CUBE, CUBE-OCTAHEDRON,

OR

RHOMBODODECAHEDRON ?

J. SERRA

Centre de Morphologie Mathématique,

Ecole des Mines de Paris

France

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The Three Cubic Systems of Grids (I)

Place a point at the center of a Euclidean unit cube. You can project it:

1/ on all of the six faces;

2/ on all of the eight vertices;

3/ on all of the twelve edges.

In each case, by symmetry, all projections have a same length. So by translation each system generates a net of vertices and edges through the whole space, that we shall call a grid.

The Three Cubic Systems of Grids (II)

In these three grids, the families of vertices admit the following interpretations :

- *1rst case* : translations of the unit cube vertices,

  ⇒ **CUBIC GRID**

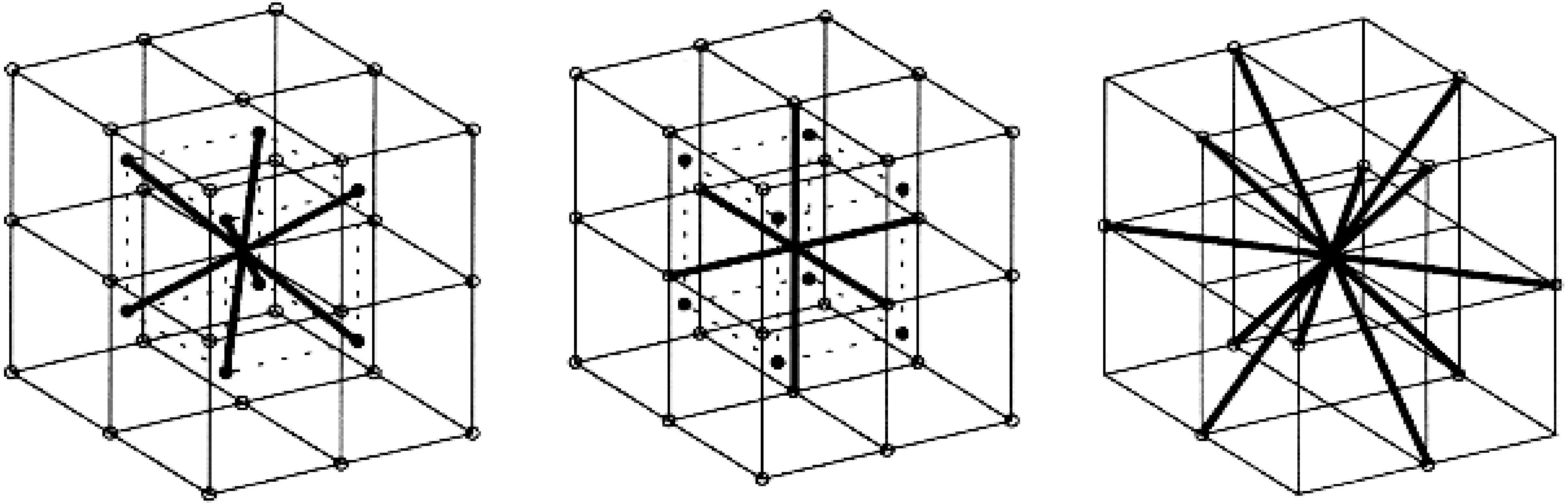
- *2nd case* : the centres of the previous cubes are added to their vertices,

  ⇒ **CENTERED CUBIC GRID (cc grid)**

- *3rd case* : the centres of the faces are added to the vertices of the cubic grid.

  ⇒ **FACE-CENTERED CUBIC GRID (fcc grid)**

First Neighbours



8 neighbours for the cc grid

6 neighbours for the cubic grid

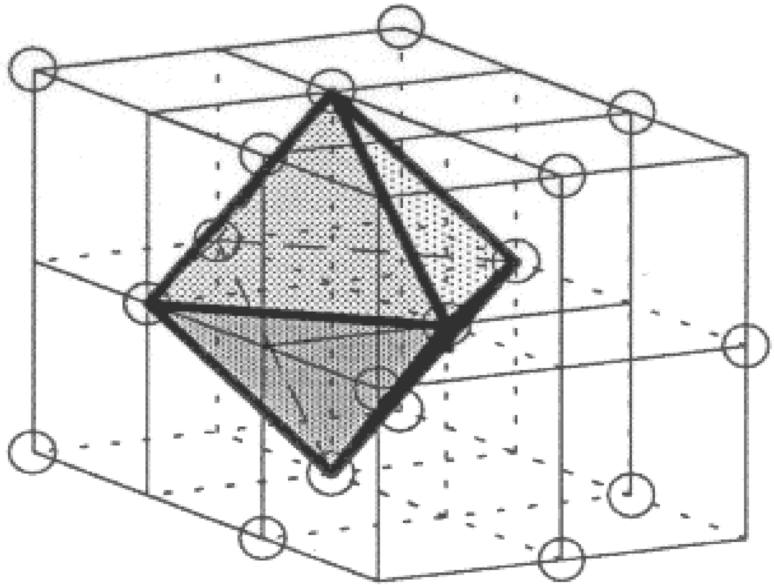
12 neighbours for the fcc grid

Cubic Grid
Neighbours Patterns

- The sixt first neighbours generate an octahedron of 7 voxels ; by adding :

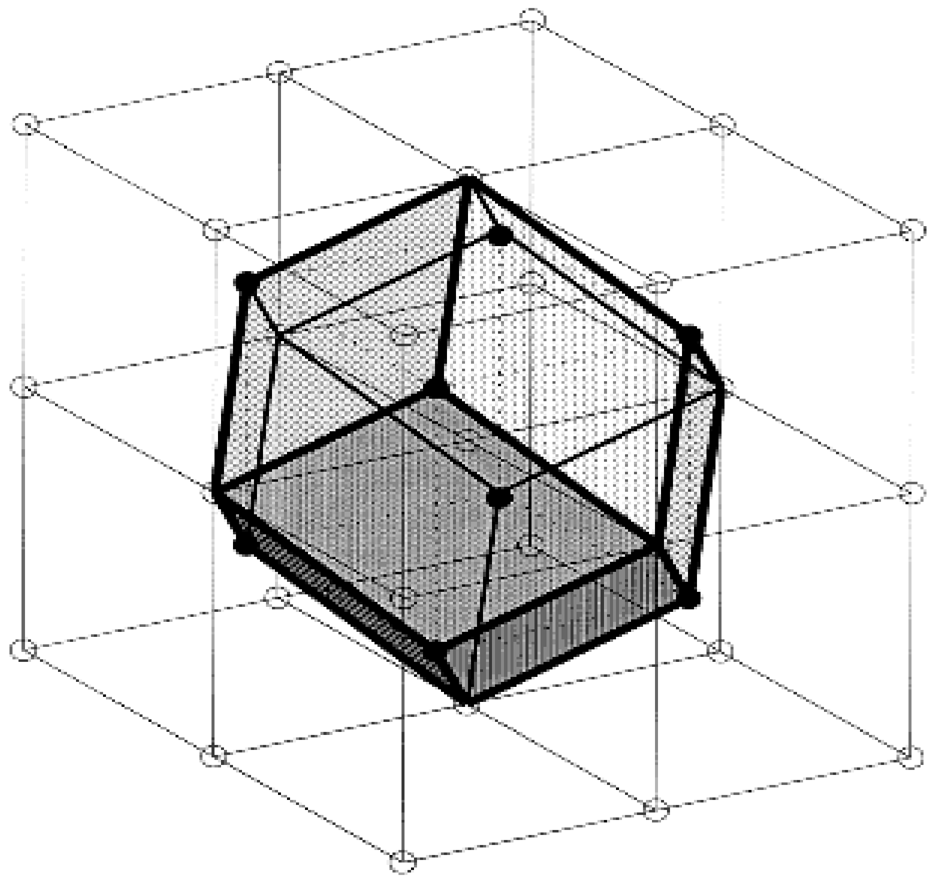
  - the 12 second neighbours, one finds a cuboctahedron of 19 voxels (dist. ratio: 1.41)

  - and again the 8 third ones, one finally obtains the cube of 27 voxels (dist. ratio: 1.73).



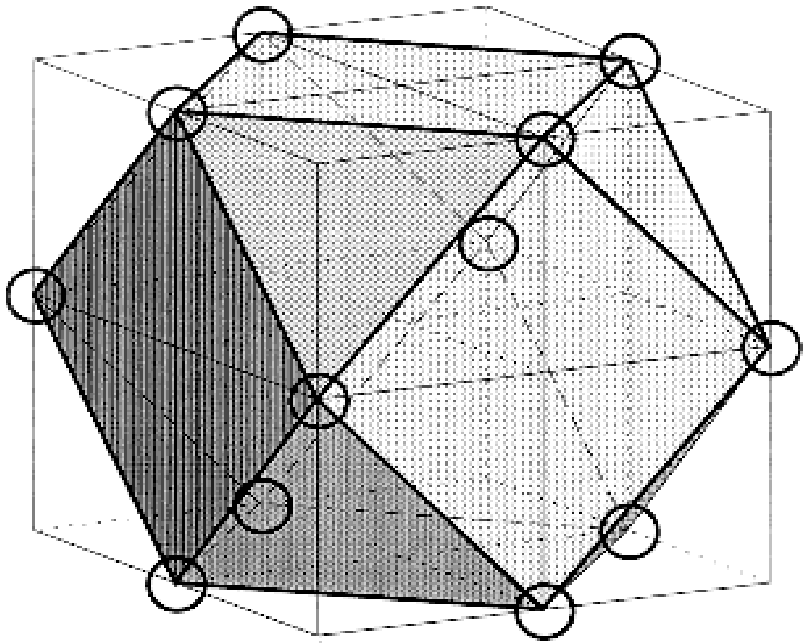
CC Grid
Neighbours Patterns

- The 8 first neighbours generate a cube of 7 voxels;
- by adding the 6 second neighbours, one finds a rhombododecahedron of 19 voxels (distance ratio 1.15);
- these rhombododecahedra partition the space ;
- The medians of the rhombs are not edges.



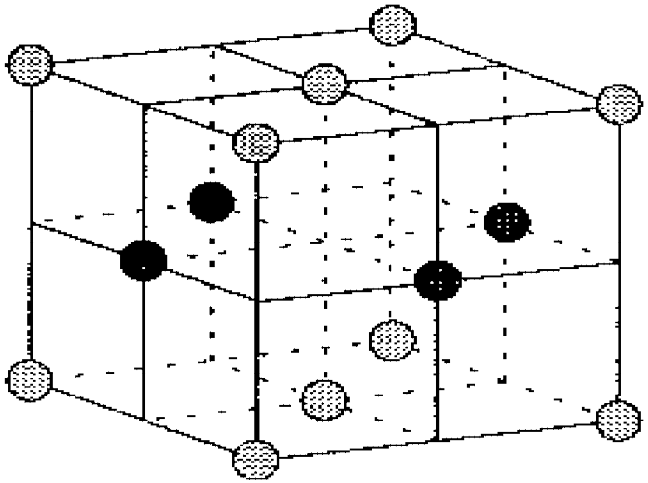
FCC Grid
Neighbours Patterns

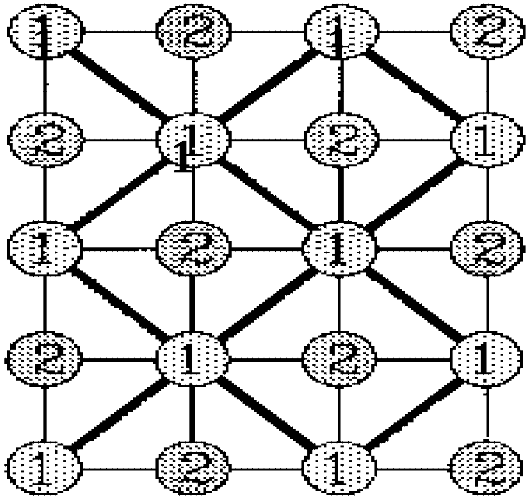
- The 12 first neighbours generate a cube-octahedron of 13 voxels.
- The cube-octahedra do not fill the space (they leave octahedric holes between them)
- However, they generate a regular net where all edges have the same lenght.



CC and FCC Grids : Square Sections

- In both cc-grid and cfc-grid, the sections normal to
  - the cube diagonals are composed of hexagons,
  - the cube edges are composed of squares.
- The ratios between square and interplane spacings have a value 2 for cc-grid, and 1.41 for fcc grid



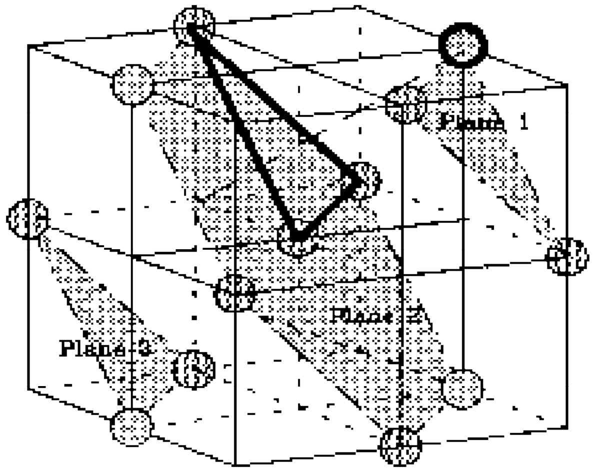


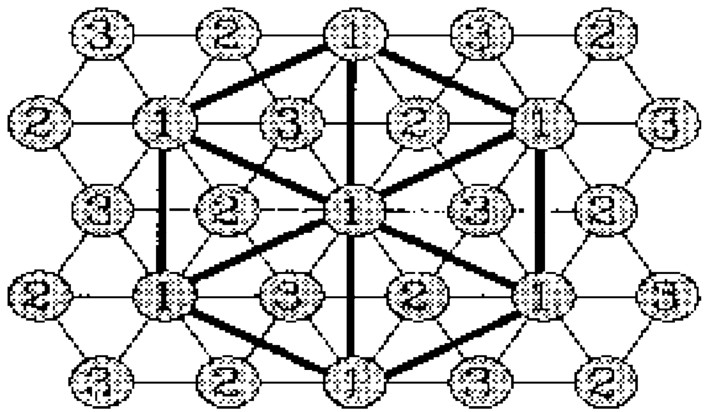
CC and FCC Grids : Hexagonal Sections

- For the sections normal to the diagonal of the cube, we find again a staggering effect, with now a perodicity of three.

- The ratios between hexagonal and interplane spacings are

  0.40 for cc-grid and 0.81 for fcc grid





Steiner Decompositions

- Consider the centre x of a cube C . By dilating
  - the three vectors from x to the faces of C, one obtains a **Cube**;
  - the four vectors from x to the vertices of C, one generates a **Rhombododecahedron** ;
  - the six vectors from x to the edges of C, one obtains a **Tetrekaidecahedron**;
  - the two tetrehedra built on the diagonals of C, one obtains a **Cube-octahedron.**

Steiner Decompositions (cont^d)

- Cube :

$$C = \begin{pmatrix} 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} (1,0) \end{pmatrix}$$

- Cube-octahedron :

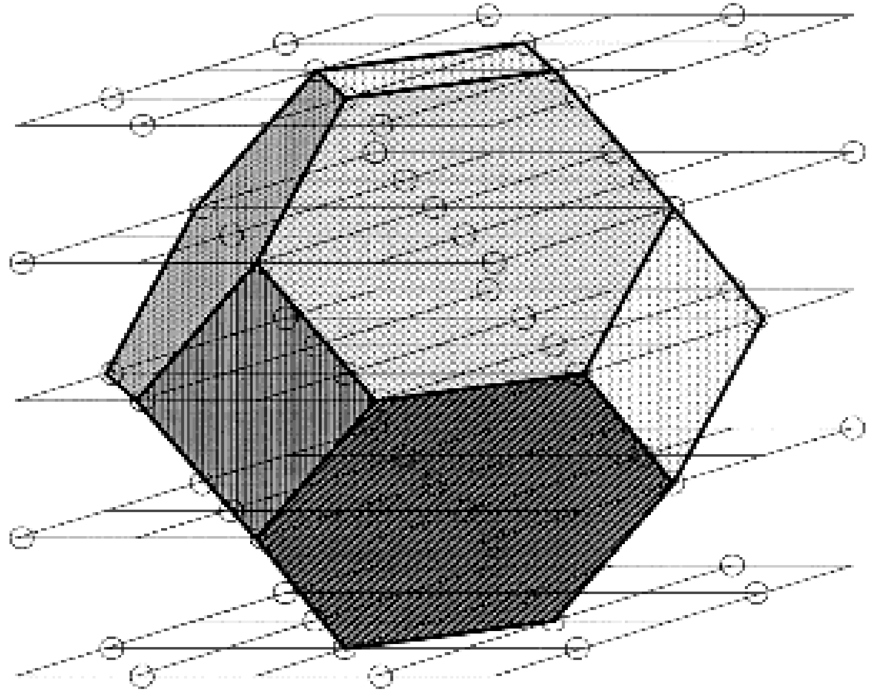
$$D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- Rhombododecahedron :

$$R = \begin{pmatrix} 1 & . \\ . & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & . \\ . & 1 \end{pmatrix} \oplus \begin{pmatrix} . & -1 \\ 0 & . \end{pmatrix} \oplus \begin{pmatrix} . & 0 \\ -1 & . \end{pmatrix}$$

Steiner Decompositions (cont^d)

By dilation of the six basic vectors of the f.c.c. grid , one obtains the tetrakaidecahedron (and not the cube-octahedron, which is not Steiner).



$$\begin{pmatrix} 0 & 1 \\ . & . \end{pmatrix} \oplus \begin{pmatrix} 0 & -1 \\ . & . \end{pmatrix} \oplus \begin{pmatrix} 0 & . \\ 1 & . \end{pmatrix} \oplus \begin{pmatrix} 0 & . \\ -1 & . \end{pmatrix} \oplus \begin{pmatrix} . & 0 \\ 0 & . \end{pmatrix} \oplus \begin{pmatrix} 0 & . \\ . & 0 \end{pmatrix}$$

Staggering the Grids

- For both polyhedra R and D, the previous decompositions generate the staggering by a 45° turn of the grid, *e.g.*

$$D = \begin{pmatrix} . & \mathbf{1} & . \\ \mathbf{1} & . & \mathbf{1} \\ . & \mathbf{1} & . \end{pmatrix} \cup \begin{pmatrix} \mathbf{1} & . & \mathbf{1} \\ . & \mathbf{1} & . \\ \mathbf{1} & . & \mathbf{1} \end{pmatrix} \cup \begin{pmatrix} . & \mathbf{1} & . \\ \mathbf{1} & . & \mathbf{1} \\ . & \mathbf{1} & . \end{pmatrix}$$

- By so doing, we leave inside holes, or, equivalently,we drop half of the voxels. Alernatively, we can build more condensed R and D as follows

upper and lower planes $\begin{pmatrix} \mathbf{1} & \mathbf{1} & . \\ \mathbf{1} & \mathbf{1} & . \\ . & . & . \end{pmatrix}$ *odd central plane* $\begin{pmatrix} . & \mathbf{1} & . \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \\ . & \mathbf{1} & . \end{pmatrix}$

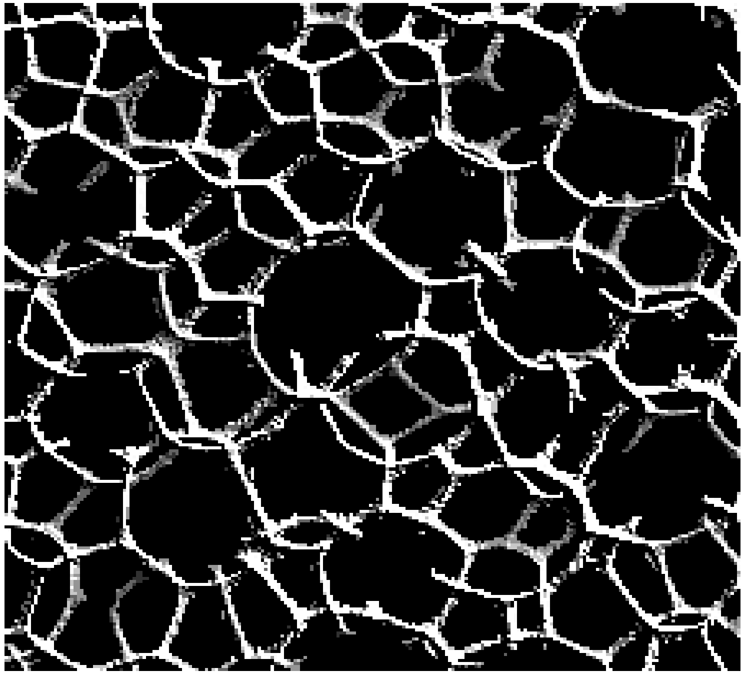
upper and lower planes $\begin{pmatrix} . & . & . \\ . & \mathbf{1} & \mathbf{1} \\ . & \mathbf{1} & \mathbf{1} \end{pmatrix}$ *even central plane* $\begin{pmatrix} . & \mathbf{1} & . \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \\ . & \mathbf{1} & . \end{pmatrix}$

Directions

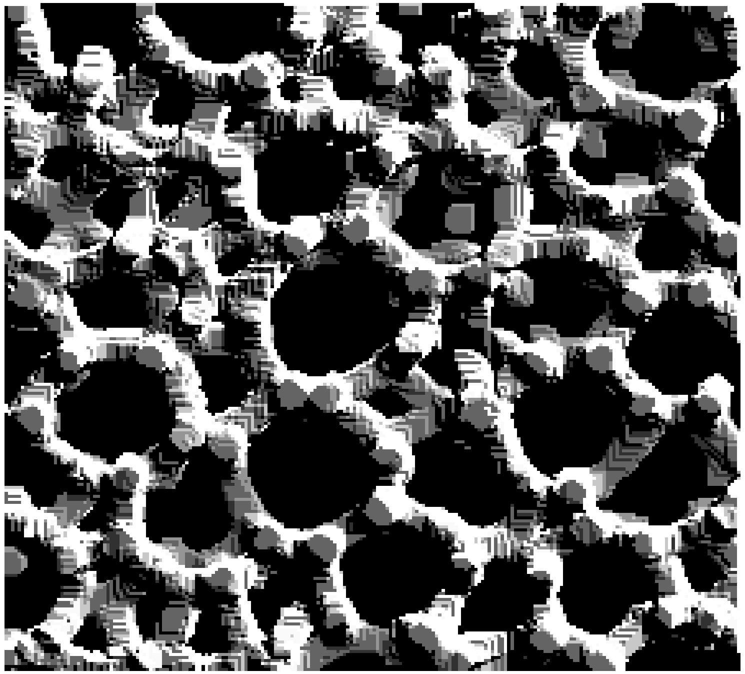
- Unlike the unit disc, the unit sphere cannot be divided into as many sectors as we want.
- The 3-D sphere can be partitionned only into 2, 4, 6, 8, 12, and 20 *equal* solid angles.
- The axes of the 6, 8 and 12 solid angles coincide with the directions of the first neighbours in the cubic, cc and fcc grids respectively. Therefore, they can be accessed **digitally**.
- The finest equidistributed digital set of directions is written :

$$\begin{pmatrix} 0 & 1 \\ . & . \end{pmatrix} ; \begin{pmatrix} 0 & -1 \\ . & . \end{pmatrix} ; \begin{pmatrix} 0 & . \\ 1 & . \end{pmatrix} ; \begin{pmatrix} 0 & . \\ -1 & . \end{pmatrix} ; \begin{pmatrix} . & 0 \\ 0 & . \end{pmatrix} ; \begin{pmatrix} 0 & . \\ . & 0 \end{pmatrix}$$

Cube-octahedral Dilation

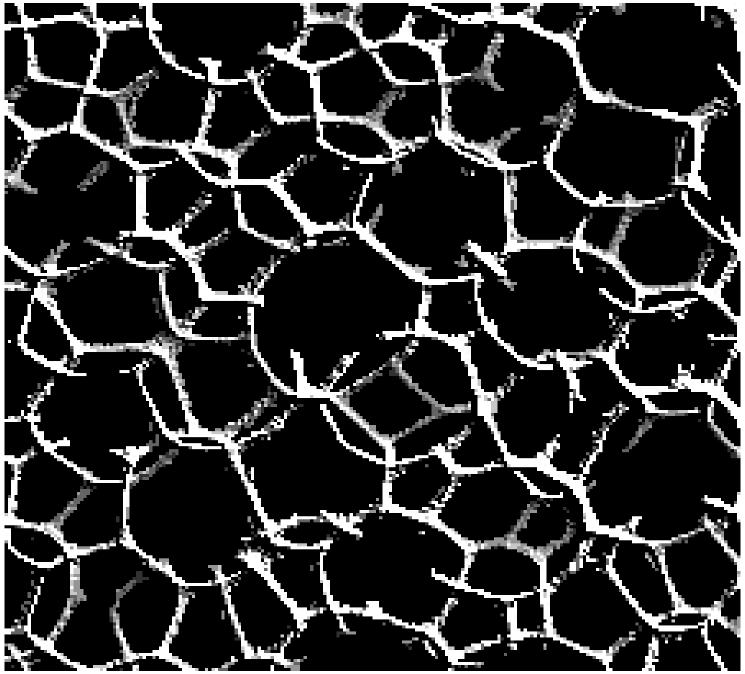


a) : Series of 74 cross sections of expanded Polystyrene (from Shell)

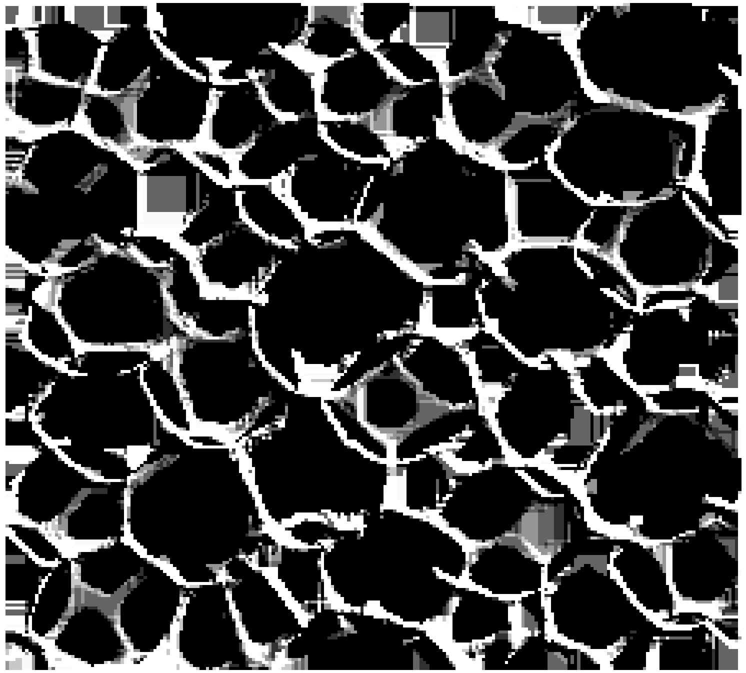


cube-octahedral dilation of size 5 of ***a)***

Cube-octahedral closing

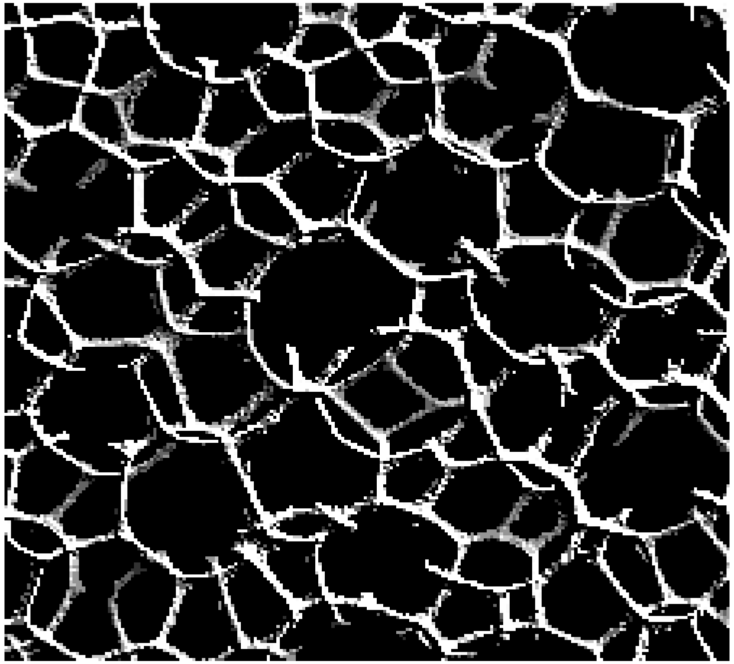


a) Initial image

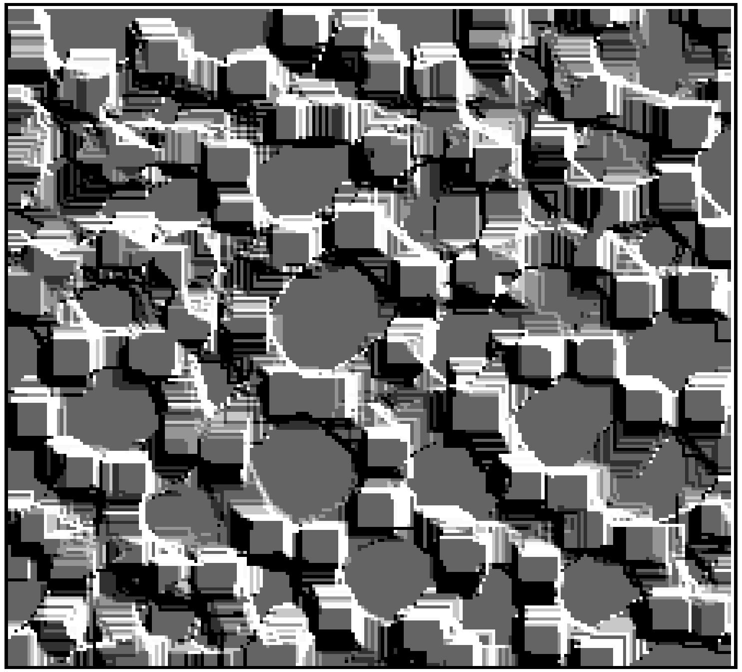


cube-octahedral closing
of size 12 of ***a)***

Cubic Dilation



a)



cubic dilation of size 5 of ***a)***

Space Graphs

- In R^3, a space graph is
  - a set *X* of ***points*** ;
  - a collection *E* of ***edges***, *i.e.* of lines homotopic to the segment [0,1] . Both ends of each edge belong to *X* , and two edges may possibly meet at their extremities only ;
  - a collection *F* of ***faces,*** *i.e.* of surfaces homotopic to the closed unit disc. The contours of the faces are exclusively edges, and two faces may meet along edges only ;
  - a set *P* of ***blocks***, formed by the connected components of the space that remain when all points edges and faces have been removed.

Euler -Poincaré Number

- Space graphs are ***the*** turning point between Euclidean and Digital spaces. Defined in R^3 , they can be reinterpreted in Z^3 , and the derived notions are meaningful in both spaces.

- It is in particular the case for the Euler-Poincaré Number $\nu(Y)$ of the set $Y = X \cup E \cup F$ formed by the points, edges and faces of space graph X . It is equal to

$$\nu(Y) = N\,(vertices) + N\,(faces) - N\,(edges) - N\,(blocks)$$

- From a digital point of view, the problem consists then to associate convenient space graphs with the sets under study.

Comments on Graphs and Connectivity

According to the type of operation, three different levels of connectivity may be involved:

⇒ Erosions, dilations, and derived filters, as well as skeletons (by erosion/opening) do not need *edges* : one always dilates a set of ***points*** by a set of *points* .

⇒ Geodesy, reconstructions, connected components (*e.g.*ultimate erosion) require ***grids***, *i.e.* vertices + edges .

⇒ thinnings, thickenings, or any operator which preserves homotopy requires the datum of a ***space graph*** on the grid.

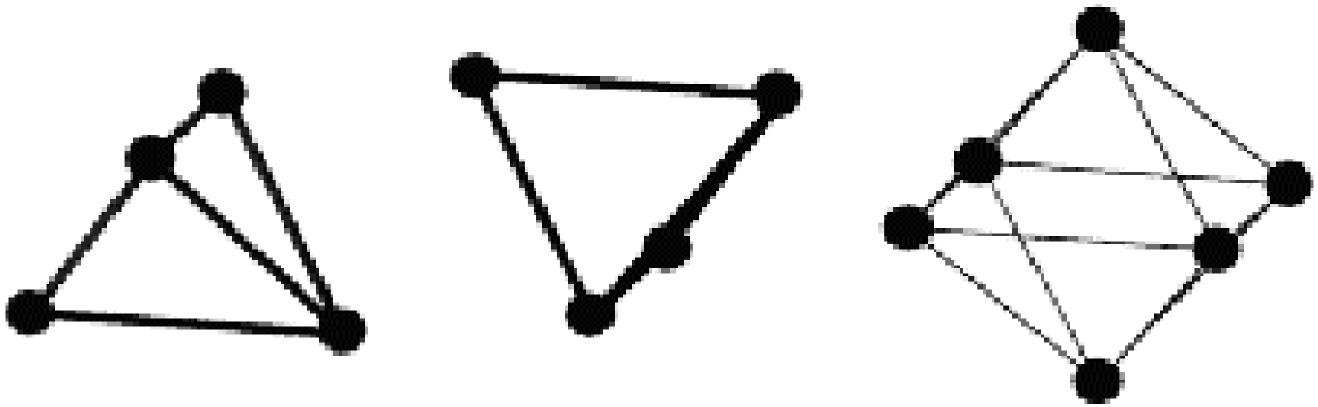
Space Graph for the FCC Grid (I)

We vill associate with the F.C.C grid the space graph whose:

⇒ vertices and edges are those of the grid,

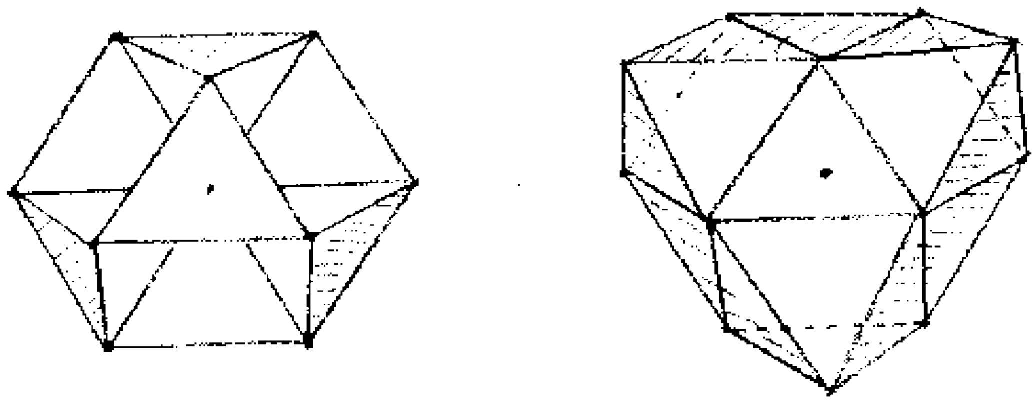
⇒ faces are the elementary triangles (but ***not*** the squares !),

⇒ blocks are the smallest tetrahedra and octahedra, *i.e.*



Space Graph for the FCC Grid (II)

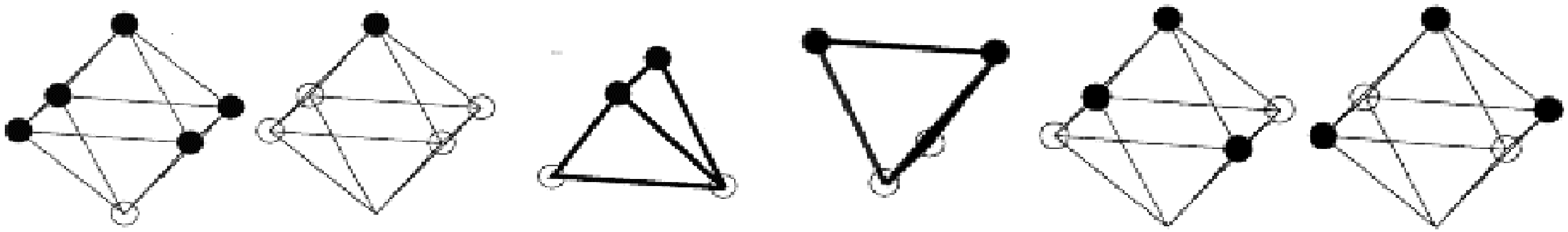
- In such a graph, the basic cube-octahedron is not a convex polyhedron, but admits six conic sinks. Its convex enveloppe for the graph, is its *stellation*, *i.e.* its circumscribed octahedron.



- Given a set X (the 1's) the vertices, and the edges, faces, and blocs whose all vertices are 1's generate the space graph of X.
- The graph associated with the 0's is obtained by applying the ***same rule*** to set X^c .

Euler-Poincaré Count on FCC Grid

- Euler-Poincaré count illustrates the role of the Space Graph. The events to be checked are the six following subsets of basic blocks



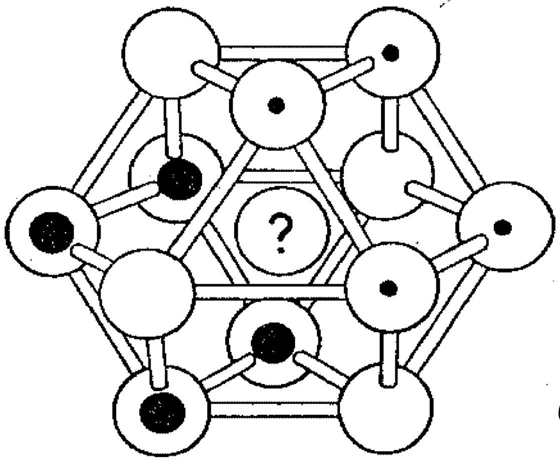
- The occurrences of each of them are computed and added, the first two ones positively, and the last four ones negatively.

Thinnings and Homotopy

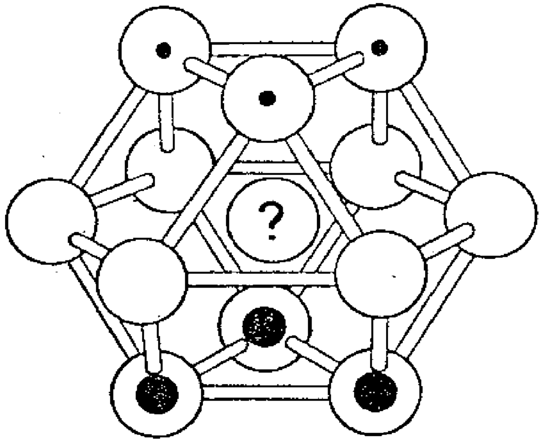
- Consider a neighborhood based thinning . It is homotopic when, by changing $1 \rightarrow 0$, we do not locally modify the genus of the boundaries, *i.e.* the change must not
  - open a hole, neither create a new particle;
  - generate a donut of grains, or of pores;
  - suppress a grain or a pore; etc..
- When the neighborhood is based on the unit digital cube-octahedron, then one finds five candidate configurations only, up to rotations and complement.

Examples of homotopic thinnings

- Here are two candidate configurations for cube-octahedral homotopic thinning



a)



b)

● the 1's ; • the 0's ; and in between, a no man's land.

For configuration *b*) three 0's have to be added in the upper plane, and three 1's in the lower plane, to get the stellations.

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