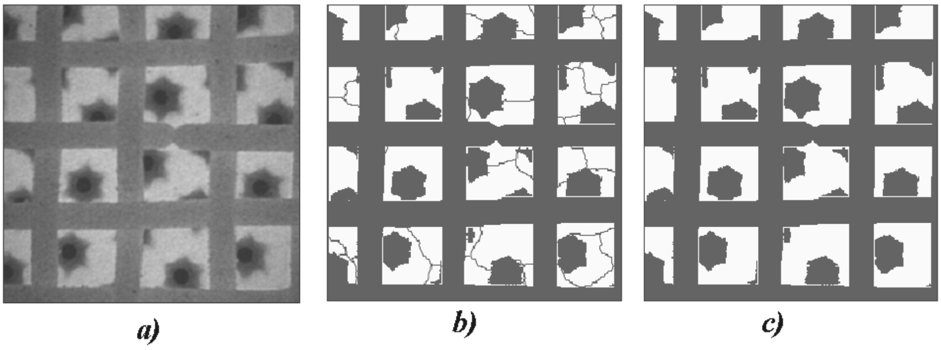


Figure 1: *a) initial burner image; b) partition of the background ; c) maximum partition of the background*

2. Whatever the perception space is, the *segmentation act* consists in partitioning this space into disjoint regions, e.g.

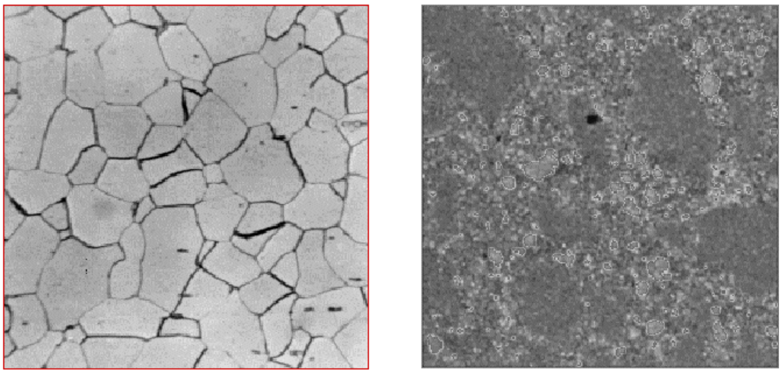
- when we recognize a handwritten word on the back of a postcard, we extract if from the rest, whatever the connectivity of the graphism is.
- when we understand a word of the speaker, it cannot overlap with the others

Such a partitioning is not necessarily the concern of the whole space. For checking that a key is in place, we only need to segment a neighborhood of the key hole. A single class of the partition often suffices for gathering all non classified regions.

3. The idea of a partition has to be completed by the further need of *maximum* partitions. When we look at a face, what we call mouth is the largest zone of points of a certain colour and shape. The example depicted in Fig. 1 says the same thing, but with less meaning since it concerns a gas burner infrared image. In the initial image, the warmest regions are the darkest ones; the clear background has a rather constant grey intensity which allows us an easy threshold. Several partitions can be drawn on the background set (Fig. 1b), but the only one which makes sense is the greatest.

4. The fourth phenomenon that human perception brings to the fore involves *hierarchies* which are travelled up and down. It is even perhaps its main feature. We recognize a face because we think that we have seen two eyes, a nose and a mouth; conversely, it is because we think that we have detected a face that we look for its expressive elements such as the eyes, the nose or the mouth.

A similar process is at work in video tracking. If a tree partly hides the house we are looking at, we still identify the latter correctly even though, when walking, we see both the house and the tree under variable orientations and occlusions. In the audio world, the hierarchies are again more evident : the phoneme induces the word, that induces the sentence, that induces the discourse (and that is not even mentioning music).



a) *b)*

Figure 2: *Two examples of microscopic images that are meaningless for the human vision of a non specialist. a) alumina, b) concrete.*

To summarize, the audio-visual perception is structured around four main themes :
- a possibly complex *perception space*, which embeds the usual physical space and time,
- *partitions* of this perceptual space into zones that exhibit some meaning for us,
- *maximization* of the said partitions,
- and most often by *hierarchical* processing.

The above description is deliberately phenomenological. It could perhaps be supported by physiological considerations, but without significant output (here at least) : computerized vision has no more reasons to copy physiology than aeronautics to mimic the flights of birds, and it should not be proved either that the logical structures that underlie computerized segmentation we hope to discover reproduce the human brain. This is particularly true when some images, because they are microscopic for example, are totally meaningless for the human perception (see Figure 2), though their studies involve random models which require segmentations [37], [38].

What is true nevertheless is that the four themes of the human vision provide us with guidelines. Quantitative approach lies on its own consistency, but it must treat the four above aspects. Therefore we shall use this guide, firstly for the brief bibliographic survey that follows, and also in the organization of the paper itself.

1.2 Segmentation in computer vision

Image segmentation methods, as well as the mathematics they bring with them, appeared during the sixties, in parallel with digital image processing, and have constantly advanced since this time. When, for the first time, somebody thresholded the electronic signal of an image scanning, the first image segmentation was born ... and also the most frequently used, even today. But clearly, it is not the only one and the number of algorithms proposed in the literature about segmentation exceeds one thousand.

1.2.1 The spaces

Just as for visual perception, physical space and time are often not sufficient for computer segmentation purposes. But unlike visual perception, in computers, the input and output spaces are always distinguished, even if they belong to the same mathematical type. Moreover, as the semantic notions might not easily be technically designed, they are replaced by *feature spaces*. In case of still images, they may be the concern of textures, orientations, colour, contrast, etc., i.e. notions that are located by coordinates in the usual physical space. But another type of feature space comes from histograms, where the coordinates refer to frequency distributions. P. Soille [89] was probably the first researcher to compute watersheds of multi-dimensional distributions in satellite imagery. A use of 2-D histogram is given below, in Section 5.3, about segmenting shadows and reflections in colour images.

In the case of video sequences, the range of the features becomes broader and includes transitions, zooms, travelling, etc. of the camera, plus the intrinsic motions of the objects themselves.

Beyond the diversity of the feature spaces arises the problem of their synthesis. Sophisticated strategies have been proposed. Here is an example of such a reduction, according to Ph. Salembier and F. Marques [73] :

"*Object contours are more accurately defined relying on color information. The combination of these feature spaces can be done following a parallel or a hierarchical strategy. The parallel approach computes a motion spatial segmentation, on one side, and a color spatial segmentation, on the other side. Then both partitions are combined. The hierarchical strategy relies on merging steps. It starts, in a first step, by a color spatial segmentation and changes its criterion to deal with motion spatial segmentation, in a second step*".

Such a precise strategy may perhaps be qualified of "ad hoc" solution, but it was finalized on large data bases, and it works. Nevertheless, it is not the only possible one, and the method proposed by F. Meyer in [59] focuses more on partitioning in association with connected filters. But again, are they general rules that govern the logical "AND" and "OR" of criteria, and the "filtering $\longleftrightarrow$ segmentation" interactions ?

1.2.2 Partitioning

In the various ways proposed in the literature, the segmenting partitions are sometimes investigated from their classes. Then, following A. Rosenfeld and A.C. Kak, one speaks of "region growing" or of "region aggregation" [71]. Alternatively, one can also focus on the borders of the segmenting classes, and design algorithms that seek these lines. Finally, the partition may be approached as such.

Examples where the emphasis is put on the partition classes are given by the thresholding operator, or by the extractions of the regions with homogeneous values. In both cases, no conditions are introduced about the limits of such regions, which play a passive role. The "region growing" and the "region aggregation" algorithms [71] belong to the same group, at least when they are generated from seeds which progressively extend, with or without non-overlaping constraints.

A pertinent alternative here is based on the observation that the human vision is more sensitive to the space changes of the high intensity or color, that to its continuity. The famous edge detection theory of E. Hildreth and D. Marr [48] [49], which dates from the end of the seventies, is one of the first, but the deepest formal expressions of this idea. In their "Primal Sketch" they convolve the image with Laplacian of Gaussian masks, then extract those points where the sign of the Laplacian changes, at each scale.

At the same time, and totally independently of his sphere of influence, S. Beucher and Ch.Lantuejoul introduced the concept of a watershed [7]. The two main points of their approach are the following

1/ to work on the gradient module of the initial image,

2/ to draw, algorithmically, the crest lines of this second image, which yields drawing *closed* contours along the inflexion points of the initial image.

Unlike most of other edge detectors, watersheding is a global procedure, where the closed contours surround the minima of the image under study (see Figure 10). From the segmentation viewpoint, the only requirement about the regions is that each of them must contain a unique minimum.

Instead of focusing either on regions or on edges, a third way was originally proposed by S.L. Horowitz and P. Pavlidis [36] under the name of "split and merge" :

"*Take an arbitrary initial partition in pyramidal data structure ; split regions with large approximation errors ; merge adjacent regions with similar approximation ; and group similar squares into irregular regions".*

The "pyramidal structure" involved here is the set of all image pixels in a square grid. Then, each level of the pyramid is obtained by grouping the previous level by clusters of four. The "split and merge" procedure moves up and down in the pyramid by comparing all neighboring regions with respect to some norm (e.g. having closed average grey). Again, just as Marr's primal sketch or Beucher's watershed, the original split and merge procedure turned out to be the ancestor of a large family of algorithms.

1.2.3 Geometric flows and pde's

It happens that a partitioning criterion, of any of the three above types, can be expressed in a differential way. In 1992, during the same year, three teams of researchers proposed re-readings of the Euclidean convex dilations by means of non linear PDEs, namely L. Alvarez, F. Guichard, P.-L. Lions, and J.-M. Morel [2] on the one hand, and R. van den Boomgaard and A. Smeuders [10],[11] on the other one, for dilations, while R. W. Brockett and P. Maragos [18] worked on dilations and openings by convex sets and concave functions.

Parallely, S. Osher and J. Sethian [64],[88] developed PDEs of Hamilton-Jacobi type to model propagations of level curves, i.e. of the contours of the horizontal sections of 2-D numerical functions. In particular, the evolution with constant normal speed models multiscale set dilation. Another beautiful achievement is the interpretation of the geodesics wave fronts in terms of geometrical optics [85]. The line of ideas that begun with C. Lantuejoul and S. Beucher [42] who introduced the geodesic methods in mathematical morphology, succeeds in P.Maragos differential morphology [45]. In optics, the function $\Phi(x, y)$ whose iso-level contours are normal to the rays is called *eikonal function* and its gradient satisfies the eikonal PDE

$$\| \bigtriangledown\Phi(x, y) \| = \sqrt{(\frac{\partial\Phi}{\partial x})^2 + (\frac{\partial\Phi}{\partial y})^2 + (\frac{\partial\Phi}{\partial z})^2} = \eta(x, y, z)$$

where is η the refractive index of the light. This PDE transcribes Fermat's principle of least time [13]. P. Maragos used this equation to set optimisation problems of geodesic morphology. In [54], in cooperation with F. Meyer, they bridged the gap between PDEs and levelling connected operators, for the arcwise connection.

Another significative achievement of these ways of thinking is provided by the geometric models of active contour that Caselles et al.[20], and Malladi et al.[44] have proposed. The PDE they introduce models the curvature motion. In a more recent original work, V. Caselles, R. Kimmel and G. Shapiro [21] reached a fine synthesis between the viewpoint of snakes by energy minimization, and geometric active contours, by means of a geodesic approach (see also [43]).

By itself, the PDE framework does not express any energy to minimize, although it may be used as a tool in some optimization problems. Basically, it generalizes the morphological operators that can be represented by iterations of elementary operations, which considerably enlarges the concept of an elementary convex dilation (or erosion), among others. Indeed, they appear in the literature just as structuring elements. A number of authors publish each a certain PDE, that correctly filters or segments the proposed examples of their publication. Sound reasons for such or such feature of the PDE are given, just as the properties of the structuring elements derive from their possible convexity, (an)isotropy, connectivity, etc.. But, what is missing here is an upper level, a point of view from which the various PDEs should be compared, and combined.

1.2.4 Optimization

We now pursue by considering the third point of the phenomenological description, namely the optimization phase. There are essentially two ways for thinking the optimization problem in segmentation. One of them associates *numbers* with partitions and makes the optimization hold on these numbers. The other one acts directly on the set of all partitions and expresses optimization in this space. Even though it is not explicitly formulated, one of the two ways often underlies each particular algorithm, and sometimes both of them.

The first way is said to be *variational*. It associates an energy which represents the amount of information left in each smooth version of the image. The most famous example of this approach was proposed by D. Mumford and J. Shah [62]. Starting from a numerical function f over a 2-D zone $\Omega \leq \mathrm{R}^2$ they wish to find simultaneously a partition of Ω, and a transform

ψf that is piecewise smooth in each class of the partition. If L stands for the set of the boundaries of the classes, they introduce the energy

$$E(\psi, L) = Length\ \ (L) + \int_{\Omega \backslash L} \left\{ |\nabla(\psi f)(x)| + [(\psi f)(x) - f(x)]^2 \right\} dx \qquad (1)$$

to be minimized. The first term imposes a minimal length for the discontinuities, the second one says that the function is smoothed in each class taken separately, and the third one that the transform (ψf) approximates at best the initial image. In particular, when the smoothing is obtained by means of piecewise constant functions, then the second term of $E(\psi, L)$ vanishes. Each term may, of course, be given a weight so that the minimization can balance contour reduction against piecewise simplification.

The idea of adding terms whose dimensions are different was daring, but it proved to be fruitful. A number of variants was proposed by the two authors and by several other ones, as well as critical comments on some mathematical points. The reader will find a very comprehensive information on the subject in J.M. Morel and S. Solimi's book [61]. They prove some deep and instructive properties of Mumford and Shah optimization; in particular that given a function f and a smoothing model, there may exist several segmentations with a minimal energy.

It would take a long time quoting all questions which are formalized by means of an energy minimization, but the "snakes" method deserves to be mentioned. Originally developed by Kass et al [39], it is based on the progressive deformation of initial seeds toward the regions of maximum gradients. The associated energy comprises one term that controls the regularity of the snake, while another one attracts the curve toward the maximum gradient regions. Just as the watershed method, the snakes evolution preserves the topology of the initial seeds; but unlike the snakes, the watershed has no constraint on the regularity of contours, which may be an advantage or not.

In the energy approach to segmentation, the operators that transform the initial function (i.e. the input image) into its optimal smoothing do not need to be explicit. Alternatively, one can place function operators ψ in the foreground and associate segmentation with lattice based operators that partition the image space. This second way for handling optimization ranges from "watersheds" to "connected operators" [77] (see, among others [94], [47], [24], [34]). But though partition optimality is usually involved in these operators, it is rarely brought to light. The segmentation goals are often distributed among different processings. For P. Salembier and F. Marques for example [73], the low segmentation level has to be divided into a generic step, common to all possible tasks, followed by a second step where specific features of the problem under study are introduced (e.g. a certain type of video sequences, such as the news). Such a versatility is more easily attained from an operator approach, where the various features can be differently allocated than from a variational method, more global.

1.2.5 Hierarchies, semi-groups and scale spaces

The last point of the phenomenological description is related to hierarchies and to multi-dimensional approach. Again two points of view compete, according to whether things are contemplated regionally, or locally. The first approach leads to semi-groups, the second to partial differential equations.

One can always make any formula, any function operator, any energy functional depend on some positive parameter. The pretension of multiscale analysis is higher, and aims at finding out some nice relationships between the various scales under study. One then speaks of "pyramid". In the weaker case, function f is submitted to a battery of transformations with respect to some scale total ordering, and next the pertinent features at each level are gathered together. But both granulometries and Gaussian pyramids, which historically appeared the first, propose more strongly structured hierarchies.

The notion of a *granulometry,* introduced by G. Matheron in 1975 [50], was initially set oriented but it extends to function operators without significant changes.

Definition 1 *Let $\mathcal{F}$ be a partially ordered set of functions and let $\psi_\lambda : \mathcal{F} \to \mathcal{F}$ be a family of operations on $\mathcal{F}$ that depends on the positive parameter λ. The mappings $\{\psi_\lambda \ , \ \lambda > 0\}$ define a* granulometry *when*

$$\begin{array}{ll} i- & \psi_\lambda(f) \leq f \text{ for any } \lambda > 0 \text{ and } f \in \mathcal{F} \\ ii- & f, g \in \mathcal{F} \text{ and } f \leq g \text{ imply } \psi_\lambda(f) \leq \psi_\lambda(g) \qquad (\lambda > 0) \\ iii- & \psi_\lambda \circ \psi_\mu = \psi_\mu \circ \psi_\lambda = \psi_{\sup(\lambda,\mu)} \end{array} \tag{2}$$

The first two axioms, plus the third one for $\lambda = \mu$, mean that each ψ_λ, taken in itself, is an opening on $\mathcal{F}$. The third axiom structures the whole family as a *semi-group*, with a commutative composition product according to which the more severe of two successive operations imposes its law.

The second pioneer hierarchy arose at the beginning of the eighties with the idea of using Gaussian convolutions for sub-band decomposition purposes [49], [41]. In the Euclidean space, denote by g_λ the isotropic Gaussian function of variance λ that is centred at the origin. As λ varies the family of the convolution operators $\{g_\lambda\}$ generate a pyramid such that

$$(f * g_\lambda) * g_\mu = f * g_{\lambda+\mu} \tag{3}$$

We find again a commutative semi-group, but different from the granulometric one, since the effects are now additive, and this new semi-group is also magnification invariant. Moreover it admits a significant differential interpretation. In 1984, J.J. Koenderink observed that to convolve an image f by the Gaussian of variance λ was equivalent to solving the heat equation

$$\begin{cases} \frac{\partial(f*g_\lambda)(x)}{\partial\lambda} = \Delta\,(f * g_\lambda)(x) \\ (f * g_0)(x) = f(x) \end{cases}$$

[41]. The Gaussian convolutions are not the only mappings that satisfy the additive semi-group of Rel.(3): it is fulfilled by the erosions and the dilations by discs of radii (λ, μ) as well.

More generally, H. Heijmans has developed a scale space theory [35] showing that the two types of commutative semi-groups, namely Rel.(2) and Rel.(3) form the two basic prototypes.

1.3 Purpose of the present study

The various ways of thinking we sketched are not incompatible, they just emphasize different aspects of the real world, and of its modelization. This said, the idea of a huge, universal and unique formula for the segmentation energy, a formula that would comprise five or six terms and whose all current methods would be particular cases may be very appealing, although somehow dogmatic. But for having all solutions depending on a single formula, they should respond to the same problem. Now, there are many. Segmenting the chromatin of cancerous cells [3], or the anomalies in submarine sonar images, or features in 30 channel satellite images, brings into play "homogeneities" that are deeply different and not specially vivid for the human vision. Perhaps we would do better to accept the plurality of situations, and to set the question of their interweaving as the *core of segmentation*, just as P. Salembier and F. Marques do [73]?

On the other hand, the variational methods and PDEs imply regularization assumptions, such as the existence of second derivatives, whose validity does not always seem justified. For example, the choice of minimizing the length of the boundaries lies on the apriorism that "*boundaries of each segment should be simple, not ragged, and must be spatially accurate*" [33]. But why boundaries would be "simple and not ragged" ? Such a criterion for a good segmentation is perfectly inadequate when I look at the tree in front of me, that stands out against houses and other trees, and for which a fractal model should be more realistic.

Here is another example. In his study about traffic automatic control [8], S. Beucher builds watersheds for composite functions made of *three* grey levels only, and this gives excellent results. Must we condemn them for "lack of correct derivability conditions"? More deeply, the purely digital branch of image processing, that arose with A. Rosenfeld in particular [70], is currently going though new and nice developments. Some topological structures discovered by G. Bertrand [6], some multiscale skeletons proposed by G. Borgefors and al [12] have no Euclidean equivalents. In [40], Ch. Kiselman undertakes an approach to mathematical morphology by means of digital concepts exclusively, and yields original new results. But in the present study, are we really obliged to face the alternative Euclidean/digital? Could we enter the segmentation problem without having to make such a choice?

Furthermore, the euclidean plane, which is the convenient framework to formalize PDEs and to express the mathematical properties of the boundary set in Rel.(1), because they involve Jordan curves, is indeed restrictive. Must we renounce feature spaces that gather together orientations, intensities, textures and colours, for example ?

Moreover, some variational methods require the datum of a zone, usually a rectangular mask in which the energy is summed up. Therefore, the contours of the objects depend on the scene under study, as well as on the mask. For example, in a T.V. travelling, if a same object occupies different locations in the mask throughout the camera motion, then its segmentation risks to differ from frame to frame. Should it be possible to design an approach where the contours of the objects would be treated as one of their *intrinsic* properties ?

Finally, by summarizing the whole image structure by the only energy value, which is a nice dense information, variational approach handles segmentation in a way that lends itself very well to minimization, and which proved itself to be practically sound. But the drawback here is perhaps an excess of synthesis, and the price for such a reduction appears sometimes, with the lack of a unique solution for example, or with the fact that an overall optimum cannot be decomposed, a priori, into a sequence of partial ones?

By exploring an alternative approach to variational methods, we do not seek a better way: we already know that it will be different, with other weaknesses. But we would like

- to provide a consistent segmentation framework to a number of existing techniques, including the PDEs;
- to link together the notions of connected filtering with that of segmentation;
- to sketch the conditions under which several optima can be mixed up;
- and propose image partitions as alternatives to the scalar energy, in optimization formalism.

These are the questions we attempt to tackle in the present study.

In all the above references, one constantly speaks of *smaller, larger and maximum* segmenting partitions, and indeed these adjectives have a precise meaning. Then, one may aim at working directly on partitions, and one may wonder which conditions have to be demanded to the criteria for yielding optimizations. This will be our starting point, and we will initially formulate it in the less restricting space framework. While developing the approach, some constraints will reduce, when needed, the range of possibilities.

The text comprises two parts, namely Sections 2 to 5, and 6 to 8. The first one is the concern of a general theory. It appears soon, in this part, that segmentation and morphological connection are closely linked notions, which means that this work takes place in the research community on connection (Ch. Ronse, H. Heijmans, F. Meyer, L. Vincent, J. Crespo, Ph. Salembier, F. Marques, P. Maragos, J. Goutsias, U. Braga-Neto ...). The notion of a *connective criterion* allows us to construct optimal partitions, and to combine them in different manners. A series of examples illustrates the theoretical results, and classifies the various modes of segmentation. The second part is devoted to the relationships between filtering and segmentation. The framework is restricted (the arrival space is supposed to be a complete lattice, and the connectivity criteria become more specific), so that morphological operators may be constructed. In particular a theory of connected filtering is sketched as a derivation of the pulsewise oriented connected opening.

2 The segmentation concept

2.1 Criteria and segmentation

Classically, one calls criterion a key which allows us to distinguish between true and false assertions. In the current context, the notion may be formalized as follows

Definition 2 *Criterion: Let E and T be two arbitrary sets and let $\mathcal{F}$ be a family of functions from E into T. A criterion σ on class $\mathcal{F}$ is a binary function from $\mathcal{F} \times \mathcal{P}(E)$ into $\{0,1\}$ such that, for each function $f \in \mathcal{F}$, and for each set $A \in \mathcal{P}(E)$, we have*
either $\sigma[f,A] = 1$ (criterion is said satisfied on A)
or $\sigma[f,A] = 0$ (criterion is said refuted on A).
Moreover, we decide conventionally that, for all functions, all criteria are satisfied on the empty set, i.e.

$$\sigma[f,\emptyset] = 1 \quad \forall\ f \in \mathcal{F}$$

For instance, the criterion which is satisfied by the singletons, and by the sets $A \in \mathcal{P}(E)$ such that

$$x \in A, \quad t_0 \leq f(x) \leq t_1 \tag{4}$$

with f, t_0 and t_1 fixed, and $t_0 \leq t_1$ defines the *threshold criterion* applied to function f.

Likewise, when space E is metric with distance d, and equipped with a connection $\mathcal{C}$, then the connected zones A such that

$$x, y \in A \Rightarrow |f(x) - f(y)| \leq kd(x,y),$$

yield the *k-Lipschitz criterion.*

Note that a criterion σ may be satisfied or refuted on set A according to reasons that involve a larger zone than the restriction of f on A. An example about Lipschitz regional connections is given by Relation (21) below. In image processing, one usually says that an image or a sequence is segmented when the domain where it is defined has been partitioned into homogeneous zones in accordance with a given criterion. This very popular definition appears as a leitmotiv in number of introductions to the matter. Since it involves partitions, we will briefly recall this last notion

Definition 3 *Partition : Let E be an arbitrary set. A partition $\mathcal{D}$ of E is a mapping $x \rightarrow D(x)$ from E into $\mathcal{P}(E)$ such that*
(i) for all $x \in E : x \in D(x)$
(ii) for all $x, y \in E : D(x) = D(y)$ or $D(x) \bigcap D(y) = \varnothing$
$D(x)$ is called the class of the partition of origin x.

The set of the partitions of an arbitrary set E is ordered as follows: a partition A is said to be *finer* (resp. *coarser*) than a partition B when each class of A is included in a class of B. This leads to a lattice which is complete, the coarsest element has one class only, namely set E itself, and the finest partition has all the singletons of $\mathcal{P}(E)$ as classes.

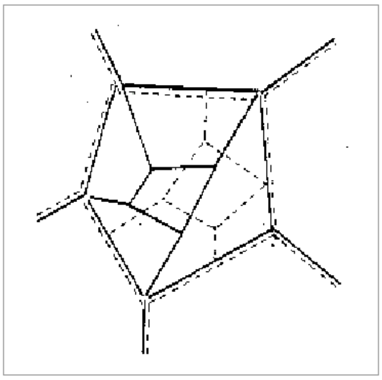


Figure 3: Supremum of two partitions

Given a family $\{\mathcal{D}_i, i \in I\}$ of partitions, the mapping $\mathcal{D}$ of E into $\mathcal{P}(E)$ which is defined via its classes

$$D(x) = \bigcap\{D_i(x), i \in I\}$$

generates obviously a partition where for all $x \in E$, $D(x)$ is the largest element of $\mathcal{P}(E)$ that is contained in each $D_i(x)$. Therefore, $\mathcal{D}$ is the inf of the $\mathcal{D}_i$, in the sense of the partition lattice. The expression of the supremum is more complex, and $\mathcal{D} = \vee\mathcal{D}_i$ means that for all $x \in E$, class $D(x)$ is the smallest set containing x, such that for every $i \in I$ and $y \in D(x)$, class $D_i(y)$ is included in $D(x)$ (see Figure 3). In [68], Ch. Ronse has explained that x and y belong to the same class of the supremum if and only if there is a chain of classes from $\{D_i(z), i \in I, z \in E\}$ beginning at x, ending at y and linking them. The lattice of all partitions of E is not distributive (which is an actual drawback), but it has the useful property to be sup-generated. The sup-generators are those partitions with one class being a pair, and the other classes singletons.

Here is a basic example of segmentation: represent the image under study by a numerical function $f : E \to \overline{R}$ where E is a set equipped with a connection, then *f is segmented into flat and connected zones* when a partition D of E is created, such that for any $x \in E$, the class $D(x)$ is the largest connected subset of E including point x and on which function f is constant and equal to $f(x)$.

All criteria do not lend themselves to such nice partitioning. Suppose, for instance, that we wish to partition E into various zones, connected or not, where function f is Lipschitz with parameter $k = 1$. Two non disjoint zones A and B may very well be found, such that the criterion is satisfied on A and on B, but not on $A \cup B$ (see Figure 4). In this case, there is *no largest zone* containing the points of A and where the criterion be satisfied. The Lipschitz criterion does not yield a segmentation.

In other words, the partitions referred to with a segmentation concept are *greatest* ones, i.e. having the greatest classes. Besides, we can always construct a smaller partition, namely

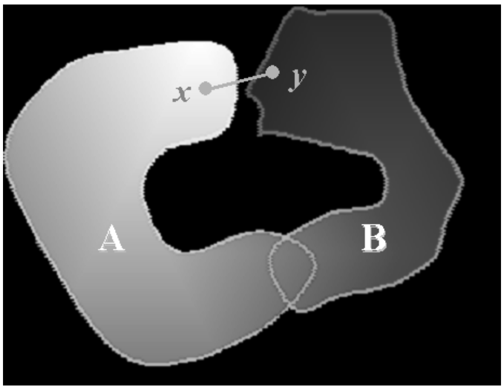


Figure 4: *This function is Lipschitz inside sets A and B but not in their union.*

the one that reduces space $\mathcal{P}(E)$ on all its singletons $\{x\}$. These remarks encourage us to replace the above popular definition by a more formal expression, therefore more precise, to the notion of a segmentation.

Definition 4 *Segmentation: Given a function $f \in \mathcal{F}$, an arbitrary set $A \subseteq E$ and a criterion σ, let $\{D_i(A),\ i \in I\}$ be the family of all partitions of set A into homogeneous zones of f according to σ. The criterion σ is said to* segment *the function f when,*

1- $\sigma[f, \{x\}] = 1$ for all $\{x\} \in \mathcal{P}(E)$;

2- family $\{D_i(A)\}$ is closed under non-empty supremum.

Then the supremum partition $\curlyvee D_i(A)$ defines the segmentation of f over set A according to criterion σ.

This definition is consistent: according to the first axiom, the partition of A into all its singletons belongs to the family $\{D_i(A)\}$, which, therefore, is never empty. In particular, when set A coincides with the space E of definition of f, then $\curlyvee D_i(E)$ is the maximum partition of the whole space. In the above k-Lipschitz criterion it is precisely this maximum partition which is missing. The axiom $\sigma[f, \{x\}] = 1$ ensures us that whatever criterion is considered, there is always at least one way to partition E into zones (the singletons) that satisfy it. This is a compulsory condition to be able to then speak of a largest partition.

2.2 Reminder on set connections

In mathematics, the concept of connectivity is formalized in the framework of topological spaces and is introduced in two different ways. First, a set is said to be connected when one cannot partition it into two non empty closed sets [23]. This first approach, because it is very general, does not derive any advantage from the possible regularity of some spaces, such as the digital ones, or the Euclidean ones. In such cases, the notion of *arcwise connectivity* turns out to be more convenient. According to this notion, a set A is connected when, for every $a, b \in A$, there exists a continuous mapping ψ from $[0, 1]$ into A such that $\psi(0) = a$

and $\psi(1) = b$. Arcwise connectivity is more restrictive than the general one; however, in $\mathbb{R}^d$, any open set which is connected in the general sense is also arcwise connected.

A basic result governs the meaning of connectivity ; namely, the union of connected sets whose intersection is not empty is still connected :

$$\{A_i \text{ connected}\} \text{ and } \left\{\bigcap A_i \neq \emptyset\right\} \Rightarrow \left\{\bigcup A_i \text{ connected}\right\} \tag{5}$$

In 1988, J. Serra and G. Matheron proposed a new approach to connectivity ([75], ch.2 and 7) where they took Eq.(5) as a starting point and not as a consequence, and where they introduced the word *connection* to mark the difference form the usual connectivities (by this time, nearly 1985-1986, G. Matheron had just formalized his theory of morphological filtering ([75], ch.6) and the two authors were looking for the design of filters that be both morphological and connected ([75], ch.7)).

Definition 5 *Connection : Let E be an arbitrary non empty space. We call connected class or* connection $\mathcal{C}$ *any family in $\mathcal{P}(E)$ such that*
(o) $\emptyset \in \mathcal{C}$
(i) for all $x \in E$, $\{x\} \in \mathcal{C}$
(ii) for each family $\{C_i,\ i \in I\}$ in $\mathcal{C}$, $\cap_i\ C_i \neq \emptyset$ implies $\cup_i C_i \in\ \mathcal{C}$.

Based on this definition, any set C of a connected class $\mathcal{C}$ is said to be *connected.* In addition, the empty set as well as the singletons $\{x\}, x \in E$ are always connected. It is clear that such a definition does not require any topological background. The classical notions (e.g. connectivity based on digital or Euclidean arcs) are indeed particular cases.

Given a set $A \in \mathcal{P}(E)$ and a point $x \in A$, axiom *(ii)* suggests that one pays a special attention to the (still connected) union $\gamma_x(A)$ of all connected components containing x and included in A

$$\gamma_x(A) = \bigcup\{C;\ C \in \mathcal{C}, x \in C \subseteq A\}. \tag{6}$$

The set operator γ_x is obviously an opening. Here the emphasis is put on the link between connections and opening operations, that makes more precise the following theorem ([75], ch.2):

Theorem 6 *The datum of a connected class $\mathcal{C}$ on $\mathcal{P}(E)$ is equivalent to that of a family $\{\gamma_x\,, x \in E\}$ of openings such that*
(iii) for all $x \in E$, we have $\gamma_x(x) = \{x\}$
(iv) for all $A \subseteq E$, $x, y \in E$, $\gamma_x(A)$ and $\gamma_y(A)$ are equal or disjoint
(v) for all $A \subseteq E$, and all $x \in E$, we have $x \notin A \Rightarrow \gamma_x(A) =\ \emptyset$.
The γ_x s are called the point connected openings *of connection $\mathcal{C}$.*

In addition, each γ_x, as an opening on $\mathcal{P}(E)$, is by definition an increasing, anti-extensive and idempotent operator. Several connections can be defined on set E, such as

(a) Given an extensive dilation δ on $\mathcal{P}(E)$ and a first connection $\mathcal{C}$ on $\mathcal{P}(E)$, such that $\delta(\{x\} \in \mathcal{C}$ for all $x \in E$, the set of all $X \in \mathcal{P}(E)$ such that $\delta(X) \in \mathcal{C}$ forms a second-class connection containing $\mathcal{C}$;

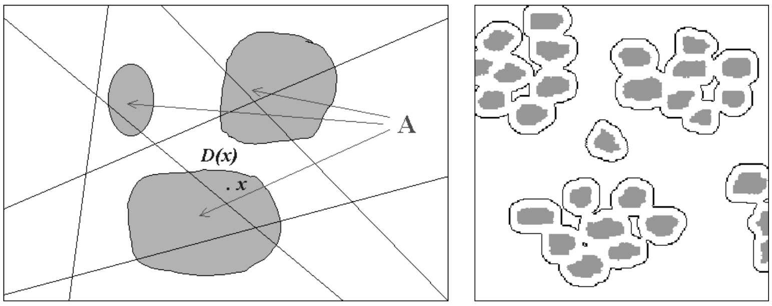


Figure 5: left *: Connection generated by a partition (the lines). The connected component of set A at point x is the union of the two* arcwise disjoint *blobs of $A \cap D_x$.* Right *: the particles of each cluster belong to a second level connection.*

(b) Given a partition D of the space, all the subsets of each class $D(x), x \in E$, of the partition generate a family closed under union ([75], ch.2). Hence we have the connection

$$\mathcal{C} = \{A \bigcap D(x), x \in E, A \in \mathcal{P}(E)\}$$

The connected component $\gamma_x(A)$ equals the intersection $A \cap D(x)$ between A and the class of the partition at point x. This technique will be constantly used in the next three sections.

Here are three basic properties of the set connections (proofs may be found in [80] for the first property, and also in [68] with a description of the supremum, and in ([75], ch.2) for the last two).

- **Lattice of the Connections :** The set of all connections on $\mathcal{P}(E)$ is closed under intersection ; it is even a complete lattice in which the supremum of a family $\{\mathcal{C}_i \ \ i \in I\}$ is the least connection containing $\cup_i \mathcal{C}_i$.

- **Connection Partitioning Theorem** : Let $\mathcal{C}$ be a connection on $\mathcal{P}(E)$ of connected openings $\{\gamma_x,\ x \in E\}$. For each set $A \subseteq E$ the openings $\{\gamma_x(A),\ x \in E\}$ subdivide A according to the largest possible partition into components of class $\mathcal{C}$. This operation is increasing in that if $A \subseteq A'$, then any connected component of A is upper-bounded by a connected component of A'.

- **Increasingness :** If $\mathcal{C}$ and $\mathcal{C}'$ are two connections on $\mathcal{P}(E)$ with $\mathcal{C} \subseteq \mathcal{C}'$, then every $\mathcal{C}'$-component of A is the supremum of the $\mathcal{C}$-component of A that it majorates. Hence, the partitions of a given set by different connections increase with the connections.

The reader will find a number of other instructive connections in [82], in [68], and in [34]. In [68], an alternative (and equivalent) axiomatic has been proposed by Ch. Ronse. A link between connection and segmentation has been proposed by D.Gatica-Perez *et al* in [29], in particular for region merging algorithms, by means of extensive operators.

Historically speaking, the number of applications or of theoretical developments which was suggested (and permitted) by Theorem 6 during the nineties is considerable. The references [65] [72] [94] [47] [56] [24], among many others, show that it has opened the way to an object-oriented approach to segmentation, compression and understanding of still and moving images.

They also show how fruitful are the exceptional properties of the connected filters. However, the very reason for such a fecundity, which lies in Theorem 8 below, was never brought to light.

2.3 Connective criteria and segmentation theorem

What are the conditions that a criterion σ must satisfy to be a segmentation tool? The need for largest partitions orients us towards a connection based approach, via the concept of a *connective criterion.*

Definition 7 *Connective criterion. A criterion* $\sigma : \mathcal{F} \times P(E) \rightarrow \{0,1\}$ *is* connective *when for each* $f \in \mathcal{F}$ *the sets* A *such that* $\sigma[f,A] = 1$ *form a connection, i.e. when*

1- σ *is satisfied on the class* $\mathcal{S}$ *of the singletons and by the empty set,*

$$\forall f \in \mathcal{F}, \quad \{x\} \in \mathcal{S} \quad \Longrightarrow \quad \sigma[f, \{x\}] = 1$$

2- and when for any function $f \in \mathcal{F}$ *and for all families* $\{A_i\}$ *of* $\mathcal{P}(E)$ *for which the criterion* σ *is satisfied, we have*

$$\bigcap A_i \neq \emptyset \quad and \quad \bigwedge \sigma[f, A_i] = 1 \Rightarrow \sigma[f, \bigcup A_i] = 1. \tag{7}$$

In other words, when a connective criterion σ is satisfied by a function f on a family $\{A_i\}$ of regions of the space, and if all these regions have one common point, then it is also satisfied on the union $\cup_i A_i$. A zone A is said to be *homogeneous* according to f and σ when $\sigma[f,A] = 1$. Similarly, when $\sigma[f, D(x)] = 1$ for all the classes $D(x)$, $x \in E$ of a partition D, then D is said to *satisfy*, or to *fulfill* criterion σ.

The following key theorem, which is a consequence of the above connection partitioning theorem, characterizes the segmentation processes by identifying them with classes of connections.

Theorem 8 *Consider two arbitrary sets* E *and* T*, and a family* $\mathcal{F}$ *of functions* $f : E \rightarrow T$ *and let* σ *be a criterion on class* $\mathcal{F}$*. Then the three following statements are equivalent:*

1. *Criterion* σ *is connective;*
2. *For each function* $f \in \mathcal{F}$*, those sets on which criterion* σ *is satisfied constitute a connection* $\mathcal{C}$
$$\mathcal{C} = \{A \mid A \in \mathcal{P}(E) \text{ and } \sigma[f, A] = 1\}. \tag{8}$$
3. *Criterion* σ *segments all functions of the family* $\mathcal{F}$*.*

Proof. Let $\mathcal{C}$ defined by Rel.(8). By Definition 2 (of a criterion), we always require that $\emptyset \in \mathcal{C}$ (item (o) of Definition 5).

Let $\Pi(\mathcal{C})$ be the family of partitions of E whose classes belong to $\mathcal{C}$. We observe that item 3, i.e. the fact that criterion σ segments f, means by Definition 4 that $\Pi(\mathcal{C})$ is closed under suprema, in particular the least possible partition, that is the one whose classes are all singletons, belongs to $\Pi(\mathcal{C})$.

($1 \Leftrightarrow 2$) Items 1 and 2 of Definition 7 are equivalent to items (i) and (ii) of Definition 5, and item (o) of Definition 5 holds anyway.

($2 \Rightarrow 3$) If $\mathcal{C}$ is a connection, then $\mathcal{C}$ contains all singletons, so the empty supremum in $\Pi(\mathcal{C})$, that is the least element of the lattice of partitions (the partition with singleton classes) belongs to $\Pi(\mathcal{C})$. Consider now a non-void family $\mathcal{D}_i, i \in I$ of elements of $\Pi(\mathcal{C})$, and let $\mathcal{D} = \bigvee_{i\in I} \mathcal{D}_i$ be their supremum. We can decompose each class of $\mathcal{D}$ into its $\mathcal{C}$-connected components, and make a partition $\mathcal{D}'$ from them; so $\mathcal{D}' \leq \mathcal{D}$ and $\mathcal{D}' \in \Pi(\mathcal{C})$. For $i \in I$, every class $D_i(x)$ is included in a class $D(y)$ of $\mathcal{D}$; but $D_i(x) \in \mathcal{C}$ so $D_i(x)$ is included in a connected component of $D(y)$, in other words in a class $D'(z)$ of $\mathcal{D}'$. Thus $\mathcal{D}_i \leq \mathcal{D}'$ for all $i \in I$, so $\mathcal{D} = \bigvee_{i\in I} \mathcal{D}_i \leq \mathcal{D}'$. We conclude that $\mathcal{D} = \mathcal{D}'$, hence $\mathcal{D} \in \Pi(\mathcal{C})$. So $\Pi(\mathcal{C})$ is closed under suprema.

($3 \Rightarrow 2$) $\emptyset \in \mathcal{C}$ anyway (see above). If $\Pi(\mathcal{C})$ is closed under suprema, then it contains the least possible partition, that is the one whose classes are all singletons; hence $\mathcal{C}$ contains all singletons. Now let $C_i, i \in I$ be a non-void family of elements of $\mathcal{C}$ with a non empty intersection: $p \in \bigcap_{i\in I} C_i$ for some $p \in E$. To each C_i we associate the partition $\mathcal{D}_i$ whose classes are C_i and all the singletons $\{x\}$ for $x \in E \backslash C_i$. Let $X = \bigvee_{i\in I} C_i$ and $\mathcal{D} = \bigvee_{i\in I} \mathcal{D}_i$. Clearly, each $\mathcal{D}_i \in \Pi(\mathcal{C})$, and as $\Pi(\mathcal{C})$ is closed under suprema, we get $\mathcal{D} \in \Pi(\mathcal{C})$. For $x \in E\backslash X$, $\{x\}$ is a class of each $\mathcal{D}_i, i \in I$, so $\{x\}$ is a class of $\mathcal{D}$. For $x \in X$, we have $x \in C_i$ for some $i \in I$, so x is in the same class as p (namely C_i) for the partition $\mathcal{D}_i$, so x and p are in the same class for the supremum partition $\mathcal{D}$; thus X is included in the class $D(p)$ of p in $\mathcal{D}$; but the points $x \in E\backslash X$ are not in $D(p)$ (see above), so $D(p) = X$, and as $\mathcal{D} \in \Pi(\mathcal{C})$, we get $X \in \mathcal{C}$. We have thus proved that $\mathcal{C}$ satisfies the 3 items of Definition 5, so $\mathcal{C}$ is a connection. ■

In general the class at point x of a supremum $\curlyvee D_i$ of partitions is not $\cup_i D_i(x)$.We are here in this favourable case because each partition D'_i contains one non-point class at most. In particular, the segmentation of function f over the whole set E and according to the connective criterion σ is the partition D_f whose class at point x is given by

$$D_f(x) = \bigcup \{A : A \in \mathcal{P}(E),\ x \in A, \sigma[f, A] = 1\} \qquad x \in E. \tag{9}$$

Note that $D_f(x)$ is the connected component $\gamma_x(E)$ according to the connection $\mathcal{C}$ defined by Eq.(8). One will notice that Theorem 8 could not be stated by means of the classical arcwise connectivity only: one can easily find segmentation criteria that do not involve paths, such as the jump connection presented below. As a matter of fact, the more general concept of a connection is exactly right for the theorem to work.

Indeed, Theorem 8 belongs to this category of go-betweens from an abstract definition, hard to implement (must we calculate all partitions for checking whether there is a larger one?) and a more "local" notion which better lends itself to easy validations. But the theorem does not provide by itself any recipe to find nice connective criteria.

2.4 Class permanency

The strenght of Theorem 8 lies in that it provides *any sub-space* $A \subseteq E$ (and non only of the whole space) with a largest partition, since the set E involved in the theorem is arbitrary. Remark also that, given any partition D of E, that is not assumed to belong to the family generated by criterion σ, Theorem 8 associates a largest partition D' of the family and such that $D' \preceq D$.

In practice, the set E over which function f is defined is often known through relatively small windows, called masks. A typical example is given by the traveling of a TV camera. Then a same object S that we wish to segment may appear inside two distinct masks, Y and Z say. By minimizing a functional over each mask the functional methods run the risk to generate two different contours for s, according to the mask. But when we try to overcome this drawback by stating, in the very Definition 4, that the segmentation concept holds on all possible sub-sets of E, are we sure that if a segmented object lies in two different masks, it exhibits the same contour at both places?

Proposition 9 *Consider a connective criterion σ, a function f, two masks Y and Z of E, and a point $x \in Y \cap Z$. Let $D_x(Y)$ (resp.$D_x(Z)$) be the segmented class of Y (resp.of Z) at point x. These two classes are equal, i.e.*

$$D_x(Y) = D_x(Z) \tag{10}$$

if and only if $D_x(Z)$ lies in Y and $D_x(Z)$ lies in Z i.e.

$$D_x(Y) \subseteq Z \qquad \text{and} \;\; D_x(Z) \subseteq Y \tag{11}$$

Proof. (11) $\Rightarrow$ (10) On the one hand we have $x \in D_x(Y) \subseteq Z$, and on the other one, set $D_x(Z)$ is the union of all subsets of Z containing x and where σ is satisfied. Therefore $D_x(Y) \subseteq D_x(Z)$. By the same reasoning applied to mask Y, we obtain the inverse inclusion, which results in Equality (10).

(10) $\Rightarrow$ (11) this implication is straightforward if we observe that $D_x(Y) \subseteq Y$ and $D_x(Z) \subseteq Z$. ■

An instructive case arises when we take for set Y the whole space E. The above proposition states then that the local segmentation $D_x(Z)$, in mask Z, equals the overall one, $D_x(E)$, as soon as $D_x(E) \subseteq Z$. Such a result turns out to be more formal than practical, since the problem occurs precisely when we cannot access f over the whole set E. Indeed, Proposition 9 ensures a good segmentation consistency inside relatively closed masks. But it needs more assumptions to be extrapolated from local to global.

We now briefly analyse the problem of the identification $D_x(Z) = D_x(E)$, when E is the Euclidean space $\mathbb{R}^n$, and when criterion σ is *conditionnally decreasing.*

Definition 10 *A connective criterion σ on a function $f : \mathbb{R}^n \to T$ (T an arbitrary set) is said to be conditionnally decreasing when*

1. *every class segmented according to σ is arcwise connected;*
2. *$\sigma[f, A] = 1$ and $B \subseteq A$, where set B is arcwise connected, imply that $\sigma[f, B] = 1$*

Such a decreasingness is not always true. For example, it is satisfied by both criteria of connected flat zones and smooth connection, but not by that of quasi-flat zones.

Proposition 11 *Let σ be a conditionnally decreasing connective criterion on a function $f : \mathbb{R}^n \to T$, let Z be a set in $\mathbb{R}^n$ and $x \in \mathbb{R}^n$. If the segmented class $D_x(Z)$ of Z at point x is strictly included in mask Z i.e. if there exists an $\varepsilon > 0$ such that*

$$D_x(Z) \oplus \varepsilon \overset{\circ}{B} \subseteq Z, \tag{12}$$

then it is equal to the class at point x for the overall space $\mathbb{R}^n$

$$D_x(Z) = D_x(\mathbb{R}^n).$$

Proof. Given x, the mapping $Z \to D_x(Z)$ from $\mathcal{P}(\mathbb{R}^n)$ into itself is increasing, hence $D_x(Z) \subseteq D_x(\mathbb{R}^n)$. We split the segmented class $D_x(\mathbb{R}^n)$ into three disjoint sets:
1/ A_1 is the arcwise component of $D_x(\mathbb{R}^n) \cap Z$ containing $D_x(Z)$,
2/ $A_2 = (D_x(\mathbb{R}^n) \cap Z) \backslash A_1$,
3/ $A_3 = D_x(\mathbb{R}^n) \backslash Z$.
By conditional decreasingness, $\sigma[f, A_1] = 1$, and as $x \in A_1 \subseteq Z$, with A_1 arc-connected, we derive $A_1 = D_x(Z)$. Suppose that $D_x(Z) \neq D_x(\mathbb{R}^n)$, that is $A_2 \cup A_3 \neq \emptyset$, and let $\tau : [0, 1] \to D_x(\mathbb{R}^n)$ be a path with $\tau(0) = x$ and $\tau(1) \in A_2 \cup A_3$. Let

$$t_1 = sup\{s \in [0, 1] \mid [0, s] \subseteq \tau^{-1}(A_1)\}.$$

Thus $[0, t_1[\subseteq \tau^{-1}(A_1)$. If $t_1 \in \tau^{-1}(A_2)$, then the restriction of τ to $[0, t_1]$ connects A_1 to A_2 in $A_1 \cup A_2 \subseteq Z$, which contradicts the fact that A_1 is a connected component of $D_x(\mathbb{R}^n) \cap Z$. If $t_1 \in \tau^{-1}(A_3)$, then $\tau(t_1)$ is a point of A_3 adherent to A_1, so at a distance 0 from A_1, contradicting Inclusion (12). Thus $t_1 \in \tau^{-1}(A_1)$. As $\tau(t_1) \in A_1$, it is at distance $\geq \varepsilon$ from A_3, and as τ is continuous, $[t_1, t_1+\varepsilon/2]$ is disjoint from $\tau^{-1}(A_3)$, thus $[t_1, t_1+\varepsilon/2] \subseteq \tau^{-1}(A_1 \cup A_2)$. Then for $t_2 \in]t_{1,}, t_1 + \varepsilon/2[$, the restriction of τ to $[0, t_2]$ is a path joining A_1 to A_2 within $A_1 \cup A_2 \subseteq Z$, again a contradiction. ■

2.5 Lattice of the connective criteria

We now study the logical structure of the various connective criteria we can apply to f. The class Σ of all connective criteria on function f is ordered by the ordering of the connections they induce on space $\mathcal{P}(E)$, and this ordering yields a complete lattice where the infimum is nothing but the logical intersection. More precisely

Proposition 12 *The family Σ of all connective criteria forms a complete lattice. In this lattice, the infimum of a family $\{\sigma_i,\ i \in I\}$ corresponds to the logical intersection of the criteria; the supremum is the smallest connective criterion larger than each σ_i, $i \in I$.*

Proof. Consider a family $\{\sigma_i,\ i \in I\}$ of connective criteria. Denote by $\mathcal{A}$ the set of those sets for which all criteria of the family are simultaneously satisfied, and by $\sigma = \wedge\sigma_i$ the infimum (or logical intersection) of all criteria σ_i i.e. for any function $f \in \mathcal{F}$ we have

$$A \in \mathcal{A} \Leftrightarrow \sigma[f, A] = \bigwedge \sigma_i(f, A) = 1. \tag{13}$$

Class $\mathcal{A}$ is the intersection of the connections generated by all σ_i, hence is itself a connection [68], [80], so that Σ is an inf semi-lattice. Now, Σ admits a largest element $\sigma_{\max}$, namely

$$\sigma_{\max}(f, A) = 1 \quad \text{for all} \quad A \in \mathcal{P}(E)$$

hence Σ turns out to be a complete lattice. If $\mathcal{C}_i$ stands for the connection associated with criterion σ_i, then the supremum σ' of family $\{\sigma_i,\ i \in I\}$ is the connective criterion whose associated connection is the smallest upper bound of all $\mathcal{C}_i$, $i \in I$. ■

The smallest element $\sigma_{\min}$ of lattice Σ is that which, for all $f \in \mathcal{F}$, is fulfilled by the empty set and by the singletons only, and the largest element $\sigma_{\max}$ that which is satisfied by all $A \in \mathcal{P}(E)$.

Consider Proposition 12 from the point of view of the partitions. We have seen that a partition D_i is smaller than another, $D_i \preceq D_j$ say, when the corresponding classes at each point $x \in E$ satisfy the inclusion

$$D_i(x) \ \subseteq D_j(x) \qquad x \in E\ ,$$

therefore, we have $\sigma_i \leq \sigma_j$ if and only if $D_i \preceq D_j$. However, we have an increasing mapping from criteria to segmentations, which is not a lattice isomorphism. The segmented class $D^*(x)$ of $\sigma = \wedge\sigma_i$ at point x may not coincide with $\cap_i D_i(x)$, although the set $\cap_i D_i(x)$ is the class (at point x) of the infimum of all greatest partitions D_i associated with the σ_i criteria. More precisely, we have

$$\begin{aligned} D^*(x) &= \bigcup\{B \mid x \in B,\ \bigwedge \sigma_i[f, B] = 1\} \\ &\subseteq \bigcup\{B \mid x \in B,\ \sigma_i[f, B] = 1\} = D_i(x) \ \text{ for all } \ i \in I. \end{aligned}$$

As this inclusion is true for all $i \in I$, we finally obtain

$$D^*(x) \subseteq \cap_i D_i(x) \tag{14}$$

i.e. the greatest partition D^* associated with $\wedge\sigma_i$ is smaller than the infimum $\curlywedge D_i$ of the greatest partitions associated with each criterion σ_i. Figure 6 depicts two examples which show that Inclusion (14) may be strict. In the first example, we have to deal with a colour image, in RGB representation, in the Euclidean plane. A first criterion extracts the arcwise connected components of the plane where the red band is above Threshold r_0 (plus the

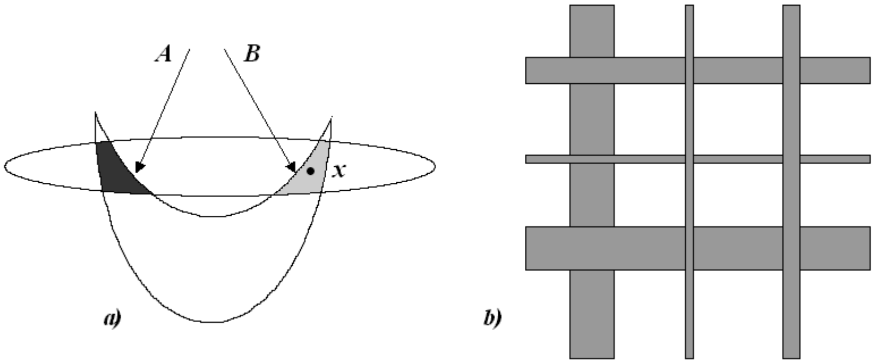


Figure 6: *a) a bi-coloured function has been thresholded. The ellipse (resp. crescent) is the connected component of the red (resp. green) threshold at point x; b) given a function f and a value a, a vertical (resp. horizontal) line is a connected component if it has one point y at least where* $f(y) \geq a$.

singletons). This gives the ellipse of Figure 6a. A second criterion acts similarly for the green band, which results in the crescent. The infimum of the two criteria corresponds to the arcwise connected components where both red and green bands are above their respective thresholds; the class $D^*(x)$ of the infimum at point x is the set B, which is strictly smaller than the intersection $A \cup B$ of the two red and green classes at the same point x.

The second example is the concern of orthogonal projections (see Figure 6b). Given a numerical function f in $\mathbb{R}^2$ and a number a, put $\sigma_v[f, B] = 1$ if B is a singleton, or if B is arcwise connected, and if each $y \in B$ belongs to a vertical straight line $D_y \subseteq B$ where one can find a point $z \in D_y$ with $f(z) \geq a$. Criterion σ_v is connective, as well the equivalent one, σ_h say, for horizontal lines. If the two partitions D_v and D_h have not the whole plane as unique class, then D_v (resp.D_h) is composed of vertical (resp. horizontal) stripes, and the classes of $D_v \curlywedge D_h$ form rectangles, whereas the partition D^* associated with $\sigma_v \wedge \sigma_h$ is that of the singletons.

Note that although space E is not a priori equipped with any connection, the datum of a connective criterion σ plus that of a function f supplies a connection to E. Now, if E is already equipped with some connection $\mathcal{C}'$, then the proposition implies that the intersection $\mathcal{C} \cap \mathcal{C}'$, which is still a connection, generates the maximum partition whose classes satisfy both constraints (Proposition 12). For instance, the criterion "function f is constant on A" leads to the partition of E where all flat zones belong to a same class. If, in addition, we demand that each class A be $\mathcal{C}'$-connected, then we find the segmentation of f into flat *and* connected zones, as previously described.

The nice property of a logical "AND" of criteria, stated in Proposition 12 (in the sense of intersection of connections), has unfortunately no counterpart for the logical "OR" (in the sense of supremum of connections) because the supremum of several connections does not always involve connected components where one criterion at least should be fulfilled.

Nevertheless, we can state the problem differently and look for the conditions under which the logical "OR" for connective criteria does exist, and makes sense. The following proposition answers this question

Proposition 13 *Given a function $f \in \mathcal{F}$, let $\{\sigma_i,\ i \in I\}$ be a family of connective criteria, of associated connections $\{\mathcal{C}_i,\ i \in I\}$. Then the following three statements are equivalent*

1. *For $J \subseteq I$ and any family $\{A_j, j \in J\}$ such that each $A_j \in \mathcal{C}_j$, we have*
$$\bigcap_{j \in J} A_j \neq \emptyset \Rightarrow \exists i_0 \in I \text{ such that } \quad \cup_{j \in J} A_j \in \mathcal{C}_{i_0}. \tag{15}$$
2. *The connection $\mathcal{C}$ associated with the supremum criterion $\sigma = \vee \sigma_i$ equals the union of the $\mathcal{C}_i$*
$$\mathcal{C} = \bigcup \{\mathcal{C}_i, i \in I\}. \tag{16}$$
3. *For any set $X \subseteq E$, each $\mathcal{C}$-component of X is also a $\mathcal{C}_i$-component of X for at least one index $i \in I$.*

Proof. $1 \Rightarrow 2$. If we prove that $\cup_i \mathcal{C}_i$ is a connection, since it is also the smallest upper-bound of all classes $\mathcal{C}_i$, we shall have proved Rel.(16). Consider a family $\{B_j, j \in J\}$ in $\cup_i \mathcal{C}_i$ of a non-empty intersection, and a point $x \in \cap_j B_j$. The family $\{B_j\}$ contains elements of each connection since the singleton $\{x\}$ belongs to all $\mathcal{C}_i$. Regroup all those sets B_j that are elements of a same connection $\mathcal{C}_i$ into their union $A_i = \cup \{B_{j(i)}, B_{j(i)} \in \mathcal{C}_i\}$. Then, according to Relation (15) we can write

$$\bigcup \{B_j, j \in J\} = \bigcup \{A_i, i \in I\} = A_{i_0},$$

where for some index i_0 we have $A_{i_0} \in \mathcal{C}_{i_0} \subseteq \cup_i \mathcal{C}_i$. Therefore $\bigcup \mathcal{C}_i$ turns out to be a connection, hence $\bigcup \mathcal{C}_i = \mathcal{C}$.

$2 \Rightarrow 1$. Let $\{A_j, j \in J\}$ be a family in $\mathcal{C}$ of a non-empty intersection. As $\mathcal{C}$ is a connection, $\cup\{A_j, j \in J\} \in \mathcal{C}$. Suppose that $\mathcal{C} = \cup_i \mathcal{C}_i$, then $\cup\{A_j, j \in J\} = A'_{i_0}$, with $A'_{i_0} \in \mathcal{C}_{i_0}$ for some index $i_0 \in I$, i.e. Relation (15).

$1, 2 \Rightarrow 3$. Let now X be an arbitrary set in E, x be a point of X and $Y_X(x)$ be the $\mathcal{C}$-component of X at point x. Rel.(16) implies that $Y_X(x)$ is given by the union

$$Y_X(x) = \bigcup \{B : x \in B \in \mathcal{C}_i,\ i \in I\} = \bigcup \{A_i(x), i \in I\}$$

where the $A_i(x)$ are the $\mathcal{C}_i$-components of X at point x. By applying Rel.(15) we see that $Y_X(x) = A_{i_0}(x)$ for some index i_0.

$3 \Rightarrow 2$. As set X spans $\mathcal{P}(E)$, and as point x spans set E, the $\mathcal{C}$-component $Y_X(x)$ spans the whole connection $\mathcal{C}$. Since $Y_X(x)$ always belongs to $\cup_i \mathcal{C}_i$ we have $\mathcal{C} \subseteq \cup_i \mathcal{C}_i$, hence $\mathcal{C} = \cup_i \mathcal{C}_i$, which achieves the proof. ■

Proposition 13 remains true when Rel.(15) is satisfied for an arbitrary family $\{A_j, j \in J\}$ in $\cup_i \mathcal{C}_i$. However, the adopted presentation makes checking easier. This proposition clarifies the possibilities and the limits of the logical "OR" in segmentation. It serves rather often in the elaboration of connected operators. But its most common use arises when the non-point class $A_{i_0}(x)$ covers point classes only of all other connections $\mathcal{C}_i$, $i \neq i_0$. Then Rel.(15) is satisfied and Proposition 13 allows us to calculate the maximum class at point x without having to calculate the other classes. The global approach to segmentation is therefore reduced to independent more regional ones. This circumstance is exploited below for generating jump connections for one-jump connections, and in the 2-D histogram of the colour guitar image. Other properties of the lattice of connective criteria and of the associated partitions are currently under study by C. Ronse, who has established several instructive results (private communication).

Cross connections Both Propositions 12 and 13 are consequences of the lattice structure of the connections over set E. Now, in Theorem 8 each connection $\mathcal{C}$ is generated by combining a criterion σ and a function f. We kept function f fixed because we wished to focus mainly on the criteria, but such a requirement becomes cumbersome when the two Propositions 12 and 13 are contemplated from the viewpoint of the connections.

According to Theorem 8, each pair (σ_i, f_i) of a connective criterion σ_i and a function $f_i \in \mathcal{F}$ determines a unique connection $\mathcal{C}_i = \mathcal{C}_i(\sigma_i, f_i)$. Therefore, by using the same proofs as in Propositions 12 and 13, we can state the following

Theorem 14 *(Cross connections theorem) Let $\{\sigma_i, f_i, i \in I\}$ be a family of connective criteria σ_i and of function $f_i \in \mathcal{F}$, of associated connections $\mathcal{C}_i$. Then*

1. *The connection $\mathcal{C} = \cap_i \mathcal{C}_i$ is the largest one such that each criterion σ_i is satisfied for the corresponding function f_i on each connected component of $\mathcal{C}$;*
2. *Given $\mathcal{C}' = \vee \mathcal{C}_i$, the following two facts are equivalent:*

   (i) $\mathcal{C}'$ is the smallest connection where on every component one criterion σ_j at least is satisfied for the corresponding function;

   (ii) $\mathcal{C}'$ satisfies item 2 of Proposition 13.

This cross connections theorem generalizes both Propositions 12 and 13, and opens the door to multivariate functions, such as time sequences of images, or colour images (an example is treated in Section 8 below).

2.6 Levels of generality

Remarkably, we have been able to identify the segmentations on functions $f : E \to T$ with some family of connections without having equipped either the starting space E or the arrival space T with any *a priori* property. This is true for Theorem 8 as well as for Propositions 12 and 13, which do not impose any condition on E and on T. When, in the following, we shall build up a theory of the connected operators, we shall have to make T a complete lattice. The same restriction also appears in the examples below, but it is not an obligation.

Indeed, Theorem 8 opens the way to all applications where various heterogeneous variables are defined over *the same space*. These circumstances arise in geography, for example, where radiometric data (satellite images) live together with physical ones (altitude, slope of the ground, sunshine, distance to the sea, etc.) and with statistical data (demography, fortunes, diseases, etc.). Among others, the versatility of space E allows us not to pay too much attention to the continuous/digital distinction. All the illustrations below are, of course, digital, but the algorithms they describe can always be stated in both continuous and digital frameworks.

Although the mapping f is presented as a function of the points of E, it always intervenes in association with elements A of $\mathcal{P}(E)$: in Definition 7 of a connective criterion, as well as in the proof of Theorem 8, the distinction between the two levels E and $\mathcal{P}(E)$ is never brought into play (singletons excepted, but their axiom is independent of f). Nevetheless, for the sake of clarity, it was preferable to fit the level of the theory with what we need for the examples below. All of these examples deal with pointwise functions.

More deeply, the whole approach might be set in a more abstract framework, by replacing $\mathcal{P}(E)$ by some complete lattice $\mathcal{L}$, say, since the theory of set connection has been extended to complete lattices in [80]. As a matter of fact, in none of Definitions 2, 4, 7, Theorem 8 and Propositions 12 and 13, is the $\mathcal{P}(E)$ specific structure compulsory. But we need the assumption that lattice $\mathcal{L}$ is sup-generated, in order to extend the first point of Definition 4. In addition, for the proof of Theorem 8, we also need to partition any element $a \in \mathcal{L}$ into all but one sup-generator classes. This is not trivial, and we conjecture that the complete distributivity of $\mathcal{L}$ (in Birkhoff's sense [9]) should provide us with a necessary and sufficient condition, as it can be identified with the monoseparation in Matheron's sense (Theorem 9-2 in [52]). One can also wonder on the practical interest of adopting the framework of the complete lattices. Here, the answer is two-sided:

1/ We may meet in practice situations where the underlying structure is not exactly of $\mathcal{P}(E)$ type. A typical example is given by the morphological graph processing, such as the Binary Partition Tree of Ph. Salembier and F. Marqués [73],[74], or the Region Adjacency Graph of F. Meyer [58],[59], or again the connectivity trees of C.S. Tzafestas and P. Maragos [92]. In these cases the basic piece of information puts together points and edges (i.e. doublets of points), and rather often the set E is a product of the image space by the resolution axis. However graphs, and scale spaces are not so far away from a $\mathcal{P}(E)$ lattice, and can be reduced to it under some changes.

2/ In machine vision, the *space image* currently under study is compared and combined with *non spatial* models, features, and memorized data. The human body, as a ground of investigations for medicine, lends itself to the same comment. Then the segmentation theorem 8 still can handle such heterogeneous combinations as soon as the non-spatial references are equipped with convenient ordering.

We now pursue the present segmentation theory by describing its validity on a comprehensive series of situations. We deliberately mix well known algorithms (watersheds, region growing) with more recent ones. Our aim is to show how broad is the scope of the segmentation concept. The examples are classified according to a double entry, according to whether they apply to the space of the function under study, or to feature spaces, and whether the criteria involve or not some seeds (the feature spaces of these last two categories are gathered together). In each of these examples, the function f under study is always fixed, therefore it is sometimes omitted in the notation, when the context is not ambiguous.

3 Seedless segmentations

The Lipschitz counter-example we used above left us in doubt about this type of functions which is, in other respects, a basic model in mathematical morphology. Now, the distance that we took for rejecting it, in the example of Figure 4 was based on an overall metric on space E. What happens if we restrict the distance values, or if we replace the overall metric by a new one whose geodesics should lie *inside* the sets A and B of Figure 4? We will now explore these approaches, after a brief reminder on ω-continuity .

3.1 Reminder on ω-continuity

E is now a metric space, of distance d, and T is the lattice of the extended real line, i.e. $\mathbb{R}$, plus the two points at plus and minus infinity. We enlarge the Lipschitz class to that of the ω-continuous functions, because they are the concern of the same approach. The ω-continuous functions are classically defined by the following condition

$$|f(x) - f(y)| \leq \omega(d(x,y)) \qquad \forall\ x, y \in E \tag{17}$$

where the mapping $\omega : \mathbb{R}_+ \to \overline{\mathbb{R}_+}$ is continuous at the origin and must be sub-additive, i.e. $\omega(a+b) \leq \omega(a) + \omega(b)$. This parameter ω is called *modulus of continuity* [23]. For each modulus ω, the ω-continuous functions form a complete sub-lattice $\mathcal{F}_\omega$ of T^E, i.e. a lattice where sup and inf turn out to be the *pointwise* supremum and infimum in each point [78]. In practice, the most useful three ω-continuous classes are

- the Lipschitz functions, where we substitute $kd(x,y)$ for $\omega(d(x,y))$ in the inequality (17),
- the functions whose variation is *bounded* by a sill r which is reached at range d_0. Their modulus ω is given by

$$\omega = \frac{rd(x,y)}{d_0} \quad \text{if} \ \ d \leq d_0 \qquad ; \qquad \omega = r \quad \text{if}\ d \geq d_0 \tag{18}$$

- and, of course the class of the constant functions

Relation (17) suggests an equivalent definition of ω-continuity, in a more morphological style. Define the *increment function* $\alpha_b(f)$ as

$$\alpha_b(f) = (\delta_b f - f) \bigvee (f - \varepsilon_b f) \tag{19}$$

where δ_b and ε_b are respectively the dilation and the erosion by the closed ball of radius b (a notion that is close to Beucher's gradient $(\delta_b f - \varepsilon_b f)/2)$. Then, a function is ω-continuous if and only if

$$\alpha_b(f) \leq \omega(b).$$

We now come back to our objective, namely elaborate ω-continuity based segmentations. In the counter-example of Figure 4, we have been too demanding when we imposed the Lipschitz condition for any value of the distance. Such a condition should be relaxed and reduced to all distances smaller than a given range a, as proposed in the *regional ω-continuity* .

Definition 15 *A function $f \in \mathcal{F}$ is* regionally ω-continuous *of range $a > 0$ on a set $A \in \mathcal{P}(E)$ when it is ω-continuous in the (non-geodesic) open ball $B_a(x)$ of radius a and centered at any point $x \in A$.*

For example, a function $f \in \mathcal{F}$ is a regional Lipschitz on set A when we have

$$x \in A,\ \ y \in E \text{ and } d(x,y) \leq a \Rightarrow |f(x) - f(y)| \leq kd(x,y) \tag{20}$$

The ball $B_a(x)$ is said to be *Lipschitz* (for function f and slope k).

3.2 Lipschitz segmentations

3.2.1 Regional Lipschitz connections

Relation (20) suggests to introduce the following two criteria

$$\sigma_a[f, A] = 1 \quad \text{iff} \quad x \in A \ \Rightarrow B_a(x) \text{ is Lipschitz} \tag{21}$$

$$\sigma'_a[f, A] = 1 \quad \text{iff} \quad x \in A \ \Rightarrow \exists y : x \in B_a(y) \subseteq A, \text{ where } B_a(y) \text{ is Lipschitz,} \tag{22}$$

or if A is a singleton (for both criteria). In the first criterion, unlike the second one, the $B_a(x)$'s are *not* supposed to be included in set A. Criterion (21) reminds us of an erosion, and Criterion (22) of an opening. Indeed, one passes from the first to the second by replacing set A by its dilate $\delta_a(A)$ according to the open ball B_a

$$\sigma_a[f, A] = 1 \Rightarrow \sigma'_a[f, \delta_a(A)] = 1.$$

Proposition 16 *Both criteria σ_a and σ'_a are connective.*

Proof. Let $\{A_i, i \in I\}$ be a family of sets that satisfy Relation (21), and let $x \in \cup_i A_i$. There exists an index $i_0 \in I$ such that $x \in A_{i_0}$ therefore $B_a(x)$ is Lipschitz and $\sigma_a[f, \cup_i A_i] = 1$. Similarly, when all members of the family $\{A_i, i \in I\}$ satisfy Criterion (22), then for each point $x \in \cup_i A_i$ one can find an index $i_0 \in I$ and a point y such that $x \in B_a(y) \subseteq A_{i_0} \subseteq \cup_i A_i$ so that $\sigma'_a[f, \cup_i A_i] = 1$. ■

Remarkably, the assumption " $\cap_i A_i = \emptyset$" does not intervene in the proof; therefore criterion σ_a leads to a segmentation partition D_a which has a unique non singleton class, $\overline{D}_a$ say, with

$$\overline{D}_a = \bigcup \{x \mid x \in E \text{ and } B_a(x) \text{ is Lipschitz}\}. \tag{23}$$

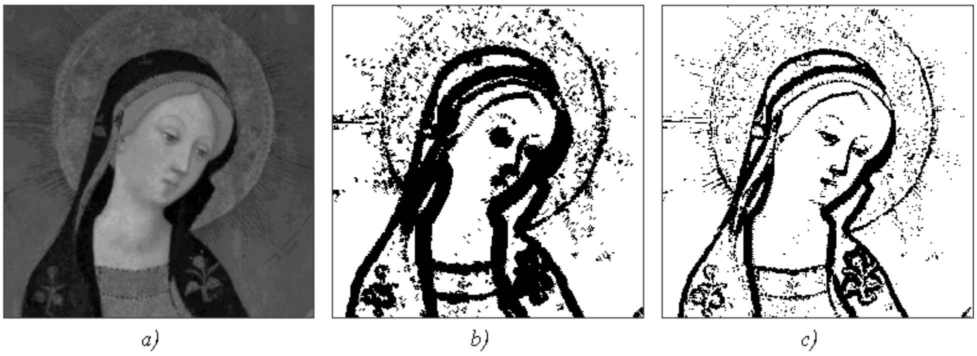


Figure 7: *a) 8-bits initial image; b) and c) in white, zones that are locally Lipschitz by erosion (b) and by opening (c) w.r.t. H(6,7); in black, the zones whose all pixels are singletons classes.*

Given a connection $\mathcal{C}$, the class $\overline{D}_a$ can contain isolated points w.r.t. $\mathcal{C}$, then in the segmentation arising from the intersection of $\mathcal{C}$ and the connection arising from σ_a these points form singleton classes, in addition to the other singleton classes in the complement of $\overline{D}_a$ (for a more detailed discussion about this ambiguity, see Section 5 in [87]). Similarly, Criterion (22) leads to segmentation partition D'_a whose unique non singleton class

$$\overline{D}'_a = \bigcup\{x \mid x \in E \text{ and } x \in B_a(y) \text{ which is Lipschitz}\} = \delta_a(\overline{D}_a). \tag{24}$$

The implementations of criteria σ_a and σ'_a are easy. In the Euclidian case, this relation (23) means that on class $\overline{D}_a$, function f is equal to both its erosion and its dilation by the vertical "pencil" $H(k,a)$ (a cylinder whose top ends by a cone) of slope k and radius a

$$x \in \overline{D}_a \Leftrightarrow f(x) = (f \ominus H)(x) = (f \oplus H)(x). \tag{25}$$

The passage to $\overline{D}'_a$ is straightforward by means of Eq.(24). Relation (23) also provides a digital algorithm to perform the segmentation. Remark that the expression of the criterion on a set A requires that one knows f on the *dilate* of A by the ball of radius a (and non only on A itself). Figure 7 illustrates the two Lipschitz type segmentations. The sets $\overline{D}_a$ and $\overline{D}'_a$ are the whites in Figure 7b and c respectively, for the parameter values indicated in the caption. They are instructive because they substitute a regional problem that admits a solution for a global that does not, but they depend on the size parameter a which is not convenient for practical uses, and which leaves thick black stripes, in case of erosions at least. We will now try and eliminate this parameter by segmenting according to the so-called *smooth connection*.

3.2.2 Smooth connection

As radius a reduces, the condition (23) becomes less demanding, and the non-singleton class $\overline{D}_a$ increases. In fact, the whole partition D_a increases since all its other classes, which are singletons, can only enlarge and we have

$$
\begin{aligned}
0 &< b \leq a \Rightarrow \overline{D}_b \supseteq \overline{D}_a \Rightarrow D_b \succeq D_a \\
0 &< b \leq a \Rightarrow \overline{D'}_b \supseteq \overline{D'}_a \Rightarrow D'_b \succeq D'_a
\end{aligned}
$$

Therefore the supremum of the partitions D_a, i. e. $D = \curlyvee\{D_a,\ a > 0\}$ has itself a unique non-singleton class

$$\overline{D} = \bigcup\{\overline{D}_a,\ a > 0\} = \{x \in E \mid \exists a > 0 : B_a(x) \text{ is k-Lipschitz for function } f\}$$

Class $\overline{D}$ is the greatest open set on which the function is locally Lipschitz. A point x belongs to class $\overline{D}$ of D if and only if one can find a $a > 0$ such that function f is k-Lipschitz in $B_a(x)$. When no such $a > 0$ can be associated with point x, then the singleton $\{x\}$ turns out to be one of the classes of the supremum partition D.

Similarly, the opening type partitions D'_a yield the supremum $D' = \curlyvee\{D'_a,\ a > 0\}$, which admits the following unique non-singleton class

$$
\begin{aligned}
\overline{D}' &= \bigcup\{\overline{D'}_a,\ a > 0\} \\
&= \{x \in E \mid \exists a > 0, \exists y \in E : x \in B_a(y) \subseteq \overline{D}' \text{ and } B_a(y) \text{ is k-Lipschitz for function } f\}
\end{aligned}
$$

In the Euclidean case, i.e. when $E = \mathbb{R}^n$, the two supremum partitions D and D' are identical. To show it, it suffices to prove that the two non-singleton classes $\overline{D}$ and $\overline{D}'$are the same. We draw from Eq.(24) that $\overline{D} \subseteq \overline{D}'$. Conversely, let $x \in \overline{D}'$.Then there exists a Lipschitz open ball $B_a(y)$ for some radius $a > 0$, and some centre $y \in E$, with $x \in B_a(y) \subseteq \overline{D}'$. Therefore $d(x, y) < a$ and if one takes b in the interval $]a-\ d(x, y), 0[$, then the open ball $B_b(x)$ is k-Lipschitz for f, so that $x \in \overline{D}_b \subseteq \overline{D}$. This results in the equality $\overline{D} = \overline{D}'$. Such an identification no longer holds in the digital case, i.e. when $E = \mathbb{Z}^n$. Then we have $\overline{D} = \overline{D}_1$ and $\overline{D}' = \overline{D}'_1$.

Clearly, segmentation D is induced by the connective criterion σ :

$$\sigma[f, A] = 1 \text{ iff } x \in A : \Rightarrow \exists a > 0 \text{ and } B_a(x) \text{ is Lipschitz, or } A \text{ is a singleton,}$$

which is noticeably different from the union $\sigma_0 = \cup\{\sigma_a\ ;\ a > 0\}$ (which is not connective). The quantity $\sigma_0(f, A)$ equals 1 when there exists a value $a > 0$ such that $B_a(x)$ is k-Lipschitz for *all points x of set A* , whereas when $\sigma[f, A] = 1$ the Lipschitz radius a can vary as point x spans set A.

Stronger results are obtained when space E is equipped with an initial connection $\mathcal{C}$. Then the intersection of the connective criterion that underlies $\mathcal{C}$ with the local k-Lipschitz property generates zones that are both smooth and $\mathcal{C}$-connected.

When connection $\mathcal{C}$ is an arcwise connectivity, then in each class A of D, function f is ω-continuous along all paths included in the interior $\overset{\circ}{A}$ of A. In case of the 2-D digital space Z^2 in particular, the smallest value of the range a of Rel.(20) is one (unit square, or hexagon).

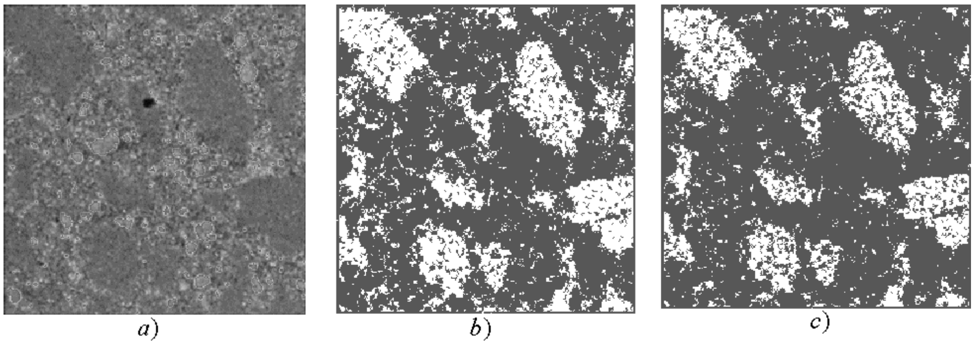


Figure 8: *a) electron micrograph of concrete; b) and c) segmentation of image a) by smooth connections of parameters 7 and 6 respectively.*

It suffices, to implement it, to erode functions f and $-f$ by the cone $H(k,1)$ whose base is the unit square or hexagon, and whose height is k, the origin being placed at the top, and then to take the intersection of the two sets where f equals its erosion and its dilation respectively. Alternatively, the point classes of partition D can be obtained by taking the threshold of the increment function (19) between 0 and k, with $b = 1$. The two algorithms are equivalent.

The smooth connection turns out to be a good segmentation tool to separate the smooth zones from the more granular ones with a similar grey level, as it often appears in electron microscopy [82]. Figure 8 depicts two segmentations of a concrete micrograph, carried out by using smooth connections.

3.2.3 Quasi-flat zones

Instead of requiring that f be ω-continuous along *all* paths included in A, we can also only require the ω-continuity for *at least one* path. This more comprehensive new criterion is still connective, and leads to the connection according to the "*quasi-flat*" zones, due to F. Meyer [57]. This time, the digital implementation involves geodesic reconstructions (see Figure 29b below).

4 Seed-based segmentations

From now on the viewpoint changes: the segmentations under study start and spread out from some specific sources. Consider a clustering process where we find on the one hand a certain function f, and on the other hand a family $\mathcal{G} = \{G_i,\ i \in I\}$ of sets $G_i \in \mathcal{P}(E)$. The sets G_i are called *seeds* because the clustering process consists in aggregating to them all points x of the space, according to some rules that involve function f. The clustering process may comprise, or not, several iterations; the seeds locations may be fixed or depend on the iterations; the seeds themselves may be endogenous (e.g. the minima of f) or carry an external information. Whatever these conditions are, and the clustering algorithm is, it results in the assignment of each point $x \in E$ to one, or more, final seed. Then every such process generates a *connective criterion.* More precisely, we can state the following, where $\mathcal{S}(E)$ stands for the set of the singletons of $\mathcal{P}(E)$

Proposition 17 *Given a function $f \in \mathcal{F}$, a seed family $\mathcal{G}(E)$, and a clustering process which leads to a final space distribution of the seeds, the criterion*

$$\begin{aligned}
\sigma[f,\{x\}] &= 1 \quad \{x\} \in \mathcal{S}(E) \\
\sigma[f,A] &= 1 \text{ for } A \in \mathcal{P}(E)\backslash\mathcal{S}(E), \text{ when all points of } A \\
& \qquad \text{are assigned to a unique final seed,} \\
\sigma[f,A] &= 0 \text{ when not,}
\end{aligned}$$

is connective.

Proof. The first axiom of definition 7 is satisfied by construction of criterion σ. For proving the second one, consider a family $\{A_i,\ i \in I\}$ in $\mathcal{P}(E)$ whose intersection is not empty. We observe that if all points of set A_i are assigned to a unique seed $G_i,\ \ i \in I$ and if x is a point of the intersection $\cap_i A_i$, then x is assigned to all seeds G_i and the uniqueness assumption implies that $G_i = G_j$ for all $i, j \in I$. ■

In the final maximum partition, the regions whose points are aggregated to more than one seed are pulverized into all their singletons (the reader may check that the criterion "the points of A are assigned to one or two seeds" is not connective, and that the criterion "the points of A are assigned to one or more seed " comes back to take set E as the unique class).

We now list a few seed based segmentations. They are classical and popular, except the last one (jump connection [82]) which is less well known, although it is particularly efficient.

4.1 Watershed contours

We begin this brief survey by mentioning one of the oldest segmentation techniques, which was initially due to S. Beucher and Ch.Lantuejoul, namely the watershed contours [7],[55].

Consider an integer function f from $\mathbb{R}^n$ or $\mathbb{Z}^n$ into $\mathbb{Z}$, and interpret the graph of f as the surface of a geographical relief. Suppose that a hole is made in each local minimum, from which the relief is flooded (Figure 9). Progressively, the water level increases. In order to prevent the merging of water coming from two different holes, at each step of the progression a dam is built on each contact point. At the end, the union of all complete dams define the so-called *watersheds*. The method is particularly adapted to contour detection. For example, we can observe in Figure 10b that the gradient module of the biological cells of Figure 10a exhibits crest lines that correspond to the maximum slope variation in the initial cells image. Therefore, when we superimpose the watershed lines of the gradient with the initial image we obtain a contouring of the cells.

As the criterion "all A points are flooded from a same minimum" is obviously connective, the watersheds partition the definition space E into arcwise connected catchment basins,

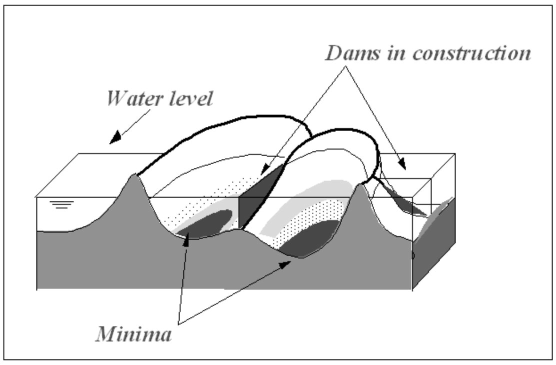


Figure 9: *Watersheds of a relief*

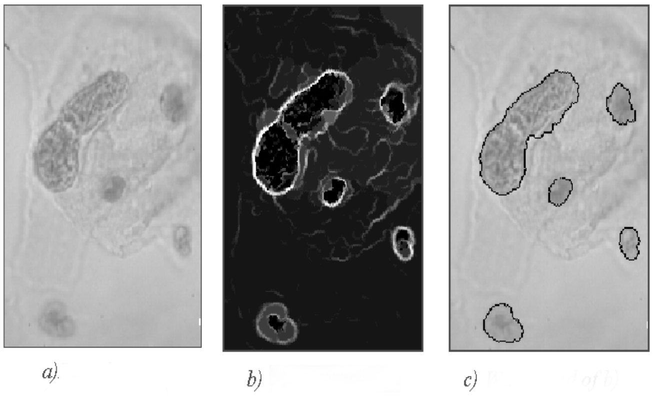


Figure 10: *a) cytological smear; b) module of the gradient of a); c) watershed of b).*

plus into all points of the watershed contours, considered as singletons in the segmentation partition.

As a matter of fact, a gradient module is often noisy, and the derived watershed partition over-segmented. One gets around the drawback by replacing the minima gradient g by the given seeds. The cytological smear of Figure 10b was processed this way. There always exists a smallest function $\varphi(g)$ larger than g and whose all minima are located at the seeds implantations. The operator $\varphi : g \to \varphi(g)$ turns out to be a closing, called *swamping* [55].

4.2 Region growing

The well known Chassery-Garbay algorithm for colour image segmentation [22] builds regions from a series of seeds that play exactly the role of minima. Each pixel is compared to the seeds according to an aggregation criterion that involves colour distances, plus other possible attributes, such as texture or convexity constraints. These comparisons yield a first segmentation of the space. Then a new seed is redefined in each class according to various motivations, and the segmentation procedure is iterated... and so on for a finite number of times. Then by applying Proposition 17 to the last iteration, we see that the whole procedure is connective.

The same comment applies to a number of classification algorithms for clouds of points, that work in multidimensional spaces, such as the Diday's popular K-means [27].

4.3 Jump criteria and connections

The human vision easily accepts small variations when they originate from the minima or the maxima of the function under study. The *jump criteria* we present now work this way, according to a few variants [80][82]. They generalize notions such as thresholds or level sets (for the sake of simplicity, we do not systematically reproduce below the axioms of connective criteria for the singletons and the empty set).

Let E be a space on which a connection, $\mathcal{C}$ say, has been defined. Suppose that the arrival space T is metric and totally ordered (e.g. $T = \overline{\mathbb{R}}$, $T = \overline{\mathbb{Z}}$ or any compact subset of $\overline{\mathbb{R}}$ or $\overline{\mathbb{Z}}$). Consider a continuous function $f : E \to T$, hence a function that has minima and maxima.

one-jump criterion σ. Let $A \subseteq E$ be a $\mathcal{C}$-connected set, and k be a positive constant. Define the one-jump criterion σ by the following requirement: $\sigma[f, A] = 1$ if A is a singleton, or the support of a minimum of f, or if for any point $x \in A$ there exists a minimum m of f of support $M \subseteq A$ and such that $0 \leq f(x) - m < k$.

The positive number k is called the range, or the jump, of σ. Criterion σ is obviously connective: when it is satisfied by all the A_i of a family $\{A_i, i \in I\}$and if $x \in \cap_i A_i$, then $\cup_i A_i$ is connected because each $A_i \in \mathcal{C}$; in addition at each point $z \in \cup_i A_i$ the value $f(z)$ is less than k above a minimum. The minimum m is not necessarily unique in A.

The segmentation partition $D_{f,k}$, of class $D_{f,k}(x)$ at point x, may be described in the following way. Let $\mathbb{M}$ be the family of the ($\mathcal{C}$-connected) supports $M \subseteq E$ of all minima m of the function f. For $M \in \mathbb{M}$ consider the set $B(M)$ such that

$$B(M) = \{z : z \in E, \quad 0 \leq f(z) - m < k\}.$$

Set $B(M)$ is composed of all pixels of E where the values of function f are less than k above level m. It obviously contains M. Denote by $S(M)$ the connected component of $B(M)$ that contains the support M. Then the set

$$S_f = \cup\{S(M), \quad M \in \mathbb{M}\} \tag{26}$$

is composed of all points x of the space where $f(x)$ is less than k above a minimum, which entails that the class $D_{f,k}(x)$ at point x is the $\mathcal{C}$-connected component $\gamma_x(S)$ of set S at point x, when $x \in S$, and $D_{f,k}(x) = \{x\}$ when not. One will notice that, unlike the watershed case, each segmented component $\gamma_x(S)$ may comprise several minimum zones M. Note also that the connection generated by the one-jump criterion σ_1 is just a sub-set of $\mathcal{C}$ since each $A(M)$ is $\mathcal{C}$-connected.

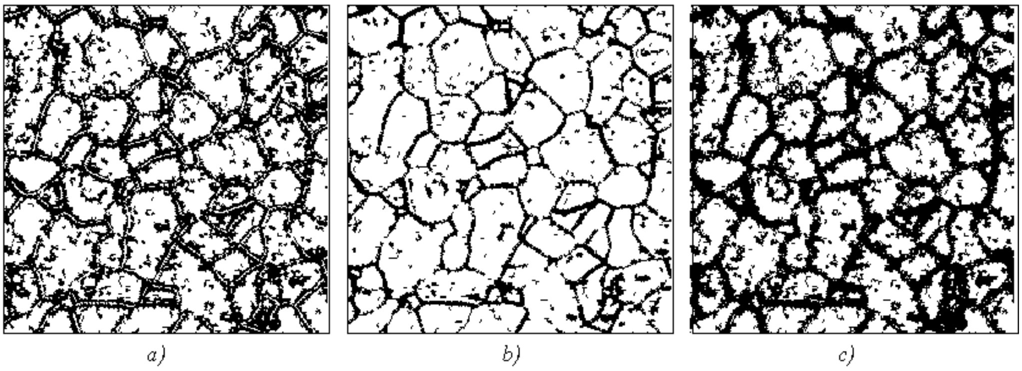


Figure 11: *Jump connections generated by the alumina micrograph of Fig. 2, for a jump* $k = 15$; *a) from below, b) from above, c) infimum of below and above. In black, the set* R *of all singletons.*

We will illustrate the one-jump criterion when f is the alumina micrograph of Figure 2a, also reproduced in Figure 12a. Figure 11a depicts the partition $D_{f,k}$ induced by f for $k = 15$. Conventionally in Figures 11a,b and c the white particles are the non singleton classes of the partitions and the sets in black are made of all singletons $S = \cup[\{x\}, D_{f,k}(x) = \{x\}]$. The flat bottoms of the valleys appear as white stripes inside black tubes (the slope regions).

As criterion σ is not self-dual, it may be instructive what it gives when we apply it to the negative image $255 - f$. The new criterion, σ' say, derives from σ by replacing *"minimum"* by *"maximum"* in the definition of σ. Its action on the alumina micrograph is depicted in Figure 11b. The bottoms of the valleys now become dark, but less continuous. These valleys, i.e. the grains boundaries of alumina, will be better extracted if we combine both criteria into a symmetrical version. The simplest symmetrization consists in taking the infimum $\sigma\wedge$ σ' since we know from Proposition 12 that $\sigma\wedge$ σ' inherits the connectiveness of $\sigma$and σ'. Due to the display convention of Figure 11, it suffices to take the intersection of the two sets Figure 11a and b. The result is depicted in Figure 11c. This intersection may seem rough, but its skeleton by influence zones yields the rather satisfactory boundary extraction shown in Figure 12b.

The concrete electron micrograph of Figure 8a gives another opportunity to illustrate the "and" Proposition 12 when a second connection is used to strengthen a first one, namely the smooth connections of Figure 8. By taking its infimum with a jump connection of a suitable parameter, we remove the noise of each connection, and obtain the rather nice Figure 13c, where there are more "blacks", i.e. more singleton classes, but a better noise reduction.

However the symmetrization $\sigma \wedge \sigma'$ by infima has a disadvantage: when one class A from σ overlaps with another, A' from σ', their intersection may become a new class in the best case, but is often pulverized into all its singletons. An alternative solution consists in splitting $A \cup A'$ into two classes only, according to whether $f(x)$ is closer to the minimum m, or to the

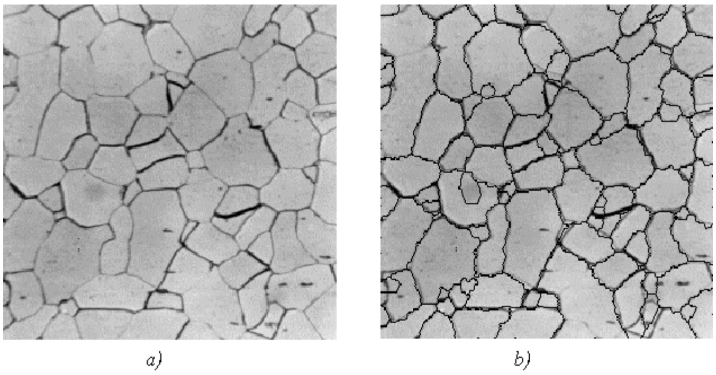


Figure 12: *a) initial alumina micrograph ; b) superimposition of a) and of the Skiz of set Fig. 11c.*

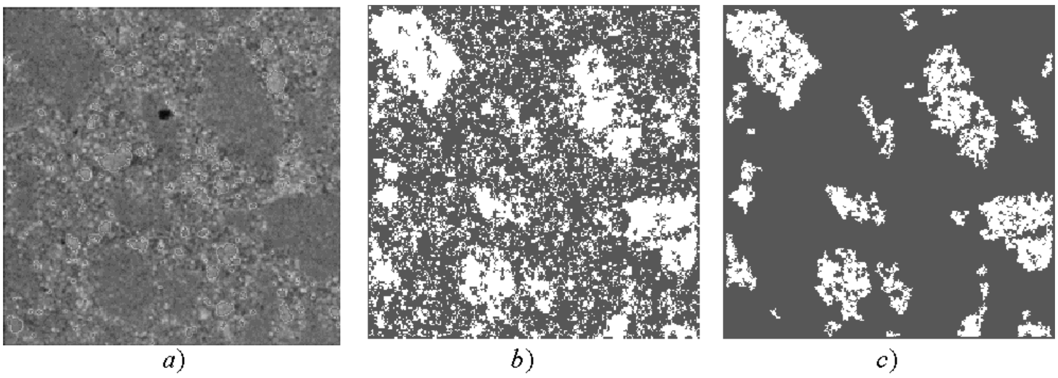


Figure 13: *An example of intersection of criteria. a) concrete electron micrograph of Fig.8a ; b) segmentation of a) by jump connection of range 12 ; c) intersection of jump connection b) and smooth connection of range 6 obtained in Fig.8c .*

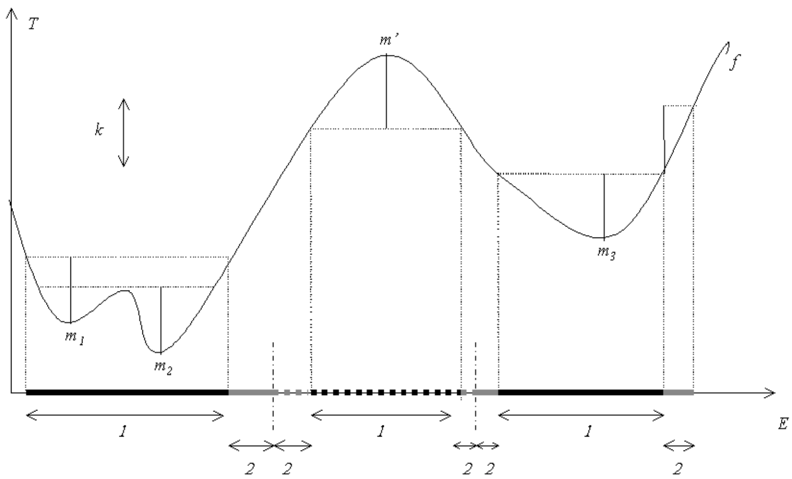


Figure 14: *Symmetrical jump connection in one dimension. The first step (one-jump) is depicted by the "1" arrows; the second, and last, step of the iteration by the "2" arrows.*

maximum m'. The corresponding symmetrical one-jump criterion σ^* is formally introduced as follows

$\sigma^*[f, A] = 1 \Leftrightarrow \{A \in \mathcal{C}$, and for each $x \in A$ there exists either

- a minimum m of f in A such that $0 \leq f(x) - m < k$,
and $\mid f(x) - m \mid \leq \mid f(x) - \mu \mid, \forall \mu \in \mathcal{M}\}$

- or a maximum m' of f in A such that $0 \leq m' - f(x) < k$
and $\mid f(x) - m' \mid \leq \mid f(x) - \mu' \mid, \forall \mu' \in \mathcal{M}'\}$

where $\mathcal{M}$ (resp.$\mathcal{M}'$) stands for the set of all values of the minima (resp. the maxima) of function f. Figure 14 depicts an example of jump connection in one dimension. The "1" arrows correspond to the the one-jump situation, and the dotted lines to the segments from the maxima..

Jump criterion σ In the alumina micrograph of Figure 11, the one-jump connection has been used more for extracting the union R of all singletons (the black set) than for differentiating the segmented classes (the white connected components). In other situations the later objective becomes primordial, so that the singleton area has to be reduced as most as possible. This goal can be obtained by iteration. Suppose that the function f is bounded, write $S_1 = S_f^c$ for the complement of the set S_f defined by Relation 26, and denote by f_1 the restriction of function f to S_1. Then the application of the one-jump criterion σ to f_1 results in a new zone $S_{f_1} \subseteq S_1$ such that in each $\mathcal{C}$-connected component of S_{f_1} the function f_1 is less than k above one of its minima.

Under iteration, successive nested sets S_2, S_3,etc. are generated. As function f is bounded, after a finite number n of iterations the set S_n becomes empty. The whole space E is then partitioned into zones in which the range of variation above inside minima is smaller than k. One proves (Proposition 3 in [87]) that this partition is the greatest one with $\mathcal{C}$-connected classes in which f (resp. $f_1, f_2, ...f_n$) is less than k above a minimum in S_f (resp. $S_{f_1}, S_{f_2}, ...S_{f_n}$). In the case of Figure 14, the whole process comprises two steps only.

The same type of iterations applies as well to the symmetrical form σ^* of the one-jump criterion. In the applications, this symmetrical iterative version turns out to be an excellent technique to segment images, thanks to the quality of the partitions it creates (few point zones, visually significant classes), and to its fast computation. As the jump k varies geometrically ($k = 1, 2, 4, 8..$) the partitions that segment f are nested in each other. This allows us to build up *multiscale segmentation schemes* by means of jump connections. Figure 15 depicts an example in colour imagery, where the only piece of information we keep is the luminance, because its extrema are visually more significant than those of the saturation or the hue. To give a better idea of the resulting segmentations, the arithmetical averages of the red, green and blue in each connected class have been calculated ; this creates a mosaic which can be compared to the original image. Notice the lack of false colors in the results.

A last comment: watersheds and jump criteria have been introduced as acting on numerical functions. We did it because such a framework models the examples presented here, and also because we implicitly chose to take minima and maxima as seeds. But when exogen seeds are given, or brought from a previous processing step, then the arrival space T no longer needs to be totally ordered; it suffices to provide it with a metric. Indeed, the actual demand is the existence of *seeds*, and if we suppress them, we lose connectiveness. For example, a criterion such as

$$\sigma[f, A] = 1 \Leftrightarrow [\sup\{f(x), x \in A\} - \inf\{f(x), x \in A\}] \leq k$$

is not connective: one could not segment a function from it.

5 Feature-spaces based segmentations

The above segmentations involve image operations for extracting the connected zones. These transformations may or may not depend on numerical parameters, but they always occur in the same space as that of the image under study. The segmentation procedures we now have in mind refer to an *external space* in which some measurements are displayed. The latter can be histograms, in one dimension or more, or the plot of the surface area of a moving wave front, or again a probability space in which optimizations are performed in an iterative manner, as the relaxation techniques do. In these approaches, the connective criteria generally admit a double interpretation according to whether they are stated in the image space or in the external one, although they still remain, basically, the concern of points or regions of the image under study.

The threshold operator is the simplest measurement based segmentation. After having briefly presented it and listed a few of its numerous variants, we devote the last two sections

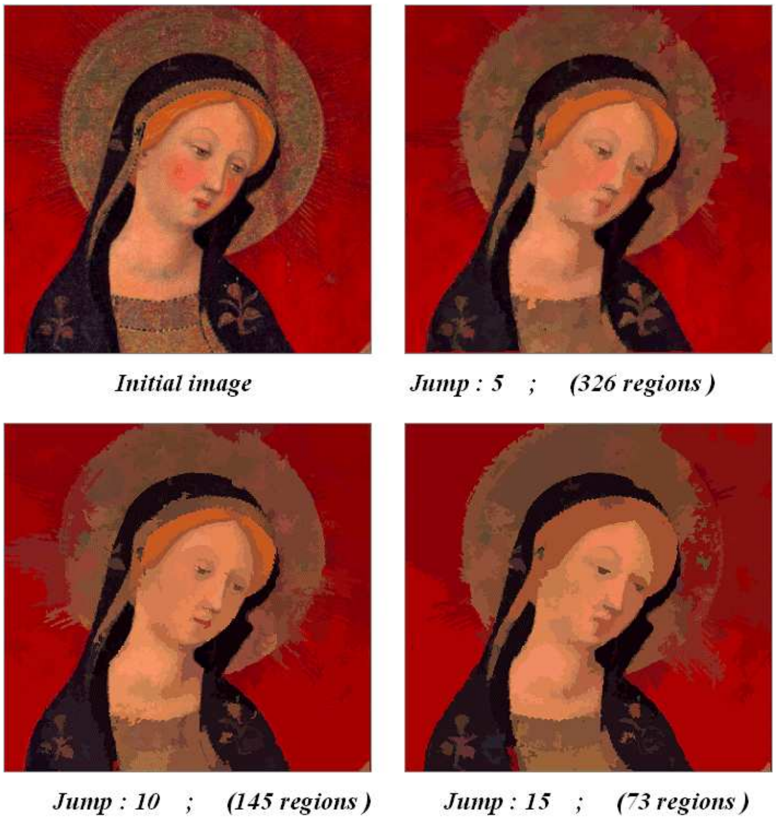


Figure 15: *a) Colour version of St Cugat painting of The Virgin* ; *b) c) and d) segmentations of a) by iterative jump connections of range 5 , 10 and 15 respectively. Note that the region number decreases exponentially.*

to two more sophisticated situations. In the first one, the image is a 3-D *set,* whose indicator function is unfortunately a constant, so that we have to find a more convenient numerical function to segment in some external space. In the second situation, which deals with a colour image, it is shown that the segmentation of the whole space is not necessary for extracting some of its classes of interest. In the whole section, it is always supposed that the criteria are satisfied by the singletons and by the empty set.

5.1 Connective criteria and threshold

In the space of a numerical image, the threshold operator is defined by the following criterion

$$\sigma_0[f, A] = 1 \Longleftrightarrow \inf\{f(x), x \in A\} \geq t_0$$

which extracts all those points where function f is above (or below) a given value t_0. The threshold criterion is obviously connective. Up to now, all criteria we have described were translation invariant for the grey tone axis T, i.e. functions f and $f + t$, $(t \in T)$ yielded the same segmentation. The threshold criterion is not of this type. Here are a few variants of the threshold idea

- One can take the intersection of criterion σ_0 with σ_1

  $$\sigma_1[f, A] = 1 \Longleftrightarrow \sup\{f(x), x \in A\} \leq t_1$$

  which provides the threshold between levels t_0 and t_1;

- On can intersect these threshold criteria with a connection $\mathcal{C}$ on $\mathcal{P}(E)$;

- moreover one can combine the previous connections with a criterion of minimum volume. For example, one can select among all thresholded classes $D(y), y \in E$, those where the integral $\int_{D(y)_i} f(x)dx$ is larger than a given value;

- one can also have integral criteria which associate upper bounds. For example: suppose space E is metric, and denote by $B(y, r_0)$ the ball of centre y and radius r_0. Start from a connection $\mathcal{C}$ on $\mathcal{P}(E)$, and take for segmented classes all connected components A such that for any $y \in A$ the integral $\int_{B(y,r_0)_i} f(x)dx \leq a_0$, where a_0 and r_0 are fixed. Clearly, the criterion extracts smooth regions;

- One can also think of all directional, textural, or hue criteria that are associated with the unit circle. In [31] A. Hanbury and J. Serra consider the unit circle as an external space that allows them to segment image regions according to texture and hue criteria.

5.2 Set segmentation

The material under study in this example, borrowed from [85], is a stack of one hundred thin sections of the shinbone of a chicken embryo. The biologists want to know whether the growth of the bone proceeds by nested cylinders. We see in Figure 16 two bone sections. They are perpendicular to the long axis of the shinbone and exhibit sorts of circular crowns,

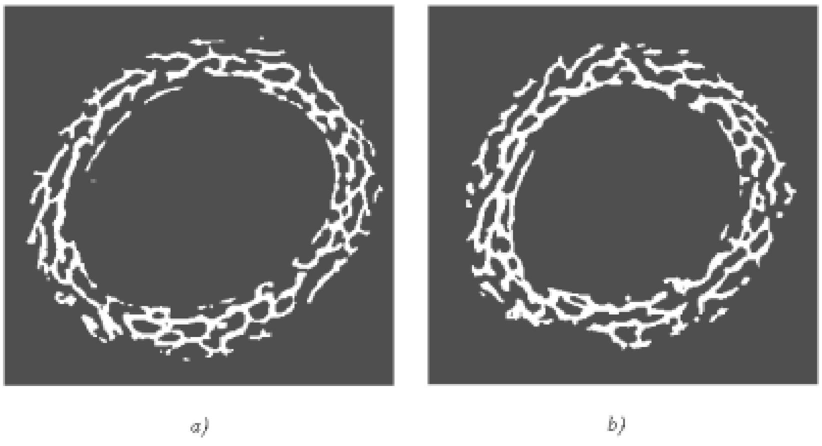


Figure 16: *Two sections of the shinbone*

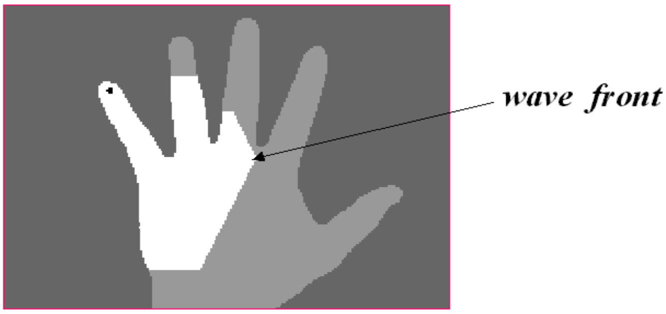


Figure 17: *An example of 2-D geodesic wavefront.*

so to say. If the assumption of the biologists is true, the successive cylinders should be linked by a few bridges, in order to ensure the robustness of the whole bone structure.

How to judge the truth of their hypothesis, and if need be, how to segment the pile of cross sections into the various nested cylindric crowns ?

The procedure is based on the idea of using the central hole of the bone (the medullary region) as a cylindric marker, and, starting from it, to invade the solid bone by 3-D geodesic dilations.

Figure 17 shows how geodesic dilation works in two dimensions. However the current case is three dimensional, and we need, for its implementation, to choose a unit sphere : we will take the unit cube-octahedron preferably to the unit cube. It is more isotropic (14 face directions instead of 6), more compact (12 voxels versus 26) and closer to an Euclidean sphere (the 12 vertices are at the same distance from the centre)[79].

The geodesic dilations create a wave front that propagates through the bone. We measure the surface area of this wave front at each step of its discrete propagation.

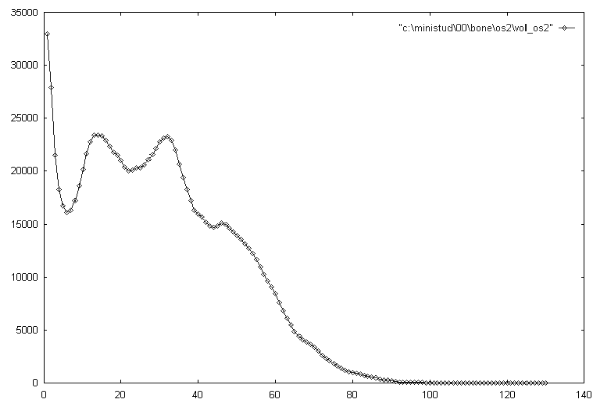


Figure 18: *Wave front surface area at each step of the propagation*

Figure 18 depicts the plot of this area measurement versus the step number. Several minima are clearly visible. Indeed, when starting from the axis of the bone, the wave invades the solid zones,

- it meets a bone cylinder first, where the front spreads out largely :

- then, the front has to cross some narrow bridges for reaching the second cylindric crown, hence its area reduces ;

- then again the front propagates over the second cylindric crown and its surface extends, etc..

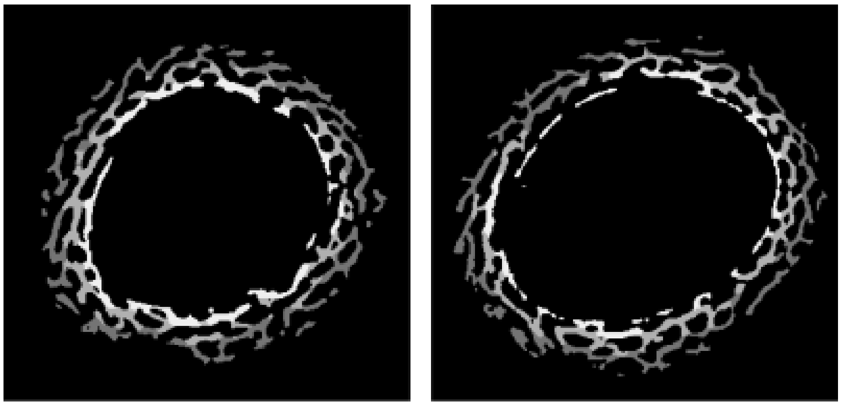
The plot of Figure 18, with its well pronounced minima, corroborates the biological assumption of nested cylindric crowns, and the abscissae of the successive minima indicate the geodesic distance from the axial marker to the bottlenecks of the bridges.

If we split the 3-D bone into the zones that the minima separate, do we perform a segmentation ? Yes indeed. Forget the set for one instant and consider the plot only. The following criterion

$$\begin{cases} \sigma[f, \{x\}] = 1 \\ \sigma[f, A] = 1 \text{ when segment } A \in \mathcal{P}\left(\mathbb{Z}^1\right) \text{ contains no minimum of } f \\ \sigma[f, A] = 0 \text{ when not} \end{cases}$$

where f is a $1-D$ numerical digital function. This obviously connective criterion divides the x-axis into segments with no minimum, and assigns point classes to all abscissae of minima. Projecting back, now, this division onto the 3-D bone set, we obtain the segmentation of the shinbone into its nested cylinders (Figure 19).

The technique in this example was to resort to the wave front areas, i.e. to replace a 3-D set which is hardly visible by a 1-D significant function, and then to define a connective criterion that makes sense on the 1-D plot as well as in the 3-D space.



a) b)

Figure 19: *Segmentation of the two above sections.*

5.3 Local segmentation by 2-D histogram

The example that follows is borrowed from J. Angulo and J. Serra [4], and leads to a model of light which is proposed in [87]. Consider the guitar image of Figure 20a.

We look for some segmentation criterion based on the homogeneity of the colour. The image, which is provided in (R,G,B) mode is transcribed into the (*brightness, saturation, hue*) mode based on the L_1 norm for the sake of consistency (for a discussion on the advantages of System (27) versus HLS type representations, see [32], and for the derivation of System (27), see [83]). The transition formulae are the following

$$\begin{aligned}
l(\mathbf{c}) &= m(\mathbf{c}) = \frac{1}{3}(\max + \mathrm{med} + \min) \\
s(\mathbf{c}) &= \begin{cases} \frac{3}{2}(\max - m(\mathbf{c})) & \text{if} \quad \max + \min \geq 2\mathrm{med} \\ \frac{3}{2}(m(\mathbf{c}) - \min) & \text{if} \quad \max + \min \leq 2\mathrm{med} \end{cases} \\
h(\mathbf{c}) &= k\left[\lambda(\mathbf{c}) + \frac{1}{2} - (1)^{\lambda(\mathbf{c})}.\frac{\max + \min - 2\mathrm{med}}{2s(\mathbf{c})}\right]
\end{aligned} \tag{27}$$

with $med = median(r, g, b)$, $\max = \max(r, g, b)$, $\min = \min(r, g, b)$, and $m = mean(r, g, b)$. The colours r, g, and b range from 0 to 1, and the integer $\lambda(\mathbf{c})$ varies from 0 to 5 according to the angular sector. System (27) shows that when the hue h is constant over some region, then the brightness l, the saturation s and the median med are linearly linked

$$s = a(l - med) \tag{28}$$

with $a \leq -3$ or $a \geq 3$. This suggests the plotting of the 2-D histogram of l (x-axis) versus s (y-axis). It is depicted in Figure 21a and exhibits two major alignments

Figure 20: *a) initial colour image; b) two segmented zones.*

- w_1, a long segment with a positive slope, on a line that passes by the origin
- w_2, a shorter segment, of negative slope and placed on a line passing through point $(1,0)$.

Introduce the following criterion in the space of the luminance/saturation histogram minus its origin:

$\sigma_1[f,A] = 1$ when all pixels $x \in A$ lie on a given line passing by the origin
$\sigma_1[f,A] = 0$ when not

Clearly, criterion σ_1 is connective. In the present case, it yields for example the alignment w_1. The back projection on the image space extracts the table of the guitar, as depicted on Figure 20b.

The alignment w_2 will be reached by another criterion σ_2, similar to σ_1, but where the line must pass through point $(1,0)$. In the image space, alignment w_2 corresponds to the clear region of the hand and finger. According to Proposition (13), the supremum σ of the two criteria σ_1 and σ_2 is still connective (in the image space the two zones do not overlap) ; σ results in two segmented regions, plus point classes elsewhere. In further steps, one can still combine new criteria with σ (e.g. lines of other slopes, clouds in the luminance/saturation space).

As long as they are connective and they generate regions that do not overlap with the previous ones, the previous segmentations remain valid. In other words, Proposition (13) allows us to replace a global segmentation by the supremum of more local ones, that can be processed independently.

To conclude, we may notice that the regionalized model for light propagation proposed in [87] allows us to interpret the linear relation between saturation and luminance, by eliminating the median from equation (28). But we did not resort to any physical model for achieving the segmentation.

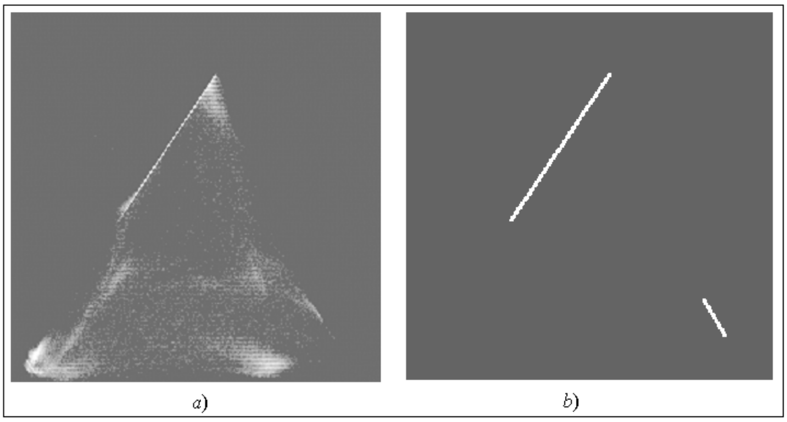


Figure 21: *a) 2-D histogram of the saturation s versus the brightness l = m for the guitar image. b) masks of two alignments in the histogram a).*

6 Segmentation and filtering

A segmentation criterion usually depends on one, or more, numerical parameters. When, for a given function, the parameter varies, the *number* of classes of the segmentation partition often varies in a monotonous manner. However, the *classes* themselves are not necessarily nested. Consider for example the three segmentations of St Cugat Virgin, in Figure 15. The jump by 10 enlarges the classes of that by jump 5 (because 10 is a multiple of 5), but the jump by 15 does not enlarge that by jump 10. On the other hand, in all these segmentations we go directly from some initial image to partitions of the space, and bypass possible filtering (i.e. mappings from image to image). Should it be possible, in the above segmentation approach, to insert some filtering between the initial image and the final partition? Such combination will make sense only if the filtering step helps us in preparing the segmentation one by regrouping, hence simplifying, the σ-classes. This section and the two following ones are devoted to the study of such combinations.

From now on, the arrival space T is supposed to be a complete lattice with universal bounds m_0 and m_1, in order to open the way to morphological operations. A natural approach here consists in starting from openings : by composition with their dual forms, the closings, they allow us to build up a broad spectrum of filters. And since the class of the openings is closed under supremum, we can start from the simplest ones, i.e. the so called *pulse connected openings,* which are associated with the pulses sup-generators of T^E. But then we have to take for the class $\mathcal{F}$ of functions the family T^E of all functions f from E into T or, possibly, a sub-lattice of T^E.

Unlike in the theory developed in Section 2, where function f was fixed and criterion σ variable, criterion σ is now fixed and various function operators map T^E into itself.

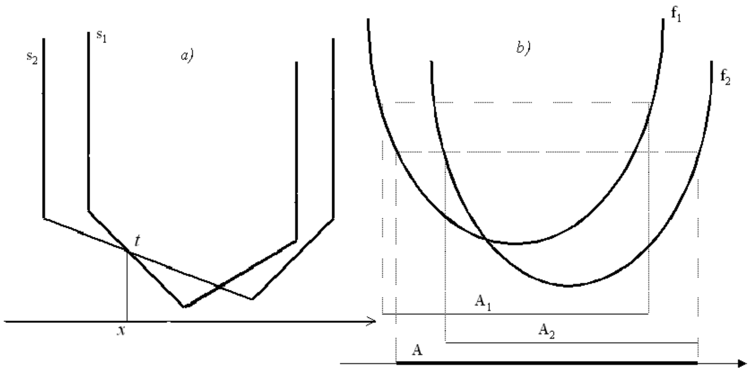


Figure 22: *Two counter examples for strong connectiveness. a) each of the two drops s_1 and s_2 going through (x,t) has a watershed class at point x which is its own base; $s_1 \wedge s_2$ has two minima and its watershed class at x shrinks (and this inf is no longer a drop). b) the jump class of the minimum of f_1 is A_1, that of the minimum of f_2 is A_2, however the class A of the minimum of $f_1 \wedge f_2$ is smaller than $A_1 \cup A_2$.*

6.1 Strong connectivity and drops

Segmentation purposes require that we reinforce Definition 7 of criterion connectiveness.

Definition 18 *Strong connectiveness: A criterion σ is said to be* strongly connective *when for all functions $f \in T^E$*

1 /criterion σ is satisfied on the class $\mathcal{S}$ of the singletons and by the empty set,

$$\forall f \in \mathcal{F}, \quad \sigma[f, \emptyset] = 1 \ an\ d\ \{x\} \in \mathcal{S} \implies \sigma[f, \{x\}] = 1$$

2/ for any function family $\{f_i, i \in I\}$ in T^E and any set family $\{A_i, i \in I\}$ in $\mathcal{P}(E)$, if there exists a point $x \in \cap_i A_i$ where the $f_i(x)$ are equal for all $i \in I$, then

$$\bigwedge \sigma[f_i, A_i] = 1 \Rightarrow \sigma[\bigwedge f_i, \bigcup A_i] = 1. \tag{29}$$

The condition $\wedge\sigma[f_i, A_i] = 1$ means that for all $i \in I$, we have $\sigma[f_i, A_i] = 1$. The strongly connective criterion σ is said to be conditionally closed under infimum. In particular, when all functions f_i are identical, then the above definition reduces to Definition 7 of a connective criterion. The second requirement of Definition 18 restricts rather severely the choice for possible criteria. For example, it accepts the smooth connection and that by flat ones or quasi-flat zones, but it rejects both watershed and jump connection (see Figure 22).

We now introduce the class of the *drop functions*, which plays an essential role in the elaboration of the connected filtering.

Definition 19 *Drops : given a criterion σ on T^E, a* drop s *of base A is a function $s : E \to T$ that is homogeneous for criterion σ on set A, and maximum elsewhere*

$$\sigma[s, A] = 1 \qquad ; \qquad x \in A^c \Longrightarrow s\,(x) = m_1$$

A drop s of base A is said to go through *the point $(x, t) \in E \times T$ when $x \in A$ and $s\,(x) = t$.*

In particular, when criterion σ is connective, a greatest base of s always exists. Every function $f \in T^E$ can obviously be decomposed in several ways into an infimum of drops, and when a criterion σ has been chosen, there always exists at least one decomposition of each $f \in T^E$ into drops on the bases of which the criterion is satisfied. This property remains true for all criteria ; if, in addition, criterion σ is strongly connective, then the family of the drops going through point $(x, t) \in E \times T$ is closed under infimum. More precisely, we consider the ordered pairs (s, A) of all drops going through (x, t) and order them as follows

$$(s, A) \leq (s', A') \Leftrightarrow s \leq s' \text{ and } \quad A \supseteq A'. \tag{30}$$

Clearly, Relation (30) is an ordering relation where the infimum is given by

$$\bigwedge_i (s_i, A_i) = (\bigwedge_i s_i, \bigcup_i A_i) \tag{31}$$

If the family $\mathbb{S}_{x,t}$ of all drops going through (x, t) forms a lattice, such infimum should be a drop.

Proposition 20 *Let σ be a strongly connective criterion on the functions $f \in T^E$. Given the pair $(x, t) \in E \times T$, the family $\mathbb{S}_{x,t}$ of all drops going through (x, t) forms a complete lattice for the ordering (30), in which the infimum is nothing but the pointwise infimum of Relation (31).*

Proof. Let $\{s_i, i \in I\}$ be a sub-family in $\mathbb{S}_{x,t}$. Denote by A_i the base of drop $s_i, i \in I$ and by s_o the infimum $\wedge s_i$. As point x belongs to all A_i and as each $s_i\,(x)$ equals t, the infimum s_o goes through (x, t). Then, by applying to σ the strong connectiveness Relation (29), we obtain $\sigma[s_o, \cup_i A_i] = 1$. Moreover, if $x \in (\cup_i A_i)^c$, then x belongs to all A_i^c hence each $s_i\,(x)$ equals m_1 and $s_o\,(x) = m_1$. Therefore, function s_o is a drop of base $\cup_i A_i$, and family $\mathbb{S}_{x,t}$ is closed under infimum. Moreover, the anti-pulse of base x

$$s_{\max}\,(x) = t \qquad ; \qquad s_{\max}\,(y) = m_1 \qquad y \neq x$$

which is the largest function of T^E going through (x, t), is also a drop, since $\sigma[s_{\max}, \{x\}] = 1$. Hence, the complete inf-semi lattice $\mathbb{S}_{x,t}$, which admits a maximum element, turns out to be a complete lattice. ■

The minimum element of lattice $\mathbb{S}_{x,t}$ is denoted by $s_{x,t}$, it has a base

$$D_{s_{x,t}} = \bigcup \{\cup \{A : x \in A, \sigma[s, A] = 1\}, \quad s \in \mathbb{S}_{x,t}\}. \tag{32}$$

Since criterion σ is connective, each function $s \in \mathbb{S}_{x,t}$ can be segmented according to Theorem 8, and its segmented class at point x is equal to $D_s = \cup \{A : x \in A, \sigma[s, A] = 1\}$, so that

$$D_{s_{x,t}} = \bigcup \{D_s, \quad s \in \mathbb{S}_{x,t}\} .$$

We now consider a fixed function $f \in T^E$, such as $t \leq f(x)$. The drops of $\mathbb{S}_{x,t}$ whose values inside the base are smaller than f form a sub-lattice of the previous one. More precisely

Corollary 21 *Given a function $f \in T^E$, a strongly connective criterion σ, and a pair $(x, t) \in E \times T$, with $t \leq f(x)$, the family $\mathbb{S}_{x,t}(f)$ of those functions $s \in \mathbb{S}_{x,t}$ such that*

$$y \in D_s \Longrightarrow s(y) \leq f(y) \tag{33}$$

where D_s is the base of drop s, is a complete sub-lattice of $\mathbb{S}_{x,t}$.

Proof. The assumption $t \leq f(x)$ implies that $\mathbb{S}_{x,t}(f)$ contains the anti-pulse at point (x, t), hence the greatest element of lattice $\mathbb{S}_{x,t}$. We have to prove that the family $\mathbb{S}_{x,t}(f)$ is closed under the pointwise infimum. This family is not empty, since it contains the anti-pulse at point (x, t). As $\mathbb{S}_{x,t}(f)$ is a part of $\mathbb{S}_{x,t}$, we draw from Prop.20 that it is closed under the pointwise infimum in $\mathbb{S}_{x,t}$, i.e. that any family $\{s_i, i \in I\}$ in $\mathbb{S}_{x,t}(f)$ of bases D_i has an infimum drop $s_o(f) = \wedge s_i$ of base $D_o = \cup_i D_i$. It remains to verify that the drop s_o is an element of $\mathbb{S}_{x,t}(f)$. If $y \in D_o$ then there exists an index $j \in I$ such that $y \in D_j$, hence $s_o(y) \leq s_j(y) \leq f(y)$, i.e. s_o fulfills implication (33), which achieves the proof. ■

The minimum element of lattice $\mathbb{S}_{x,t}(f)$ is denoted by $s_{x,t}(f)$, and its a base, by $D_{s_{x,t}}(f)$.

6.2 Pulse opening

If we look at the things from the point of view of the operators, it becomes convenient to introduce the family $\{\gamma_{x,t} \ , (x, t) \in E \times T\}$ of the mappings from T^E into itself (for a given criterion σ) that are defined as follows

1. if $t \nleq f(x)$, then $\gamma_{x,t}(f)$ equals the constant minimum function m_0;
2. if $t \leq f(x)$ then

$$\begin{aligned} [\gamma_{x,t}(f)](y) &= [s_{x,t}(f)](y) = [\bigwedge \{s, \ s \in \mathbb{S}_{x,t}(f)\}](y) && \textit{when } y \in D_{s_{x,t}}(f) \quad (34) \\ [\gamma_{x,t}(f)](y) &= m_0 && \textit{when } y \notin D_{s_{x,t}}(f) \end{aligned}$$

3. In addition we suppose that the minimum function m_0 is a drop, i.e. that $\sigma[m_0, E] = 1$.

Proposition 22 *When σ is a strongly connective criterion, then each $\gamma_{x,t}$ turns out to be an opening, called the* pulse opening *at pulse $i_{x,t}$.*

Proof. The operator $\gamma_{x,t}$ is obviously anti-extensive, we have to prove that it is also increasing and idempotent. Both properties are trivially satisfied when $t \nleq f(x)$, since the minimum function m_0 is a drop.

Increasingness: let f_1 and f_2 be two functions, with $f_1 \leq f_2$. We suppose $t \leq f_1(x) \leq f_2(x)$, so that none of the two families $\mathbb{S}_{x,t}(f_1)$ and $\mathbb{S}_{x,t}(f_2)$ is empty. Let s_2 be a drop of $\mathbb{S}_{x,t}(f_2)$ of base A_2. Consider the family of those sets $Y \in \mathcal{P}(E)$ such that

$$\sigma[s_2, Y] = 1 \quad \text{and} \quad y \in Y \Rightarrow s_2(y) \leq f_1(y). \tag{35}$$

The $\{Y\}$ family is not empty since it contains the singleton $\{x\}$. Denote by A_1 the union of all sets Y that satisfy the two relations (35). By connectiveness of criterion σ, we have $\sigma[s_2, A_1] = 1$. Hence the function

$$s_1(y) = s_2(y) \text{ when } y \in A_1, \quad \text{and} \quad s_1(y) = m_1 \text{ when } y \notin A_1,$$

is a drop of $\mathbb{S}_{x,t}(f_1)$ which goes through point (x,t). Now, over A_1 and A_2^c we have $s_1(y) = s_2(y)$, and over $A_2 \backslash A_1$ it comes $f_1(y) < s_2(y) \leq f_2(y)$ and $s_1(y) = m_1$, so that we find for all $y \in E$

$$s_1(x) \bigwedge f_1(x) \leq s_2(x) \bigwedge f_2(x).$$

As we have associated, with each element of the infimum (34) involved by strong connectiveness in $\gamma_{x,t}(f_2)$, a smaller element of the infimum of $\gamma_{x,t}(f_1)$, we can write

$$s_0(f_1) \bigwedge f_1 \leq s_0(f_2) \bigwedge f_2. \tag{36}$$

If we now consider an arbitrary drop s_1 of base D_1, of $\mathbb{S}_{x,t}(f_1)$, we observe that it is also a drop of $\mathbb{S}_{x,t}(f_2)$ which yields, according to Rel.(32),

$$D_{s_{x,t}}(f_1) \subseteq D_{s_{x,t}}(f_2). \tag{37}$$

The two inequalities (36) and (37) imply that $\gamma_{x,t}(f_1) \leq \gamma_{x,t}(f_2)$, i.e. that operator $\gamma_{x,t}$ is increasing.

Idempotence: Consider a drop $s' \in \mathbb{S}_{x,t}(\gamma_{x,t})$ of base $D_{s'}$.We have, on the one hand $s'(x) = t$, and on the other one

$$\begin{aligned} y &\in D_{s'} \Rightarrow s'(y) \leq \gamma_{x,t}(f) \leq f \\ y &\notin D_{s'} \Rightarrow s'(y) = m_1 \end{aligned}$$

hence s' is a drop of $\mathbb{S}_{x,t}(f)$, so $s' \geq s_{x,t}(f)$. But as an element of $s_{x,t}[s_{x,t}(f)]$ the drop s' is smaller than $s_{x,t}(f)$ on the domain $D_{s'}$. Therefore all the drops of $\mathbb{S}_{x,t}(\gamma_{x,t})$ are equal to $s_{x,t}(f)$on their bases, so that their infimum is nothing but $s_{x,t}(f)$, which achieves the proof. ■

A pulse opening is rarely used alone, but rather in association with a family of other pulses openings. Observing that any function $g \in T^E$ is the supremum of all pulses $i_{x,t}$ smaller than it, and that a supremum of openings is still an opening, we can parametrize pulse openings by means of functions g and put

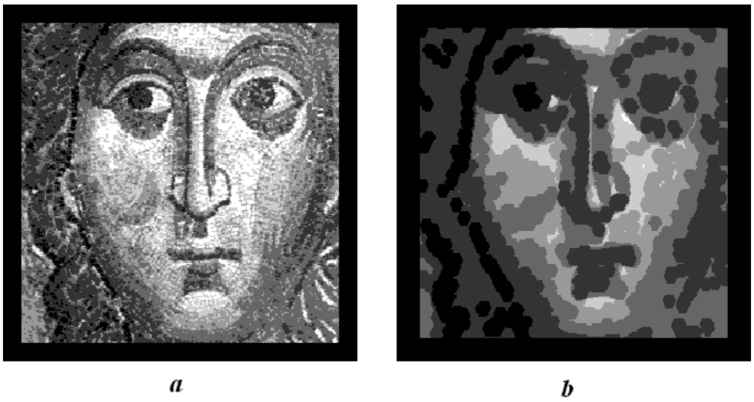


Figure 23: *a) Initial image, f say, of a mosaic; b) marker image, i.e. function g of rel. (38). Function g is the erosion of the initial image by a hexagon of size four.*

Definition 23 *When criterion σ is strongly connective, and when $g \in T^E$ is an arbitrary function, then the supremum*

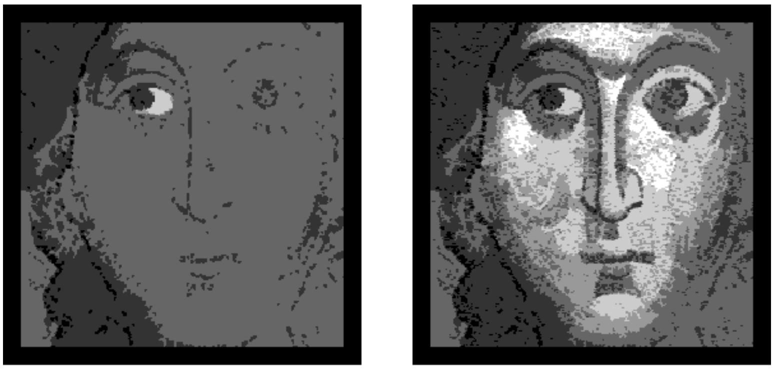
$$\gamma_g = \bigvee \{\gamma_{x,t},\ i_{x,t} \leq g\} \tag{38}$$

is an opening called opening of marker *g.*

A particular case of this operator is the geodesical reconstruction by dilation from the marker g under the mask f. This popular operation dates from the eighties, and its first systematic study may be found in [76]. Here the criterion σ is that of the connected flat zones (i.e. $\sigma[f, A] = 1$ for A connected and f constant over A), which is strongly connective, and the elementary opening $\gamma_{x,t}$ is the geodesical reconstruction from the marker $i_{x,t}$. The four photographs of Figure 23 and Figure 24 illustrate the notion of a connected opening. The initial image f is here Figure 23a, and its eroded version Figure 23b will serve as marker g. For a pulse (x, t) taken in the white region of the right eye, in Figure 23b, the corresponding pulse opening $\gamma_{x,t}(f)$ is depicted in Figure 24a; the supremum of all pulse openings of the marker g is represented in Figure 24b. Such a reconstruction of function f, "from below", by expanding marker g provides us with an opening that either preserves, or eliminates, the contours, but without creating new ones [72],[24], [17]. By idempotence of the opening, if we start from Figure 24b, erode it and re-expand the erosion, we shall obtain again Figure 24b, without changes.

6.3 Connected operators

Consider an arbitrary operator ψ from T^E into itself, or more generally from a function space $\mathcal{F}$ into itself, and a criterion $\sigma : \mathcal{F} \times \mathcal{P}(E) \to \{0, 1\}$. The pair $\{\sigma, \psi\}$, viewed as a whole, becomes itself a criterion based on the quantity $\sigma[\psi(f), A]$, since this quantity equals either 0 or 1. Moreover, criterion $\{\sigma, \psi\}$ inherits some basic properties that σ may



a *b*

Figure 24: *a) pulse opening $\gamma_{x,t}(f)$ for a pulse (x,t) taken in the white part of the right eye of Fig. 23b. b) supremum $\gamma_g(f)$ of all pulse openings for the pulses smaller or equal to Fig.23b*

satisfy. For example, if criterion σ is connective (resp. strongly connective), then criterion $\{\sigma, \psi\}$ is connective (resp. strongly connective) as well. To prove it, we observe that the connectiveness axiom (7) applies for all functions $f \in \mathcal{F}$. Since $\psi(\mathcal{F}) \subseteq \mathcal{F}$, it suffices to replace f by $\psi(f)$ in axiom (7). For the same reason, the passage $\sigma \rightarrow \{\sigma, \psi\}$ preserves also the strong connectiveness, and the pulse decreasingness of Definition 27.

The new property that we will add now to criterion $\{\sigma, \psi\}$ no longer holds on σ, but on operator ψ. In image processing, one classically uses the adjective "connected" to signify that an operation ψ on T^E enlarges the segmentation partition. We keep the term, but redefine it within the background of connective criteria.

Definition 24 *An operator ψ on T^E is said to be* connected *with respect to a connective criterion σ when for all $f \in T^E$ and for all $A \subseteq E$ it satisfies the relationship*

$$\sigma[f, A] = 1 \Rightarrow \sigma[\psi(f), A] = 1. \tag{39}$$

In other words, the connection generated by σ and $\psi(f)$ is larger than that due to σ and f. The reason of implication (39) is clear. As soon as the segmentation according a connective criterion σ is preceded by a connected filter, according to σ as well, one can interpret the sequence of the two operations as a *unique* segmentation. It usually depends on a size positive parameter, which provides it with semi-groups scale space properties (see next section below, and [81],[14], [35]).

In particular when a connection $\mathcal{C}$ is defined over $\mathcal{P}(E)$ and when $\sigma[f, A] = 1$ for A connected and f constant, then Implication (39) reduces to the classical definition of a connected operator (i.e. f constant over the connected set $A \Rightarrow \psi(f)$ constant over A, [76] [72],[24],[34]).

Consider now the segmentation class $D_f(x)$ at point x, i.e.

$$D_f(x) = \bigcup \{A : x \in A, \quad \sigma[f, A] = 1\}.$$

Since σ is connective, Implication (39) shows that at every point x we have $\sigma[\psi(f), D_f(x)] = 1$, hence $D_f(x) \subseteq D_{\psi(f)}(x)$ so that $D_f \preceq D_{\psi(f)}$, i.e. the operator ψ enlarges the segmentation partitions. As such an enlargement can only be reached by merging classes, the overall boundary set of the classes reduces. We can state

Proposition 25 *Any connected operator with respect to a connective criterion σ on T^E enlarges the segmentation partitions, and reduces the boundaries of the associated classes.*

In particular, if the boundaries are regular enough to have a finite perimeter, then this perimeter also reduces. Note that an operator that enlarges the partitions may not be connected. A supplementary assumption is necessary for the converse statement, such as the set decreasingness. If, when function f is fixed, the marginal mapping $\sigma[f, A] : \mathcal{P}(E) \to \{0, 1\}$ is decreasing, i.e. if

$$\sigma[f, A] = 1 \text{ and } B \subseteq A \Rightarrow \sigma[f, B] = 1 \tag{40}$$

and if implication (40) remains true for all functions $f \in \mathcal{F}$, then one says that criterion σ is *set decreasing.* The threshold operator for example is of this type, as well as the segmentation into possibly non connected flat zones (if f is constant over set A, it is a fortiori constant over any set $B \subseteq A$).

Proposition 26 *When the connective criterion σ is set decreasing, then any operator that enlarges the partitions is connected .*

Proof. If $\sigma[f, A] = 1$ then for any point $x \in A$ we find $A \subseteq D_f(x) \subseteq D_{\psi(f)}(x)$. Now, by definition of $D_{\psi(f)}(x)$ we have $\sigma[\psi(f), D_{\psi(f)}(x)] = 1$, which implies $\sigma[\psi(f), A] = 1$ by set decreasingness of criterion σ, i.e. implication (39). ■

Here is a counter-example due to C. Ronse (private communication). Given a set connection $\mathcal{C}$, let σ be defined by $\sigma[f, A] = 1$ iff $A \in \mathcal{C}$ and f is constant on A. Let σ^* be the criterion verifying $\sigma^*[f, A] = 1$ iff A is a maximal flat zone of f. For ψ connected in the usual sense, for a maximal flat zone A of f we have $\sigma^*[f, A] = \sigma[f, A] = 1$, then $\sigma[\psi(f), A] = 1$, but not necessarily $\sigma^*[\psi(f), A] = 1$, because $\psi(f)$ may be constant on a larger flat zone containing A.

One will notice that set decreasingness is a particularly demanding condition. C. Ronse (private communication) proposes to weaken it as follows. Start from a connection $\mathcal{C}$, then a criterion σ is $\mathcal{C}$-decreasing if

i) whenever $\sigma[f, A] = 1$, we must have $A \in \mathcal{C}$

ii) for $A, B \in \mathcal{C}$ such that $\sigma[f, A] = 1$ and $B \subseteq A$, we have $\sigma[f, B] = 1$

Proposition 26 remains true under this weaker assumption of $\mathcal{C}$-decreasingness. Anyway, we have avoided to use Proposition 26 in the present theory.

Coming back to the pulse opening (34), we observe that though $D_{s_{x,t}}(f)$ is the largest set on which criterion σ is satisfied for function $s_{x,t}$, the pulse opening $\gamma_{x,t}(f)$ may not be necessarily connected. However this nice property will be obtained when criterion σ is *pulse decreasing*.

Definition 27 *(Pulse decreasingness) A connective criterion σ is said to be* pulse decreasing *when, for any function f, any set A such that $\sigma[f, A] = 1$ and for any family $\{(x_i, t_i), i \in I\}$, such that $\forall i \in I, x_i \in A$ and $t_i \leq f(x_i)$, one can find a family $\{f_i, i \in I\}$ of functions f_i such that*

1/ each f_i is smaller or equal to f over A and $f_i(x_i) = t_i$;

2/ each f_i is homogeneous over A

$$\sigma[f_i, A] = 1, \quad i \in I; \tag{41}$$

3/ the pointwise supremum of the f_i's is homogeneous over A

$$\sigma[\bigvee\{f_i, i \in I\}, A] = 1; \tag{42}$$

4/ In addition, when $f = m_0$, then criterion σ is satisfied for all subsets of E

$$\forall A \in \mathcal{P}(E), \quad \sigma[m_0, A] = 1. \tag{43}$$

Despite this long definition, pulse decreasingness is not a very demanding prerequisite. When $T = \overline{\mathbb{R}}$, the first three axioms are fulfilled by all criteria that are invariant under downward translation, i.e. for which both f and $f - d$, $d > 0$, yield the same segmentation. As for the fourth axiom, it just gives a status to the zero function. Note also that the threshold criterion is not pulse decreasing. Pulse decreasingness works "vertically", whereas set decreasingness is purely "horizontal".

Proposition 28 *: Let σ be a strongly connective and pulse decreasing criterion, and $f \in T^E$ be an arbitrary function. Then for any pulse $i_{x,t}$ such that $t \leq f(x)$ the pulse opening of Equation (34) is a connected operator.*

Proof. Let $D_{s_{x,t}}$ denote the segmented class of $s_{x,t}$ at point x, where $s_{x,t}$ is the infimum defined by Rel.(34). Consider a homogeneous zone Z_f of f , i.e. such that $\sigma[f, Z_f] = 1$.We have to prove that $\sigma[\gamma_{x,t}(f), Z_f] = 1$. If $D_{s_{x,t}} \cap Z_f = \emptyset$ then $\gamma_{x,t}(f) = m_0$ over Z_f and from Relation (43), we have $\sigma[\gamma_{x,t}(f), Z_f] = 1$. A second obvious case arises when $Z_f \subseteq D_{s_{x,t}}$, since then $\gamma_{x,t}(f)$ equals $s_{x,t}$ on Z_f [Rel.(34)], so that $\sigma[\gamma_{x,t}(f), Z_f] = 1$.

Consider now a homogeneous zone Z_f that intersects $D_{s_{x,t}}$, but with possibly outside points, and take a point $y \in D_{s_{x,t}} \cap Z_f$. As criterion σ is pulse decreasing, there exists a function $f' \leq f$ with $f'(y) = [s_{x,t}(f)](y)$ and $\sigma[f', Z_f] = 1$. Introduce the following drop h :

$$\begin{cases} h(y) = f'(y) & y \in Z_f \\ h(y) = m_1 & y \notin Z_f. \end{cases}$$

Both functions $s_{x,t}(f)$ and h belong to class $\mathbb{S}_{y,s_{x,t}(y)}(f)$, therefore their inf $s_{x,t}(f) \wedge h$ is still a drop of class $\mathbb{S}_{y,s_{x,t}(y)}(f)$ and the set $D_{s_{x,t}}(x) \cup Z_f$ is a homogeneous zone of $s_{x,t}(f) \wedge h$. Now we can always make the drop $s_{x,t}(f) \wedge h$ go through (x, t): if $x \in Z_f$, it suffices to take $y = x$, and if $x \notin Z_f$ then $[s_{x,t}(f)](x) \wedge h(x) = [s_{x,t}(f)](x) = t$. Therefore function $s_{x,t}(f) \wedge h$ is a drop of the family $\mathbb{S}_{x,t}$, hence $s_{x,t}(f) \wedge h = s_{x,t}(f)$ and we have $D_{s_{x,t}} \cup Z_f \subseteq D_{s_{x,t}}$, or equivalently $Z_f \subseteq D_{s_{x,t}}$. According to Rel.(34), it implies that $s_{x,t}$ equals the transform $\gamma_{x,t}(f)$ over Z_f, which achieves the proof. ■

The geometrical interpretation of Proposition 28 is clear: either the base x of the pulse belongs to the homogeneous zone Z_f, and then $Z_f \subseteq D_{s_{x,t}}$, or $x \notin Z_f$ and then $Z_f \subseteq [D_{s_{x,t}}]^c$. This remark allows us to easily extend the proposition to the marker based openings of Rel.(38) i.e. to

$$\gamma_g(f) = \bigvee \{\gamma_{x,t},\ i_{x,t} \leq g\} = \bigvee \{\gamma_{x,t}(f),\ t \leq g(x)\}.$$

where marker g is an arbitrary function. Indeed, we have to prove that any homogeneous zone Z_f of f is still homogeneous for $\gamma_g(f)$. Let J be the family of all pulses (x_j, t_j) whose bases fall in Z_f and for which $t_j \leq f(x_j)$, and g^J be their supremum. Similarly, let g^K be the supremum of the remaining pulses of the family $\{(x,t),\ t \leq g(x)\}$. We have $g = g^J \vee g^K$ so that $\gamma_g(f) = \gamma_{g^J}(f) \vee \gamma_{g^K}(f)$. We draw from the above geometrical interpretation that $[\gamma_{g^K}(f)](Z_f) = m_0$, hence $\gamma_g(f) = \gamma_{g^J}(f)$ over Z_f. Since criterion σ is pulse decreasing, each minimum drop $s_{x_j,t_j}(f), j \in J$, is smaller, in Z_f, than a function f_j such that $\sigma[f_j, Z_f] = 1$. Therefore $\gamma_{g^J}(f) \leq \vee f_j$ over Z_f. Now, still by pulse decreasingness we also have $\sigma[\vee f_j, Z_f] = 1$ and if we take a point z in Z_f we can find a function f^* going through $[\gamma_{g^J}(f)](z)$ and with $\sigma[f^*, Z_f] = 1$. Then we draw from the strong connectiveness of criterion σ that

$$1 = \sigma[\gamma_{g^J}(f) \bigwedge f^*, A] = \sigma[\gamma_{g^J}(f), A] = \sigma[\gamma_g(f), A].$$

Therefore we can state the following

Proposition 29 *Let $\sigma : T^E \times \mathcal{P}(E) \to \{0,1\}$ be a strongly connective and pulse decreasing criterion and g be a marker function. Then the opening*

$$\gamma_g(f) = \bigvee \{\gamma_{x,t}(f),\ i_{x,t} \leq g\} \tag{44}$$

of marker g is a connected operator.

No special condition holds on marker g (if $g \nleq f$, one can always replace g by $g \wedge f$). When the segmentation criterion σ is that of the flat zones, then the conditions of Proposition 29 are satisfied and the opening γ_g is connected. This operation is often called "*the reconstruction of function f by dilation from function g*". The above example of the mosaic, in Figures 23 and 24 depicts such connected opening. Although this illustration is digital, and involves a totally ordered lattice T, the two propositions 28 and 29 remain valid for finite or continuous spaces E and arbitrary complete lattices T. Note also that connected openings, the dual closings, and their composition products such as the levellings offer an anchorage point to PDEs (e.g. P. Maragos and F. Meyer's work on levelling PDE in [54]).

7 Duality and hierarchies

7.1 Duality

The connected opening must be viewed as a starting point from which the dual closing is easily derived. By various combinations of these two operators, a broad spectrum of morphological operators is then accessible. We have thus to introduce now the duality, and to analyze the new constraints it demands.

The dual T^* of lattice T is obtained by reversing the sense of the ordering relation ($\geq$ becomes $\leq$) and by exchanging supremum and infimum. What do the connectiveness and decreasingness constraints become under duality ?

· *connectiveness* : Definition 7 does not require any particular structure for T which is just supposed to be a set (see also Definition 2); if criterion σ is connective for some function $f : E \to T$, it remains connective for the same f, considered as function $f : E \to T^*$.

· *Set decreasingness* : the same remark applies to Rel.(40) as well : set decreasingness is preserved under duality.

· *Strong connectiveness* : we now find a notable change, since condition (29) becomes

$$\bigwedge \sigma[f_i, A_i) = 1 \qquad \Longrightarrow \qquad \sigma[\bigvee f_i, \bigcup A_i] = 1 \qquad f \in T^E \tag{45}$$

· *Pulse decreasingness* : when we want the properties of Definition 27 to be valid on class T^{*E}, we must replace the condition :

$$t \leq f(x) \qquad \text{by} \qquad t \geq f(x)\,,\; f \in T^E \tag{46}$$

As one can see, the general results about segmentation, such as optimal partitioning, inf semi-lattice, etc.. are directly transposable from a function $f : E \to T$ to the function $f : E \to T^*$. As for the properties that involve connected operators, and above all for Proposition 29, the supplementary assumption that the two relations (45) and (46) are true has to be made. Note that the second relation is less an assumption than the demarcation of a certain domain for parameter t. On the other hand, the implication (45) is an actual hypothesis of duality.

When these two conditions are satisfied the connected openings on T^* are nothing but *connected* closings on T. Then the pulse connected opening turns out to be the root of a number of connected filters which derive from it by duality, by composition products and by sup's and inf's. The reader interested in the subject will find an abundant literature on connected filtering for numerical functions, which mainly dates from the nineties (see for example the works of J. Crespo, H. Heijmans, F. Meyer, Ph. Salembier, L. Vincent and J. Serra among others, [76], [72], [24], [34], [59]). The marker-based openings are not the only possible connected openings: other ones are designed from integrals of function f [94], [93], but the former allow us to construct, by composition with dual closings, nice self-dual hierarchies of operators [57], [81], [16].

A popular class of connected operators, that are obtained by duality, are the so-called "alternating filters". Here is an example of such operators. The function f, in Figure 25 a,

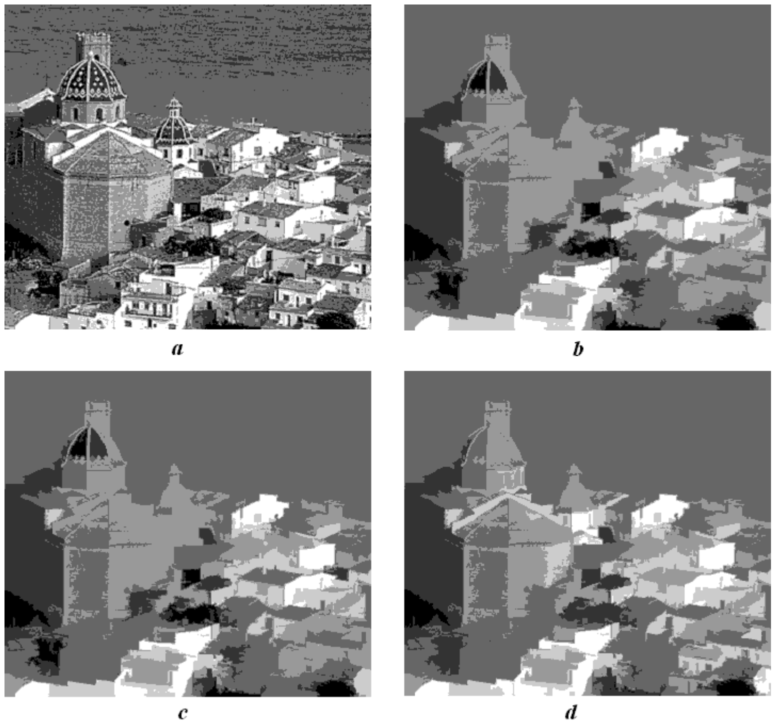


Figure 25: *a) Initial image f : Altea (Spain). b) Connected opening of f by taking for marker its eroded version of size four. c) Composition product of the opening b) by the dual closing. d) Composition product in the reverese order, i.e. closing followed by opening. Note the lack of commutativity.*

depicts a panoramic view of the small city of Altea, in Spain. This initial image is rather noisy, and requires some filtering prior to its segmentation. Fig 25b is a connected opening $\gamma_g(f)$ of f . It is constructed exactly as we did for the mosaic image, in Figure 24. Starting from this opening Fig 25b we then transform it by the closing dual of γ_g with respect to the negative, which results in Figure 25c. Finally, if we invert the order of these two operations, and begin with the closing, then we obtain Figure 25d. One proves theoretically that Figure 25d is lighter than Figure 25c, and this can be observed here by comparing the roofs of the church.

7.2 Hierarchies and connected operators

When a connective criterion σ has been chosen, how is it possible to construct a hierarchy $\{\psi_\lambda, \lambda > 0\}$ of connected operators? The answer to this question depends on the way it is decided to structure the said hierarchy, but we can already remark that we enjoy an advantage: the class of the connected operators is closed under composition. Indeed, if ψ and ψ' stand for two connected operators, then Rel.(39) implies that

$$\sigma[f, A] = 1 \Longrightarrow \sigma[\psi(f), A] = 1 \Longrightarrow \sigma\left[\psi'\psi(f), A\right] = 1. \tag{47}$$

Both composition products $\psi'\psi$ and $\psi\psi'$ are still connected operators.

7.2.1 Weak hierarchies

The minimal possible structure is that of the pre-ordering . It underlies most of the merging algorithms.

Definition 30 *Let ψ and ψ' be two connected operators, and σ a connective criterion . Operator ψ' is said to be larger than ψ for σ when we have for all $f \in \mathcal{F}$*

$$\sigma[\psi(f), A] = 1 \Longrightarrow \sigma\left[\psi'(f), A\right] = 1.$$

One can say, equivalently, that the connection generated by (σ, ψ') contains that of (σ, ψ). It implies that the segmentation partition according (σ, ψ') is coarser than that according to (σ, ψ) . Here is an example, where $f : \mathbf{R}^2 \to \overline{\mathbf{R}}$. Operator ψ consists in the following steps:

- segmenting f according to σ, which yields partition D_f
- replacing f by its average in each class $D_f(x)$ of D_f
- merging the two flat zones where grey level absolute difference is the smallest one (if they are several such pairs, new conditions, on their locations for example, may be added)
- in the two merged zones, replacing the grey values by their area weighted average.

With such an operator, the iterations $\psi^{(2)} = \psi \circ \psi; \psi^{(n)} = \psi \circ \psi^{(n-1)}$, etc., yield a hierarchy [59]. The major drawback of reducing a hierarchy to a total ordering comes from the impossibility to propagate optimality through iterations. In the previous example, the energy E is the sum of all jumps (in absolute value) along each boundary, where each simple arc of contour is counted for 1. The above procedure has consisted in minimizing the energy reduction when one boundary is removed. There is of course no reason to restrict the variation of energy to a single removal, and we have to introduce, as ψ_n, the operator that minimizes the overall jump variation when n boundaries are removed. Unfortunately, this optimal operator ψ_n does not necessarily coincide with the n^{th} iteration $[\psi_1]^{(n)}$ of ψ_1. There is a conflict between n considered as the parameter value (here the number of boundaries) and as the iteration number.

An example: the waterfall algorithm However when the elementary operator ψ_1 does not involve any parameter, then the iteration number does not compete with anything else and a significant hierarchy is obtained by iterating ψ_1. A typical example is given by the Beucher-Meyer waterfall algorithm [55], which is presented here on a colour case that we developed with J. Angulo [5].

Function f stands for a numerical or a colour image, and *gradf* for the module of its gradient. In the colour image case, there are several definitions of the colour gradient. Here,

Figure 26: *Initial image of "parrots"*

we take again the L_1 norm representation of the *Luminance/Saturation/Hue* triplet, as introduced by System (27). For the hue, which is an angle, define the increment $|h(x) \div h(y)|$ as the value of the acute angle of extremities x and y on the unit circle [31]. Denote by δ and ε respectively the unit dilation and erosion in the space of definition of image f (i.e. unit square or hexagon), and put

$$\begin{aligned} gradl &= (\delta(l) - l) \bigvee (l - \varepsilon(l)) \\ gradh &= (\delta(h) \div h) \bigvee (h \div \varepsilon(h)) \end{aligned}$$

The colour gradient module is then introduced by the following relation

$$gradf = s \times gradh + (1 - s) \times gradl.$$

In this barycentric relation, the higher the saturation, the more colour variation is taken into account. The watershed of *gradf* yields a first segmentation. Each of its classes is allocated the median values of the green, the red and the blue of the initial image in the class, which results in a new colour image. And we iterate the process. For example, if we start from the "parrots" standard image of Figure 26, four successive steps of waterfall yield the mosaic images that are depicted in Figure 27. At a rough guess the major shapes seem to be preserved. Number of non significant regions appear at the first three levels, and one has to reach the fourth one for a better, but not really convincing, partition. Small features as the eyes vanish, unrealistic frontiers appear in the feathers, etc..

7.2.2 Strong hierarchies

The simplest way to overcome the iteration drawback consists in reinforcing hierarchies by demanding that the composition product $\psi_\lambda \circ \psi_\mu$ become itself an element of family $\{\psi_{\lambda,} \lambda > 0\}$ that *acts more* than both ψ_λ and ψ_μ taken separately. Such a condition just means that the ψ_λ's form an *increasing semi group* defined by the equation

$$\psi_\mu \psi_\lambda = \psi_\nu \qquad \lambda, \mu \geq 0, \tag{48}$$

Figure 27: *Four iterations of the waterfall hierarchy, involving mosaic images.*

and by two conditions that translate the "more acting" effect, namely

1. $\forall \lambda, \mu \geq 0$, the value ν of Rel.(48) satisfies the inequality $\nu \geq \sup(\lambda, \mu)$;
2. $\forall \nu \geq \lambda \geq 0$, there exists a μ, with $\nu \geq \mu \geq 0$, such that Rel.(48) is satisfied.

The value $\lambda = 0$ is always associated with the identity operator, so that we can equivalently speak of positive, or of non negative, domain of variation for λ. Image processing uses basically the two *additive* (i.e. $s(\lambda, \mu) = \lambda + \mu$) and *granulometric* (i.e. $s(\lambda, \mu) = \sup(\lambda, \mu)$) semi groups that have been introduced in Definition 1. The first one may coincide with an iteration procedure. This occurs for example when ψ_1 is the erosion or the dilation by a unit digital disc (square, hexagon), or the convolution by the unit Gaussian. But this is not always true, as we saw previously, and anyway remains specific to the additive semi-groups.

By combining Relation (48) with Proposition 25 we find the following nice feature

Proposition 31 *To say that a family $\{\psi_{\lambda,} \lambda \geq 0\}$ forms an increasing semi-group of connected operators from $\mathcal{F}$ into itself implies that the family $\{(\sigma, \psi_\lambda), \lambda \geq 0\}$ is an increasing chain (i.e. a totally ordered set) in the lattice of the connective criteria. Then the segmentations of any function $f \in \mathcal{F}$ according to the (σ, ψ_λ)'s increase with λ, and their boundary sets decrease with λ .*

Proof. Let $f \in \mathcal{F}$ and $\nu \geq \lambda \geq 0$. The condition associated with Rel.(48) implies the existence of a $\mu \leq \nu$ such that $\psi_\mu \psi_\lambda = \psi_\nu$. As ψ_μ is a connected operator, we have from Rel.(39)

$$\sigma[\psi_\lambda(f), A] = 1 \Rightarrow \sigma[\psi_\mu \psi_\lambda(f), A] = \sigma[\psi_\nu(f), A] = 1.$$

Therefore the segmentation of $\psi_\nu(f)$ is larger than that of $\psi_\lambda(f)$, hence from Proposition 25, the corresponding boundary sets decrease. ■

In other words, the semi-group$\{\psi_{\lambda,}\lambda \geq 0\}$ generates a connected pyramid in U. Braga-Neto's sense [16]. On the other hand, it turns out to provide an answer to the epistemological question "by composing a (non optimal) operator with an optimization procedure, do we still obtain something that can be called optimal?" The answer is positive, in the conditions of the proposition, and makes precise the sense of variation of the product optimum as a function of λ. In the functional approach, a similar parameter λ appears when balancing the term "$\lambda\times$ perimeter of the partition" against a space integral that increases with the amending effect of ψ. But the resulting partitions are not ordered. Here, the reduction holds on the *boundary set itself*, and not on its measure, which can be infinite. The practitioners often take advantage of such a pyramidal structure to create synthetic partitions that borrow their contours to different levels of the hierarchy [73], (for example, a small detail is kept at a low level, and propagated to a higher one: this can work well because the boundaries of the small detail miss those of the larger partition).

How to choose the semi-group? Proposition (29) about connected openings suggests to deal with granulometric type semi-groups (Rel.(2)), since we entered the analysis by focusing on opening. Moreover, as the connected operators are closed under composition (Rel.(47)), three hierarchies of connected filters can be drawn from connected openings. They are the following

1. the sequences $\{\gamma_\lambda,\ \lambda > 0\}$ of decreasing openings, or Matheron's granulometries [50], and by duality the sequences of increasing closings $\{\varphi_\lambda,\ \lambda > 0\}$, or anti-granulometries;
2. the pyramids $\{M_i = \varphi_i\gamma_i \ldots \varphi_2\gamma_2\varphi_1\gamma_1,\ i$ a positive integer$\}$ of alternating sequential filters (or A.S.F.), where $\{\gamma_i\}$ is a granulometry, and $\{\varphi_i\}$ an anti-granulometry, which are due to S.R. Sternberg [91] and J. Serra (Ch. 10 in [75]);
3. F. Meyer's levellings [57]. A levelling $level = \gamma\varphi$ is the composition product of a marker based connected opening (38) by a marker based connected closing, with some slightly restrictive conditions [81]. The nice property here is that this composition product is commutative (the property is proved in [81] for connections such as the flat zones, but it is more general ; we do not tackle this point here). The direct consequence of this commutativity is that it provides levelling with a granulometric semi-group structure. A PDE representation of levelling, by F. Meyer and P. Maragos, may be found in [54].

We can illustrate the difference between the last two types by carrying on with "Altea" example. Figure 28 depicts the two levels of an A.S.F. pyramid $\{M_i,\ i > 0\}$, for $i = 4$ and $i = 7$. In the sequence M_i, the opening γ_j, with $j \leq i$, consists of the erosion ε_j by an hexagon of size j followed by the reconstruction that uses ε_j as marker. By comparing Figure 25c with Figure 28a, we can observe that for a same size $i = 4$ the details are treated more finely by the alternating sequential filter M_4. For example, the smaller dome of the church, near the centre of the image, is better delineated in Figure 28a. Also, the angles in the walls in front of the church remain visible.

Figure 28: *Sequential alternating filters by reconstruction, acting on "Altea" image. Both sequences begin by the unit opening; a) size four ; b) size seven.*

A drawback of the above (sequential or not) alternating filters is their propensity to reduce the contrasts. This effect is more due to the choice of the marker than to the filter itself. A second feature of these filters is that they favour sharp white zones or sharp dark ones according as they begin with an opening or a closing. That can be an advantage or not, depending on the context. In contrast, the levelling hierarchies become self-dual as soon as their markers satisfy this property.

These comments orient us towards levellings hierarchies, and suggest that we take as markers for "Altea" image the most significant extrema. The self dual marker m_{80}, used for both Figures 29b and 30a, is obtained by replacing f by zero everywhere excepted on its maxima and minima whose dynamics is $\geqslant 80$, (over 256 gray levels), and by leaving f unchanged on these extrema (see the HMIN and HMAX operators in P. Soille's book [90]). The maxima of f with a dynamics $\geqslant k$ are those which emerge at more than k above the first saddle point when going down. They are given by the connected opening $\gamma_{\text{rec}}(f)$ of f by marker $f - k$, where k is a positive constant.

We then choose as connective criterion σ that of the quasi-flat zones of slope 1, defined in Section 3.2.3, and we compute the connected opening Eq.(38), followed by its dual closing. This results in the *levelling by quasi-flat zones* depicted in Figure 29b. We can observe on Figure 29b that the sea, whose grey is almost constant on a large zone, has been considerably smoothed. Indeed, the whole smoothed region forms a single class of the segmentation.

Alternatively, we can keep the same marker m_{80} and replace the criterion by that of flat zones. The corresponding levelling is depicted in Figure 30a The texture of the sea is more preserved, but the sea class is over-segmented. The reader may find the precise algorithms in [26], or in [81].

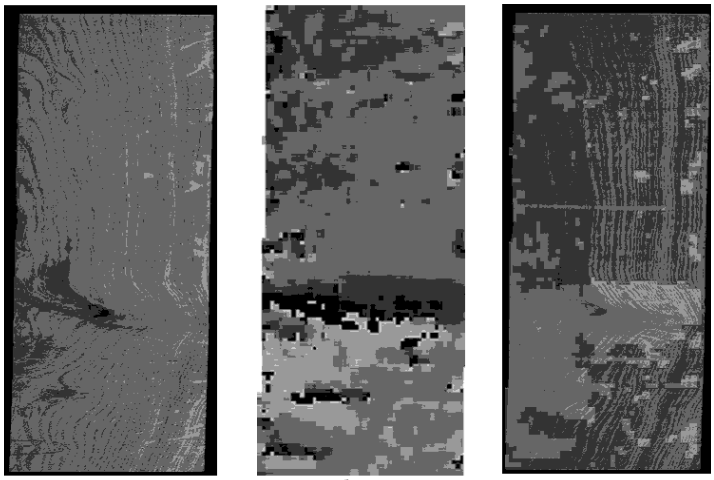
The levelling hierarchy may be illustrated, as follows, by means of this example. Take Figure 30a as the initial image, and compute its levelling by flat zones, for the marker m_{230}

a) b)

Figure 29: *a) Initial image, city of Altea ; b) leveling of a) for the criterion σ of the quasi -flat zones (marker m_{80}).*

a) b)

Figure 30: *Levelings of Altea image for the flat zones criterion σ. In a) the marker is m_{80}, and in b) m_{230}.*



a *b* *c*

Figure 31: *a) For wood texture identification, the small light patches of the oak specimen a) must be detected. b) Image of the veins orientations. c) best double opening.*

which is made of all extrema with a dynamics $\geqslant 230$. It results in Figure 30b. As levellings satisfy the granulometric semi-group equation (2) the image depicted in Figure 30b would have been the same as if we had directly calculated the levelling of the *initial image* f for marker m_{230}.

$$levelling(f/m_{230}) = levelling[levelling(f/m_{80})/m_{230}] = levelling[levelling(f/m_{230})/m_{80}].$$

The progressive levelling action appears clearly when confronting Figure 30a and b. Notice the relatively correct preservation of some fine details and border lines, which are kept because of their high dynamics, unlike other elements (textures, or a house near the sea, for example) that completely vanish.

7.3 Total ordering and hierarchies

The idea of a *total* ordering associated with a scale parameter underlies the hierarchical approaches. This physical meaning, though obvious and well adapted to scale space (magnification factors are scalar) has no particular reason to be imposed to other situations. Nothing, in a semi-group structure, demands a single label parametrization, as the following example illustrates it.

For his PhD work, A. Hanbury had to design an on-line classifier of oak boards destined to make furniture, and to measure the sizes and the orientations of small light patches [30] [31]. They are generally of a similar color to other parts of the wood, but tend to cut veins causing a disruption in the dominant local texture orientation [Figure 31a].

The first step of his processing replaces the wood image (608 X. 955 pixels) by that of the local main direction on a neighborhood of 64 X. 64 pixels, according to an algorithm developed by A.R. Rao [67]. This generates a direction image of 50 X. 112 pixels, i.e. a function θ from E into the unit circle C. In a second step the texture identification is obtained by means of a granulometry $\{\gamma_\lambda, \lambda \geq 0\}$ that works as follows :

- selecting those points whose angular value lies in the acute sector $[\alpha, \alpha+\omega]$

$$A_{\alpha,\omega}(\theta) = \{x : x \in E, \quad \theta(x) \in [\alpha, \alpha+\omega]\} \qquad 0 \leq \omega \leq \pi \tag{49}$$

- applying γ_λ to set $A_{\alpha,\omega}(\theta)$
- isotropizing the result with respect to the angular origin α by taking the supremum

$$\overline{\gamma}_{\lambda,\omega}(\theta) = \bigcup \{\gamma_\lambda [A_{\alpha,\omega}(\theta), \quad 0 \leq \alpha \leq 2\pi]\} \tag{50}$$

If we attribute conventionally the angular values π and 0 to the points of set $A_{\alpha,\omega}(\theta)$ and of its complement respectively, then operator $A_{\alpha,\omega}$ turns out to be an opening that increases with ω. It easy to see that the composition product $\gamma_\lambda \circ A_{\alpha,\omega}$, and after it the supremum average $\overline{\gamma}_{\lambda,\omega}$, are both decreasing openings of λ and of $\pi - \omega$.We can write

$$\overline{\gamma}_{\lambda,\omega} \circ \overline{\gamma}_{\lambda',\omega'} = \overline{\gamma}_{\sup(\lambda,\lambda'),\ \sup(\omega,\omega')}$$

i.e. a semi-group *without underlying hierarchy,* in the sense of total ordering (How could a hierarchy be established over a round table as the unit circle?). Such a double semi-group should not be considered as a mathematical curiosity, it is an actual tool that enables industrial companies to classify thousands of pieces of oak (see also [14]).

Independently of their 1-D feature, the hierarchies seem sometimes to constrain too much because *at each step*, the ordering must be satisfied. The jump connection proved to be a remarkable segmentation criterion that depends on the height λ of the jump. Now, the associated partition D_λ is larger than that $D_{\lambda'}$ uniquely when λ is a multiple integer of λ'. In between, most of the classes enlarge, but their frontiers may slightly vary. Is the insurance that we periodically find a larger partition not sufficient for using the criterion hierarchically?

Another variant is instructive. In a remarkable study, U.M. Braga-Neto and J. Goutsias [16] approach hierarchies of connected filters by introducing a connection measure. Their background involves the same notion of a connection as in this paper, but their original idea of measuring connections allow them to elaborate pyramids, and also to accept a fuzzy version of the connections.

8 Multi-dimensional segmentation: an example

Most of the previous examples concern grey tone images. Even in the waterfall example of Section 7.2.1, the three colour bands have been reduced to a scalar gradient from the first step of the processing. But precisely, after having condensed information so drastically, and so soon in the process, we were not able to reach fine and accurate segmentations. How could the three colour bands be involved more thoroughly in the segmentation process?

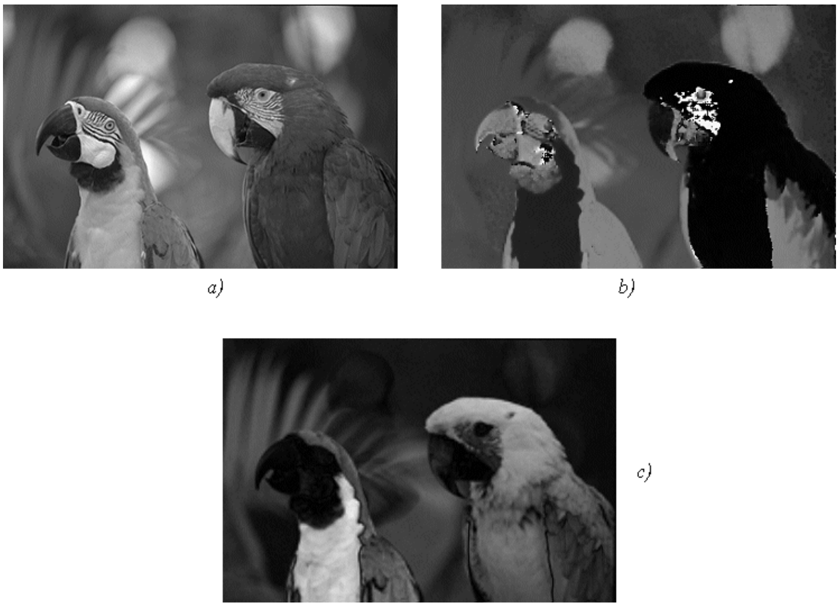


Figure 32: *Parrots image: a) luminance ; b) hue ; c) max-min saturation.*

It often seems that the dominant segmentation parameter is the hue in the zones of high saturation, and becomes more and more the luminance as the saturation reduces. This observation is re-formulated in quantitative terms by J. Angulo and J. Serra in [5]. Their approach can be split up into three major steps that involve various parameters and choices, namely :

1. To choose a representation for the triplet "luminance/saturation/hue", (HLS, HSV, etc.) ;
2. To *independently* segment the hue, luminance and saturation bands (this depends on one parameter at least for each band);
3. To combine the luminance and hue partitions into a unique *composite* partition by using the segmented saturation as a cursor (again one scalar parameter is added).

8.1 Luminance/saturation/hue representation

We will apply this approach to the parrots image of Figure 26, by keeping the same representation (27) of the colour triplet as previously. The three bands are depicted in Figure 32.

8.2 Luminance and hue segmentations

Various techniques can be applied to segment the two numerical images of luminance and of saturation. Since this paper is devoted to lattice based segmentation, we will choose the

a) b)

Figure 33: *Segmentations by jump connection of value $k = 20$. a) luminance, b) hue. Both partitions are displayed in false colour.*

jump connection, as introduced in Section 4.3. It depends on a positive parameter, namely the amplitude k of the jump.

As the parameter increases, the over-segmentations reduce, but in compensation heterogeneous regions appear. A good balance between these two effects seems to be reached for a jump $k = 20$ over 256 levels. Even then, small regions still need to be removed. This will be done by means of another connective criterion, denoted by τ, and that works on partitions, instead of functions. Given a partition D of $\mathbb{Z}^2$, take for seeds all classes with an area > 50. Let $\{M_{j,}j \in J\}$ denote this seeds family. Put

$$\begin{aligned}\tau[D, A] \quad = \quad & 1 \;\text{ when } A \in \mathcal{C} \text{ and when all points of } A \\ & \text{are closer to one seed than to the others,}\end{aligned} \tag{51}$$

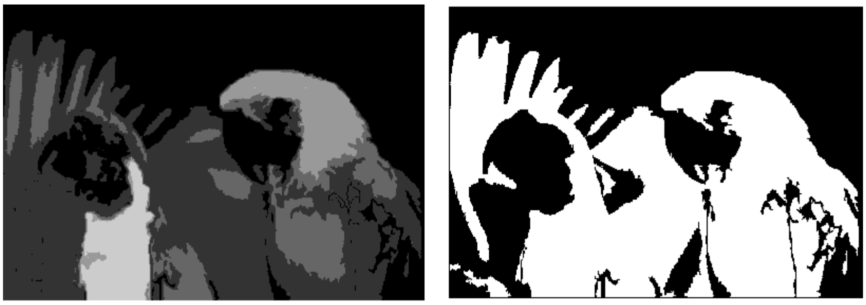
(plus the conditions on the singletons and the empty set). We obtain the criterion of Lantuejoul's skeleton by zones of influence, or SKIZ. It is obviously connective (see Prop.17) and results in Figure 33a, where all regions with an area < 50 pixels have been eliminated.

The approach followed for the luminance applies as well to the hue. The only difference in the method lies in that, for the hue, an arbitrary origin has to be fixed: the red has been chosen, as usual. The two segmentations are depicted in Figure 33.

8.3 The saturation band and composite partition

We now have in hand two representative partitions, for the luminance and the hue respectively. How to combine them ? The idea here is to reduce the saturation image to a set X, that corresponds to those pixels of a high saturation. Then, the hue partition is restricted to set X and the luminance one to set X^c. When a class of these partitions crosses the boundary ∂X, one splits it into two classes along the boundary.

The simplest way to determine set X should consist, of course, in thresholding the saturation image at a certain level, t say. But such a selection by thresholding, though simple, does not permit to control the shape of set $X(t)$, which can be irregular with various holes and small particles. An improvement consists in partitioning the saturation by processing it just as we did for the luminance and the hue.



a) b)

Figure 34: *a) Mosaic image of "parrots" saturation band, with the same sequence of operations as for Fig. 33. b) pixels image a) with a value* ≥ 56.

a) **b)**

Figure 35: *Parrots image segmentation a) by waterfall (level 4 of Fig.27), b) by composite partition.*

Figure 34a depicts the results for the criterion of jump connection. By allocating to each class the average saturation of its pixels, we generate a mosaic image which turns out to be more regular, hence more robust, than the initial saturation of Figure 32c. By thresholding this mosaic image we obtain the set $X(t)$ of all pixels of level $\geq t$, whose frontier is a selection of the edges of the saturation partition. If we now plot the total edge length of $X(t)$ versus the threshold level t a minimum appears around abscissa 60. A class of the synthetic partition is either the intersection of a luminance class with the low saturation zone $X^c(60)$, or the intersection of a hue class with the high saturation zone $X(60)$. Therefore, if the classes of the luminance, the hue, and the synthetic partition at point x are denoted by $A_m(x)$, $A_h(x)$, and $A(x)$ respectively, we have

$$\begin{aligned} A(x) &= A_m(x) \cap X^c(60) \quad \text{when} \quad x \in X^c(60) \\ A(x) &= A_h(x) \cap X(60) \quad \text{when} \quad x \in X(60). \end{aligned}$$

The corresponding composite partition is depicted in Figure 35b. The previous segmentation of the parrots, by waterfalls (Figure 27, level 4), with the drawing of the partition, has been reported on the left side of Figure 35. As expected, the multidimensional approach of Figure 35b yields better contouring.

8.4 Segmentation connections used in the example

Let us make the inventory of the various criteria involved in this example, of the corresponding segmentation connections, of their logical relationships, and also of the parameters we have introduced.

- Two jump criteria, σ_h^* and σ_l^*, have been used for the hue and the luminance respectively, with a common jump value $k = 20$. Then we took the infimum of each of them and of the connective criterion $\sigma_{50}(f, A)$ which equals 1 when $area(A) > 50$ are 0 when not (whatever function f is). This results in

$$\sigma_h^* \bigwedge \sigma_{50} \quad ; \quad \sigma_l^* \bigwedge \sigma_{50}$$

  of associated connections $\mathcal{C}_h^*$ and $\mathcal{C}_l^*$ respectively.

- We then have used the SKIZ skeleton τ as an operator on the lattice of the connections associated to all connective criteria on f (Proposition 12). $\tau(\mathcal{C})$ is the smallest connection where each point of a non singleton class is closer to one non singleton class of $\mathcal{C}$ than to all the other non singleton classes of $\mathcal{C}$. Operator τ is extensive and idempotent, so that the two connections $\mathcal{C}_h = \tau(\mathcal{C}_h^*)$, $\mathcal{C}_l = \tau(\mathcal{C}_l^*)$ turn out to yield maxima.

- For the saturation, the jump criterion σ_i^*, as defined by Rel.(**??**), has been combined by suprema according to whether the classes were below level 60 or not, i.e. for $1 \leq i \leq 3$ or not. We obtain the two connective criteria

$$\begin{aligned} \sigma_s^1 &= \bigvee \{\sigma_i^*[f, A], 1 \leq i \leq 3\} \\ \sigma_s^2 &= \bigvee \{\sigma_i^*[f, A], 4 \leq i \leq 13\}, \end{aligned}$$

which generate the two connections $\mathcal{C}_s^1$ and $\mathcal{C}_s^2$.

- the final connection $\mathcal{C}$ composes the previous ones in the following manner, that illustrates Theorem 14 on cross connections

$$\mathcal{C} = (\mathcal{C}_l \bigwedge \mathcal{C}_s^1) \bigvee (\mathcal{C}_h \bigwedge \mathcal{C}_s^2) \tag{52}$$

All in all, the whole process depends on three positive parameters, one for the jump criterion (k), another for the areas cut off (50) and the last one for the saturation cursor (60). But it does not involves parameters only, but also choices. In the present case, we met as major choices :

- the color representation (among RGB, LAB, HLS, L_1 norm, etc.)

- the connective criteria for segmenting the luminance and the hue band (jump σ^*, quasi-flat zones, or others), which can be different for the two bands,

- the decision to enlarge both luminance and hue partitions by means of criterion τ

- the way to obtaining the saturation cursor set, either by a simple threshold or via a mosaic image.

In his epistemology of Probabilities, G. Matheron establishes a clear distinction between *estimating* and *choosing*, [51]. As a consequence, there is no such thing as an overall optimization, since the global process mixes up choices and parameters. The question of estimating in an optimal way holds on the parameters only. Alternative correct segmentations of the various bands could have been reached by minimizing convenient functionals here. But the final synthesis of Rel.52 involves too many choices and to many logical operations to be the matter of *one* functional optimization only.

9 Conclusion

What has been written so far is a thought about the notion of an optimization. It is an attempt to surpass the contradiction between the globality of an optimum, and the fact that it often appears as a step in a larger process. In which sense then can we speak of optimality?

We asked ourself the question about the segmentation of a space where a physical variable spreads out, i.e. about the largest partition of this space into homogeneous regions. We showed that the problem admits a largest solution when the criterion according to which the regions are said homogeneous is connective, i.e. when it generates *connections* over all the subsets where it is satisfied, and uniquely in that case (Theorem 8).

Surprisingly, the idea of a connection, which was born in the different context of morphological filtering, turned out to be the exact convenient tool for our purpose. We could derive from this identification that the segmentation criteria form a complete lattice where sup and inf provided optimal partitioning with a precise meaning, and allowed us to combine segmentations in various manners (Theorem 14). In addition, by introducing connected operators, we could manage composition products between segmentations and filters (Proposition 29).

Optimality remains always preserved, although the lattice approach relativizes the notion and the role of an extremum.

By so doing, we did not try and discover a general segmentation procedure, which probably does not exist. Rather, we focused on the *conditions* criteria must fulfill for decomposing a complex situation into serial or parallel tasks. Downstream, the choice of such or such criterion brings with it the various specifics.

Some of the questions asked in the introduction received here positive answers :

- In the lattice approach we adopted here, the uniqueness of the segmentation, as an optimal partition, is ensured as soon as the involved criteria are connective. Moreover this segmentation holds not only on the field of the image under study, but also on each of its sub-zones, as stated by Theorem 8, the central result of this paper.
- The Euclidian/Digital alternative no longer exists. Given a totally arbitrary set E, the set $\mathcal{P}(E)$ turns out to be a sufficiently defined input space. The advantages or the drawbacks of $\mathbb{R}^2$ versus $\mathbb{Z}^2$ are transferred to the specific criteria.
- The segmented class at point x is always reached by extension, i.e. by unions of of homogeneous zones that contain the point x. Therefore, the segmentation classes are independent of their location in the measuring field, assuming that a convenient neighborhood is experimentally accessible (Proposition 9).
- By coupling convenient filters with segmentation, scale space hierarchies are built up, with respect to non-linear semi-groups in particular. They are identified to sequences of increasing criteria (Proposition 31).

In the applied sections, we have classified the segmentations according to the involved spaces (image space versus feature spaces) and the existence of seeds. But from the viewpoint of the operators, a more pertinent discrimination rests on whether a criterion is strongly connective or not (Definition 18).

The above work is far from being exhaustive. We did not really try to establish links between geometric flow PDE's and lattice based segmentations. Proposition 17 about seeds partly enlights the question, and the example on 3-D wave front propagation into a shinebone clearly shows how much segmentation and PDE propagation are distinct notions. However, these remarks are just inputs. We did not wonder either about general conditions on criteria to give rise to minimization functionals. We did not think about a random version of the theory, or a fuzzy one, that would permit us to weaken notions such as "the largest partition". We hope that the segmentation approach by maximum partitions, and its role in image processing, will open doors and suggest the exploration of new paths.

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