Connections for Sets and Functions

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Connectivity in Mathematics

- ***Topological Connectivity* :** Given a topological space E, set $A \subseteq E$ is connected if one cannot partition it into two non empty closed sets.

- ***A Basic Theorem* :**

  If $\{A_i\}$ $i \in I$ is a family of connected sets, then

$$\{ \cap A_i \neq \emptyset \} \Rightarrow \{ \cup A_i \text{ connected} \}$$

- ***Arcwise Connectivity*** (more practical for $E = R^n$) **:** A is **arcwise connected** if there exists, for each pair $a,b \in A$, a continuous mapping ψ such that

$$[\alpha, \beta] \in R \qquad \text{and} \qquad f(\alpha) = a \;\; ; f(\beta) = b$$

  This second definition is more restrictive. However, for the open sets of R^n, both definitions are **equivalent**.

Criticisms

Is* topological connectivity *adapted to Image Analysis ?

- Digital versions of arcwise connectivity are extensively used:
  - in 2-D : 4- and 8- connectivities (square), or 6- one (Hexagon);
  - in 3-D : 6-, 12-, 26- ones (cube) and 12- one (cube-octaedron).

However :

- Planar sectioning (3-D objects) as well as sampling (sequences) tend to disconnect objects and trajectories, and topological connectivity does help so much for reconnecting them;
- More generally, in Image Analysis, a convenient definition should be operating, *i.e.* should introduce specific operations ;
- Finally, the topological definition is purely set oriented, although it would be nice to express also connectivity for functions...

Lattices and Sup-generators

- A common feature to sets $\mathcal{P}$(E) (E an arbitrary space) and to functions f: E→T (T, grey axis) is that both form **complete lattice that are «well» sup-generated**.

- A **complete lattice** $\mathcal{L}$ is a partly ordered set where every family $\{a_i\}$ $i \in I$ of elements admits
  - a smaller upper bound $\vee \mathbf{a_i}$, and a larger lower bound $\wedge \mathbf{a_i}$.

- A family $\mathcal{B}$ in $\mathcal{L}$ constitutes a **sup-generating** class when each $a \in \mathcal{L}$ may be written $\mathbf{a = \vee \{b \,;\, b \in \mathcal{B} \,,\, b \leq a \}}$.

- In $\mathcal{P}$**(E)** - $\vee$ and $\wedge$ operations become union and intersection;
  - the elements of E, *i.e.* the points, are sup-generators.

Lattice of Numerical Functions

In order to ovoid the continuous/digital distinction, the real lines $\overline{R}$ and $\overline{Z}$, or any of their compact subsets, are all denoted by T. Axis T is a totally ordered lattice, of extreme elements 0 et m.

- The class of functions f : E $\rightarrow$ T, E an arbitrary space, forms a totally distributive **lattice**, denoted by T^E, for the product ordering

$$\mathbf{f \leq g \quad \text{iff} \quad f(x) \leq g(x) \quad \text{for all} \quad x \in E},$$

In this lattice, the so called numerical $\vee$ and $\wedge$ are defined by :

$$(\vee f_i)(x) = \vee f_i(x) \quad \text{and} \quad (\wedge f_i)(x) = \wedge f_i(x) \qquad x \in E .$$

- Moreover, in T^E the **pulses functions**:

$$k_{x,t}(y) = t \quad \text{when} \quad x = y \quad ; \quad k_{x,t}(y) = 0 \quad \text{when} \quad x \neq y \quad ,$$

are **sup-generating**, *i.e.* every function f is written as

$$f = \vee \{ k_{x,t} , \ x \in E, \ t \leq f(x) \}$$

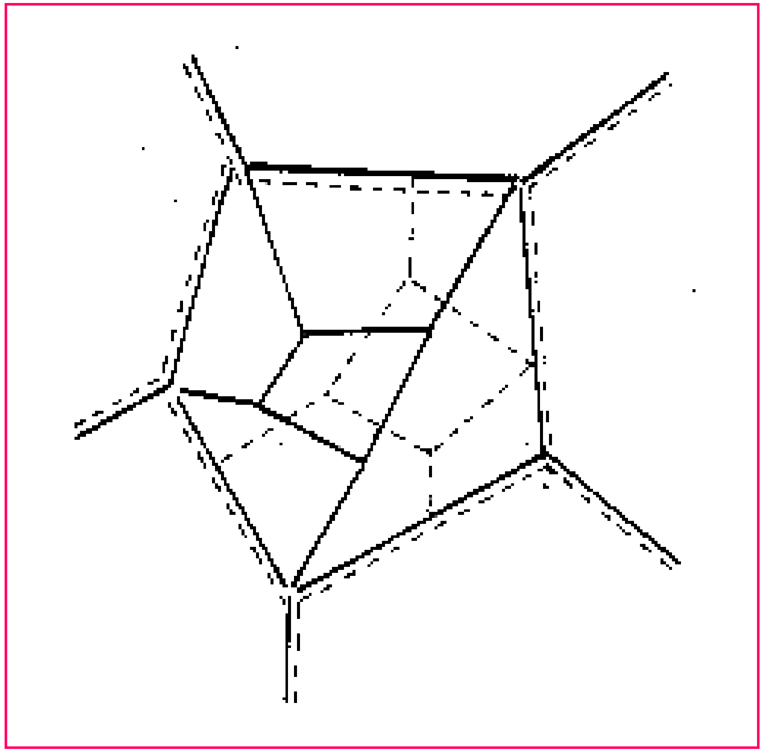
Lattice of the Partitions

- ***Reminder :*** A **Partition** of space E is a mapping D: E $\rightarrow \mathcal{P}$(E) such that

  (i) $\forall$ **x** $\in$ **E ,** **x** $\in$ **D(x)**
  (ii) $\forall$ **(x, y)** $\in$ **E,**

  either **D(x) = D(y)**
  or **D(x)** $\cap$ **D(y) =** $\varnothing$

- The partitions of E form a **lattice** $\mathcal{D}$ for the ordering in which D ≤ D' when each class of D is included in a class of D'. The largest element of $\mathcal{D}$ is E itself, and the smallest one is the pulverization of E into all its points.



The sup of the two types of cells is the pentagon where their boundaries coincide. The* inf*, simpler, is obtained by intersecting the cells.

Connections on a Lattice

Since the basic property of topological connectivity involves set $\cup$ and $\cap$ only, we can forget all about topology and take the basic property, expressed in the lattice framework, as a starting point.

Connection **:** Let $\mathcal{L}$ be a complete lattice. A class $C \subseteq \mathcal{L}$ defines a **connection** on $\mathcal{L}$ when

(i) $\mathbf{0} \in C$;

(ii) C is **sup-generating ;**

(iii) C is **conditionally closed** under supremum, *i.e.*

$$\mathbf{h_i} \in C \quad \textbf{and} \quad \wedge\, \mathbf{h_i} \neq 0 \quad \Rightarrow \quad \vee\, \mathbf{h_i} \in C\ .$$

- In particular, points belong to all possible connections on $\mathcal{P}$(E) and pulses to all connections on functions T^E. Thus they are said to constitute **canonic families** S.

Connected Opening

- ***Connected opening :*** Let $\mathcal{C}$ be a connection on lattice $\mathcal{L}$ of canonic family $\mathcal{S}$. For every s ∈ $\mathcal{S}$, the operation $\gamma_s : \mathcal{L} \to \mathcal{L}$ defined by

$$\gamma_s(f) = \vee (p \in \mathcal{C} , s \leq p \leq f) \qquad f \in \mathcal{L} ,$$

is an ***opening*** :

  – of (point, pulse) **marker** s

  – and of **invariant sets** $\{p \in \mathcal{C},\ s \leq p\} \cup \{0\}$.

  Moreover, when $r \leq s$, with $r,s \in \mathcal{S}$, then $\gamma_r \geq \gamma_s$.

- *N.B.* Operation γ_s belongs to the class of the so called **openings by reconstruction**, where each connected component is either suppress or left unchanged. However, such openings can also be based on criteria other than set markers (*e.g.* area, diameter) .

Characterization of a Connection

Conversely, the γ_s's induced by connection $\mathcal{C}$ do **characterise** it :

- ***Induced Connection :*** let $\mathcal{C}$ be a sup-generating family in lattice $\mathcal{L}$. Class $\mathcal{C}$ **defines a connection** iff it coincides with invariant sets of a family $\{\gamma_s , s \in \mathcal{S}\}$ of openings such that

  (iv) for all $s \in \mathcal{S}$, we have $\boldsymbol{\gamma_s(s) = s}$,

  (v) for all $f \in \mathcal{L}$, and all $r, s \in \mathcal{S}$, the openings $\gamma_r(f)$ and $\gamma_s(f)$ are either identical or disjoint, *i.e.*

  $$\gamma_r(f) \wedge \gamma_s(f) \neq 0 \quad \Rightarrow \quad \gamma_r(f) = \gamma_s(f) ,$$

  (vi) for all $f \in \mathcal{L}$, and all $s \in \mathcal{S}$, $s \nleq f \;\Rightarrow\; \gamma_s(f) = 0$

- ***Optimal Segmentation:*** the family of the maximal connected components $\leq f$, $f \in \mathcal{L}$, **partitions** f into elements de $\gamma_s(f)$, and one cannot segment f with **less** elements of $\mathcal{C}$.

Properties of the Connections

- ***Lattice of the Connections*** **:** The set of the connections that contain the canonic sup-generating class $\mathcal{S}$ forms a complete lattice where

$$\mathbf{inf}\,\{C_i\} = \cap C_i \qquad \text{et} \qquad \mathbf{sup}\{C_i\} = C\{\cup C_i\}$$

- ***Connected Dilations*** **:** Let $\mathcal{C}$ be a connection and $\mathcal{S} \subseteq \mathcal{C}$ a sup-generating class. If an extensive dilation δ preserves connection on $\mathcal{S}$, it preserves it also on $\mathcal{C}$.
  - Ex: in $\mathcal{P}$(E), if the (extensive) dilates of the points are connected, that of any connected component is connected too.

- ***Corollary*** **:** The erosion and the opening adjoint to δ treat the connected components of any a∈ $\mathcal{L}$ independently of each other.

Application: Filtering by Erosion-Recontruction

- Firstly, the erosion $X \ominus B_\lambda$ suppresses the connected components of X that cannot contain a disc of radius λ;
- then the opening $\gamma^{rec}(X ; Y)$ of marker $Y = X \ominus B_\lambda$ «re-builds» all the others.

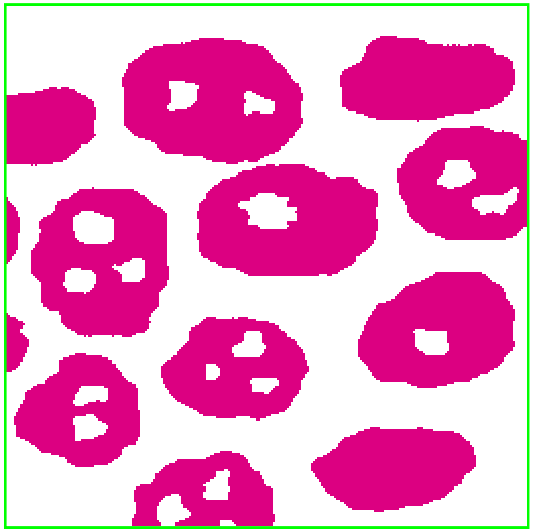
a) Initial image

b) Eroded of a) by a disc

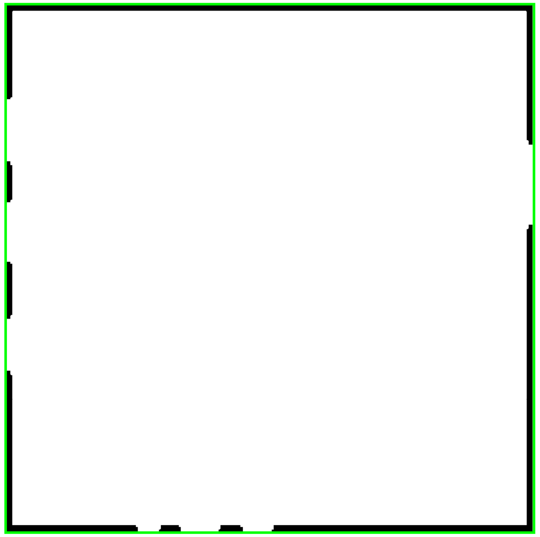
c) Reconstruction of b) inside a)

Application: Holes Filling

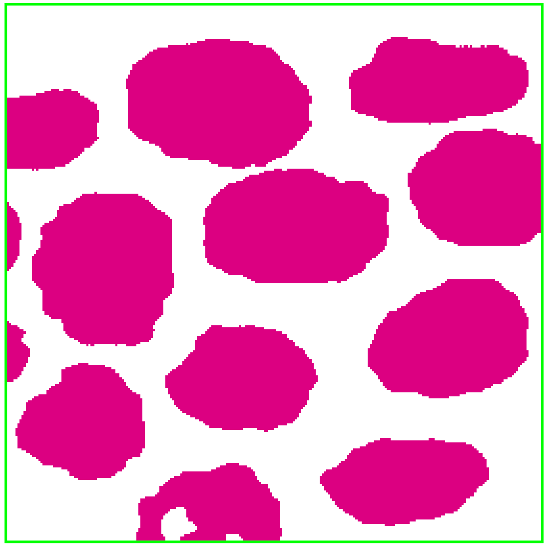
Comment : *efficient algorithm, except for the particles that hit the edges of the field.*



initial image
X



A = part of the edge that hits X^C



reconstruction of A inside X^C

Connected Operators

Definition :

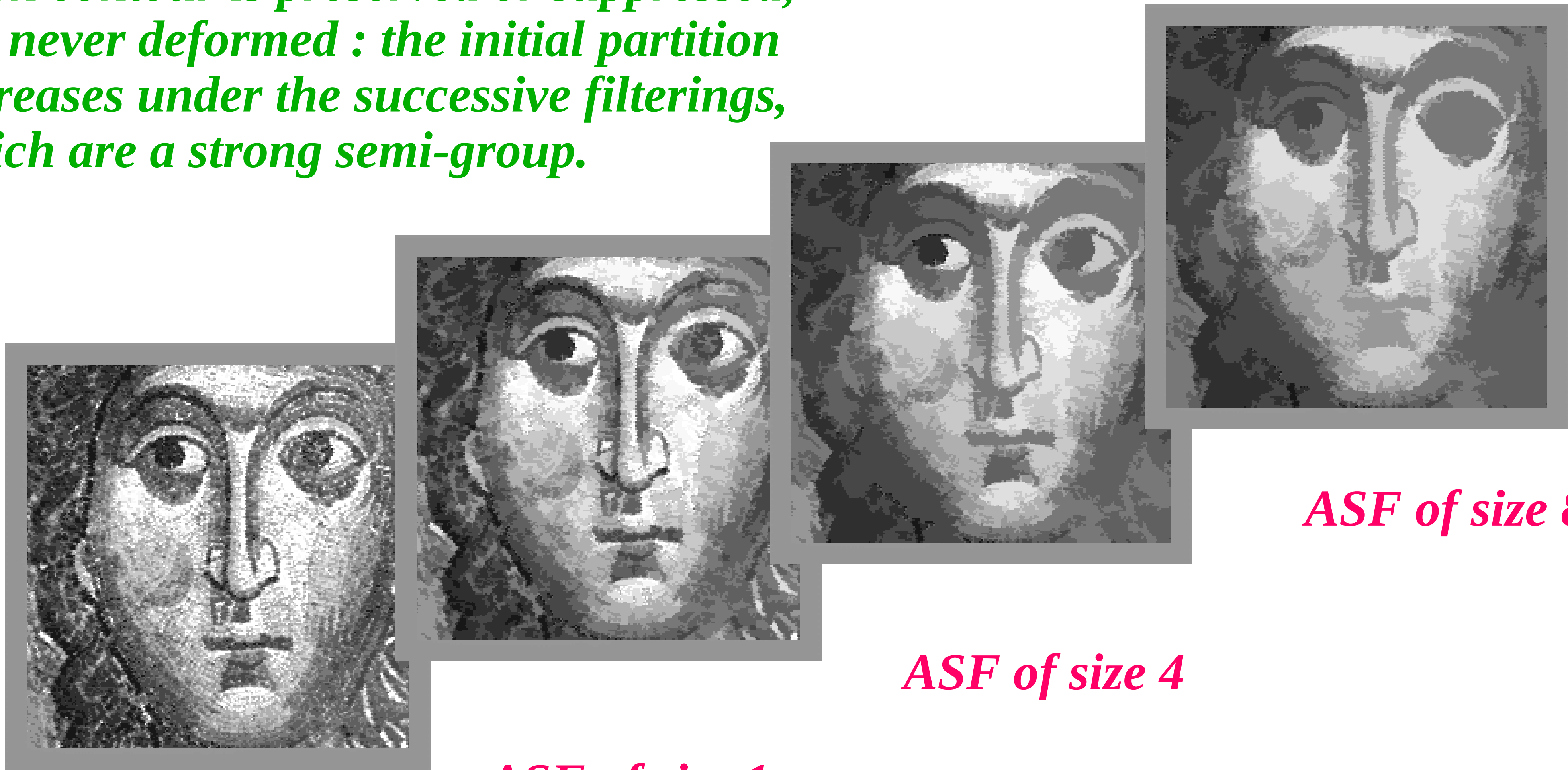
- An operator $\psi : \mathcal{L} \rightarrow \mathcal{L}$ is said to be **connected** when its restriction to $\mathcal{D}$ is extensive. The most useful of such operations are those which, in addition, are **increasing** for T^E .

Properties when $\varphi = 0$:

- All **binary** reconstruction increasing operations induce on $\mathcal{L}$, via the cross sections, increasing connected operators on $\mathcal{L}$.
- The properties to be strong filters, to constitute semi-groups, etc.. are also transmitted to the connected operators induced on $\mathcal{L}$.
- Note that a mapping may be anti-extensive on L^E, and extensive on $\mathcal{D}$ (*e.g.*reconstruction openings). However, the reconstruction closings on L^E are also closings on $\mathcal{L}$.

An Example of a Pyramid of Connected A.S.F.'s

Flat zones connectivity, (i.e. $\varphi = 0$).
Each contour is preserved or suppressed,
but never deformed : the initial partition
increases under the successive filterings,
which are a strong semi-group.



ASF of size 8

ASF of size 4

ASF of size 1

Initial Image

Second Generation Connection

We will now use a dilation δ to create a new connections C' from a first one C (of associated opening γ_x).

- ***Inverse Images* :** Let $\delta : \mathcal{L} \rightarrow \mathcal{L}$ be an extensive dilation that preserves connection C (*i.e.* $\delta(C) \subseteq C$). Then, the inverse image $C' = \delta^{-1}(C)$ of C is still a **connection** on $\mathcal{L}$, which is **richer** than C, *i.e.* $C' \supseteq C$.

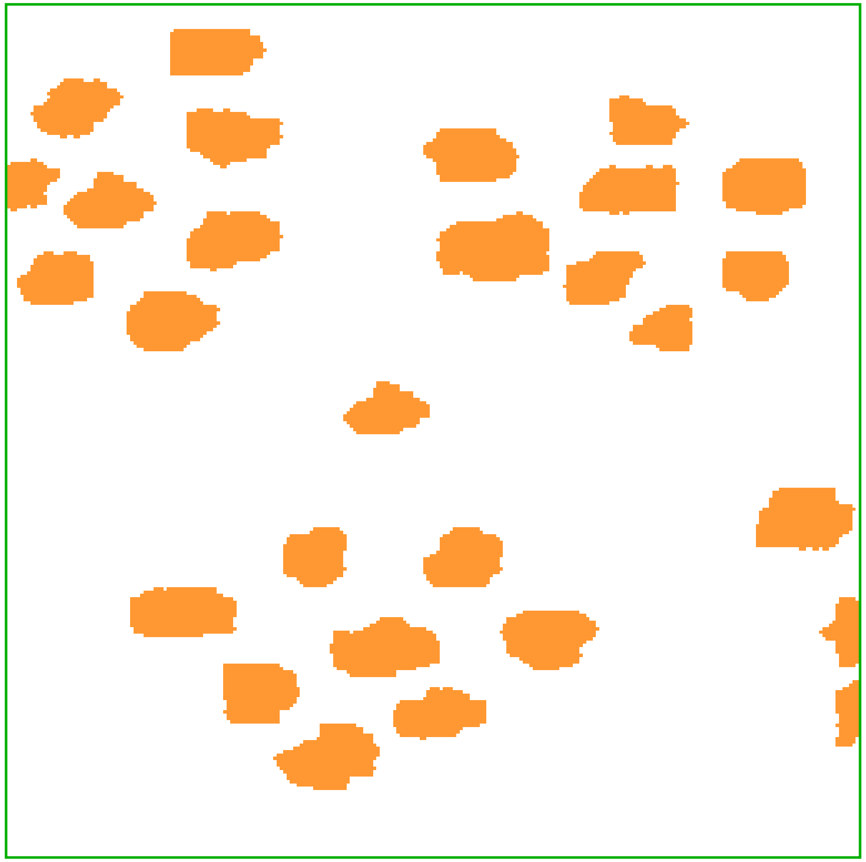
- ***Connected Opening* :** If, in addition, $\mathcal{L}$ is infinitely $\vee$-distributive, then the C-components of δ(a) are exactly the **images of the *C'*-components** of a. The opening ν_x of C' is given by

$$\nu_x(a) = \gamma_x \delta(a) \wedge a \qquad \text{when } x \leq a \quad ;$$

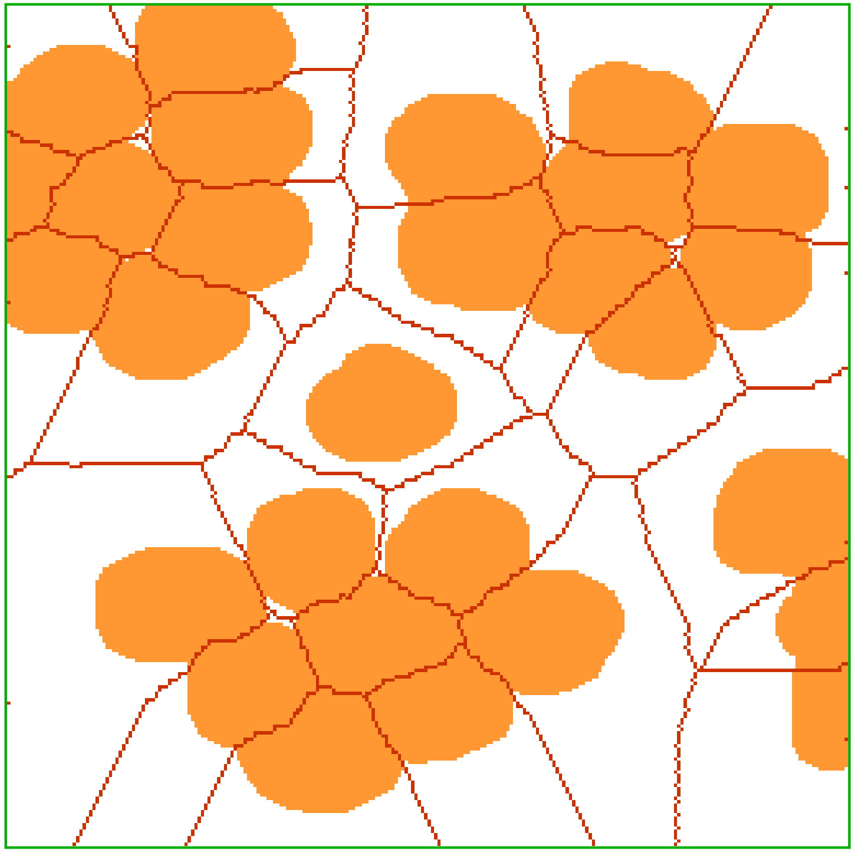
$$\nu_x(a) = 0 \qquad \text{when not .}$$

Application : Search for Isolated Objects

Comment: *One want to find the particles from more than 20 pixels apart. They are the only connected componets to be identical in both C and C' connections , i.e. the particles whose dilates of size 10 miss the SKIZ of the initial image.*



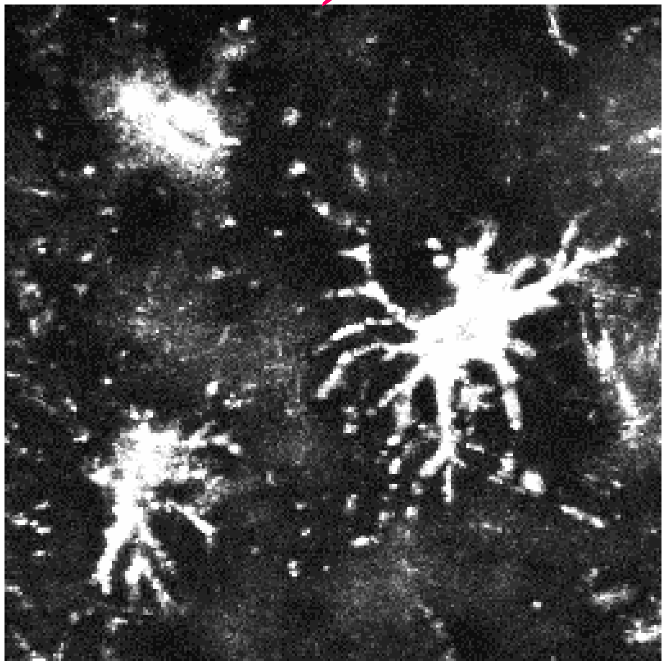
a): Initial Image



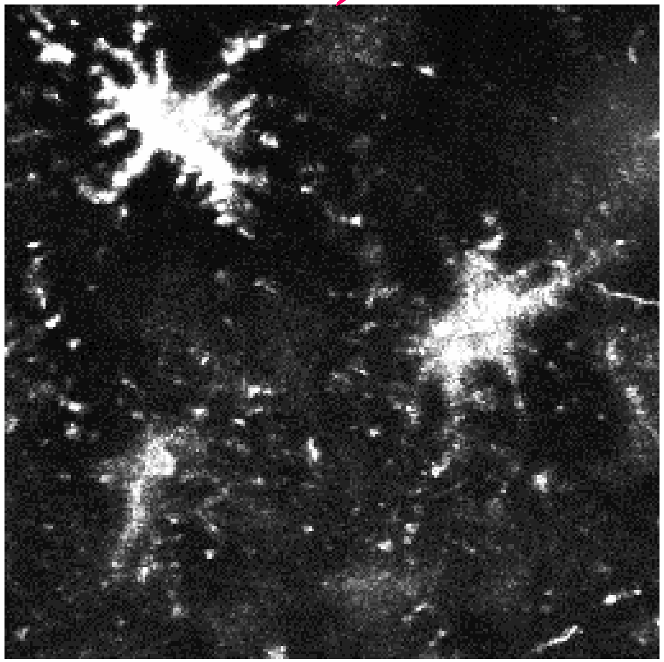
b) : SKIZ and dilate of a) by a disc of radius 10.

Application : 3-D Objects Extraction (I)

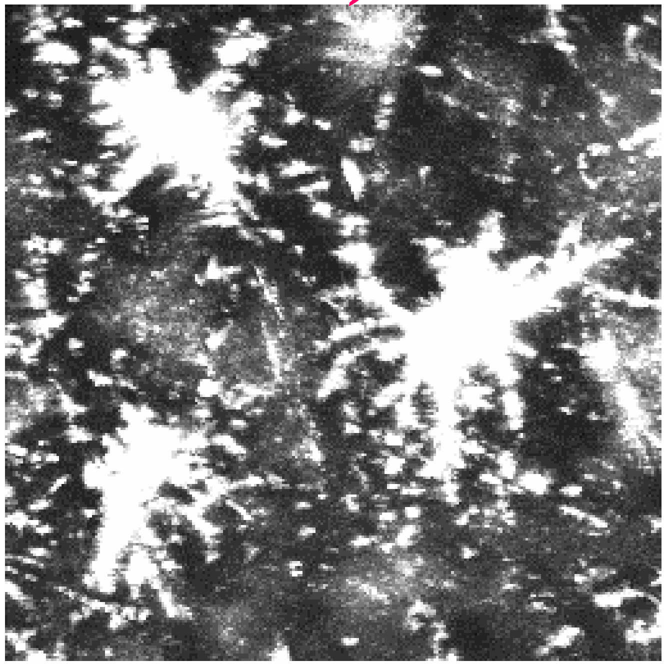
a)



b)



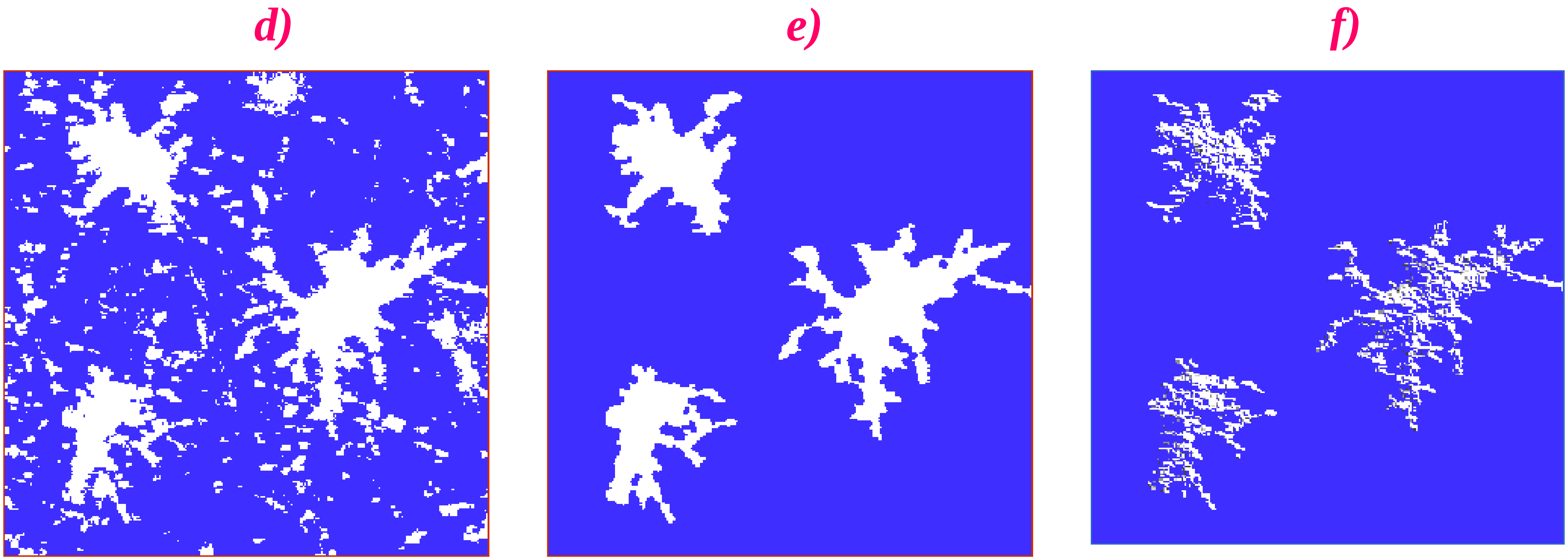
c)



Goal : Extract the osteocytes present in a sequence of 60 sections from confocal microscopy

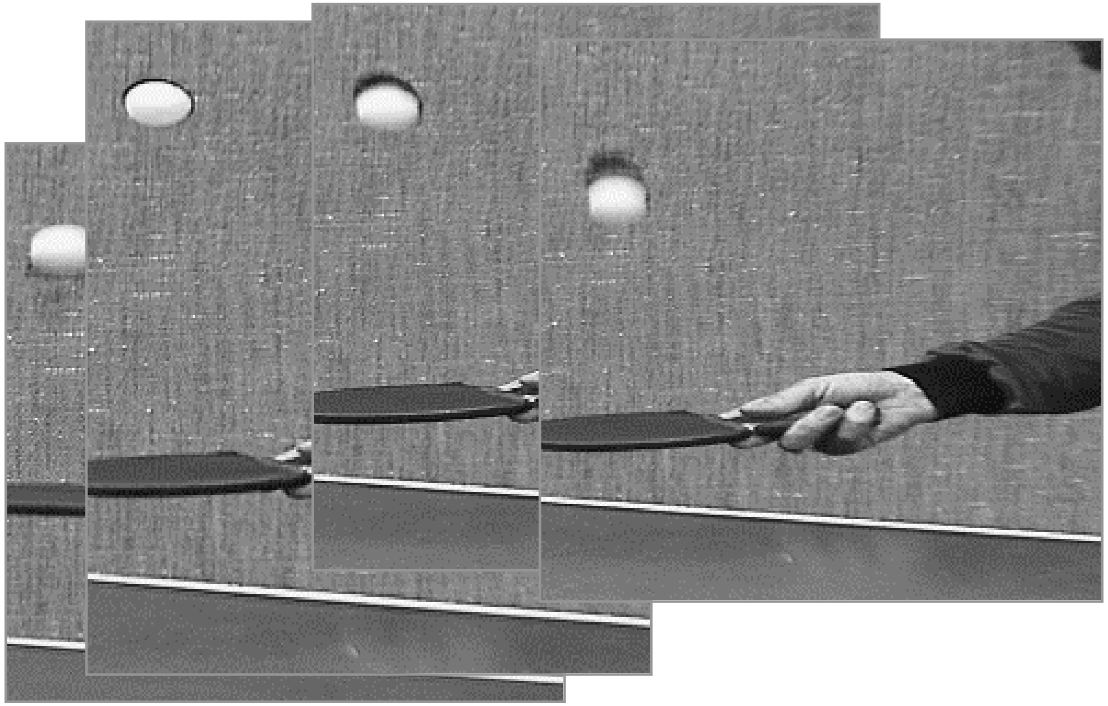
- *Photographs a) and b) : sections 15 and 35 respectively ;*
- *Image c) : supremum M of the 60 sections.*

Application : 3-D Objects Extraction (II)

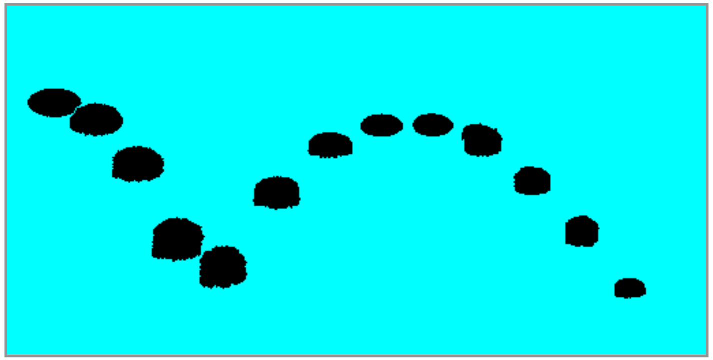


- *d) : Threshold c) at level 60 ; e) : Connected opening of d)*
- *f): Infinite geodesic dilation of the thresholded sequence (level 200) inside mask e) - perpective display -*

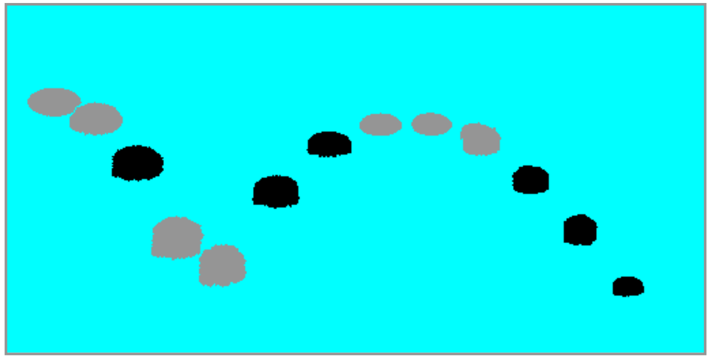
Another example: Connections in a Time Sequence



Part of the sequence



Representation of the ping-pong ball in Space$\otimes$ Time



Connections obtained by cube dilation of size 3 in Space$\otimes$ Time (in grey, the clusters)

Lattice of Equicontinuous Functions

- ***Definition :*** E is a (discrete or continuous) metric space. Choose a positive function $\varphi : R_+ \rightarrow R_+$ be which is continuous at the origin. A function $g : E \rightarrow T$ is said to be equicontinuous of module φ when

$$\left| g(x) - g(y) \right| \leq \varphi [\, d(x,y)] \qquad (\, d = \textit{distance in } E)$$

The class of these functions is denoted by G_φ

- **G_φ *Lattices:*** For each φ , G_φ turns out to be a totally distributive sub-lattice of T^E. All its elements are finite, except possibly its two extrema.

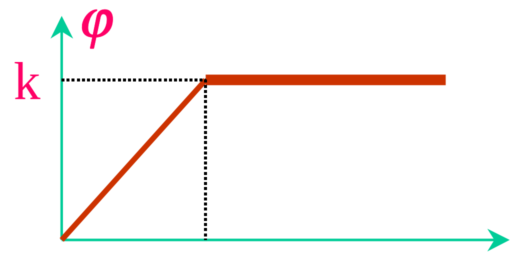
- ***Convergence :*** In each G_φ the convergences in Matheron sense and Hausdorff sense (when E is compact) coincide with the pointwise convergence, which, in addition, is uniform.

(*i.e.* « $g_n \rightarrow g$ as $n \rightarrow \infty$ » just means « $g_n(x) \rightarrow g(x)$, $x \in E$ » **).**

Examples of Modules

- ***Constant*** Functions **:**

$$\varphi = 0 \;;$$

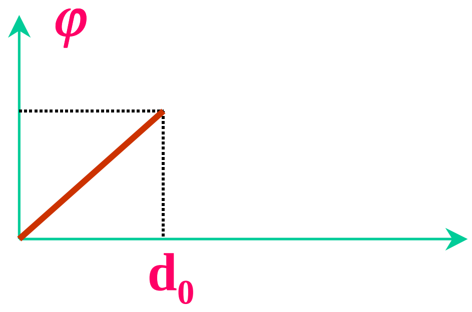


- Functions with a ***bounded variation*** k **:**

$$\forall\, \mathbf{d} : \qquad \varphi\,(\mathbf{d}) \leq \mathbf{k}$$

- ***Lipschitz*** Functions :

$$\varphi\,(\mathbf{d}) = \mathbf{k}\,.\mathbf{d}$$



- ***Geodesic Lipschitz*** Functions :

$$\mathbf{d} \leq \mathbf{d_0} \quad \Rightarrow \quad \varphi\,(\mathbf{d}) = \mathbf{k}\,.\mathbf{d}$$

Properties of Equicontinuous Classes

- ***Every*** G_φ ***:***

  - contains all constant functions ;

  - is self-dual ($\mathbf{g} \in G_\varphi \Leftrightarrow -\mathbf{g} \in G_\varphi$) ;

  - is closed under addition by any constant.

- ***Dilations:*** G_φ is closed under the usual dilations and erosions (Minkowski , geodesic), and all these operations are continuous ;

- ***Filters:*** hence G_φ is also closed under all derived filters (openings, closings, ASF, etc..), which turn out to be continuous operations ;

- ***Continuity*** is enlarged into module preservation, a stronger notion, which is valid for both continuous and digital cases .

Weighted Sets

- ***Definition:*** Given a module φ , with each pair (A, g) of the product space $\mathcal{P}(E) \times G_\varphi$ associate the restriction g_A of $g \in G_\varphi$ to A, *i.e.* the function

$$g_A(u) = g(u) \qquad \text{if } u \in A$$

$$g_A(u) = 0 \qquad \text{if } u \notin A .$$

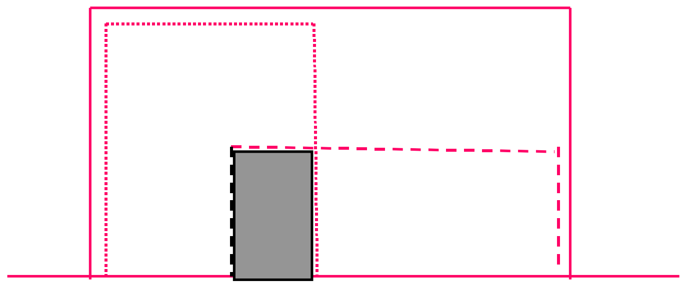
By so doing, we replace the indicator function of set A by a (variable) weight g which belongs to G_φ . Hence g_A turns out to be a **weighted set**. As the pair (A,g) spans $\mathcal{P}(E) \times G_\varphi$, the g_A's generate the set $\mathcal{P}_\varphi(E)$.

- ***Lattice of the Weighted Sets :*** Set $\mathcal{P}_\varphi(E)$ is a complete lattice for the usual ordering ≤ ; in this lattice,

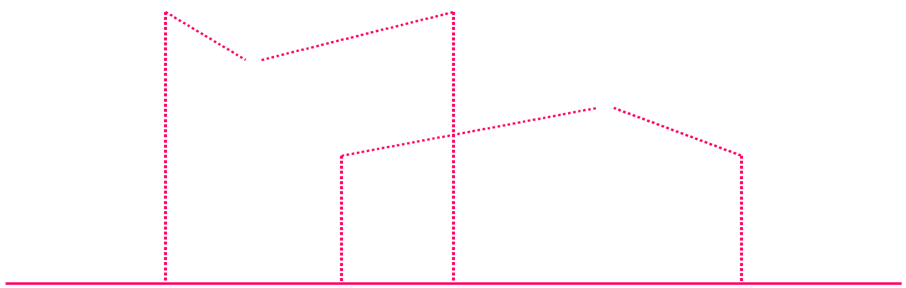
  - the supremum $\sqcup(g_A)_i$ of a family $\{(g_A)_i \, , \, i \in I\}$ is the smaller element of G_φ which is larger than $\vee(g_A)_i$ on $\cup A_i$.
  - the infimum, simpler, is given by $\sqcap(g_A)_i = (\wedge g_i)_{\cap Ai}$.

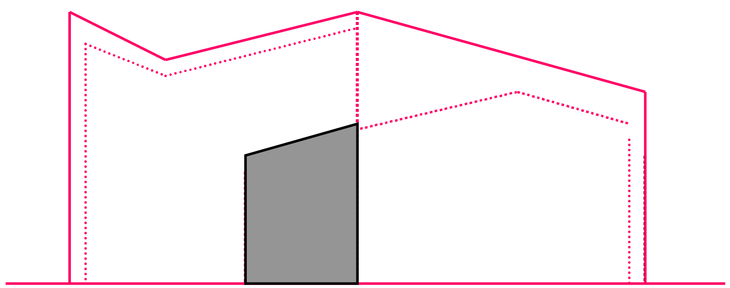
Examples of Weighted Sets

- ***First example :*** for φ = 0; the two sets are flat, but with different heights :
  their φ-sup is their flat envelope (continuous lines),
  their φ -inf is just the intersection of the two functions (dark zone)



- ***Second example :*** φ is a straight line :





Weighted Partitions

The weighted approach extends directly to partitions.

- ***Definition :*** A **weighted partition** $x \rightarrow (g_D)_x$ is a mapping $E \rightarrow \mathcal{P}_\varphi(E)$ such that

  (i) $\forall\ x \in E$, $x \in D(x)$

  (ii) $\forall\ (x, y) \in E$, ***either*** $(g_D)_x = (g_D)_y$ ***or*** $(g_D)_x \wedge (g_D)_y = 0$

- ***Sub-mappings :*** Clearly, the sub-mappings
  - $x \rightarrow D(x)$ is a usual partition, *i.e.* $D \in \mathcal{D}$
  - $x \rightarrow f(x) = \vee\{(g_D)_y\ , y \in E\}(x)$ is a usual function of T^E,

  so that a weighted partition may be denoted by $\Delta = (D, f)$.

- ***Function Representation :*** Every function $f : E \rightarrow T$ can be represented, in different ways, as a $\vee\{(g_D)_x\ , x \in E\}$. It suffices to partition f into zones on which f admits module φ (for example, on which f is constant).

Lattice $\mathcal{L}$ of the Weighted Partitions

- ***Theorem*** *(J.Serra)* **:** Denote by $\mathcal{L}$ the set of the weighted partitions. Then, the relation

$$\Delta \preceq \Delta' \Leftrightarrow \{ D \leq D' \text{ in } \mathcal{D}, \text{ and } f \leq f' \text{ in } T^E \}$$

defines an **ordering** on $\mathcal{L}$ to which is associated a **complete lattice**.

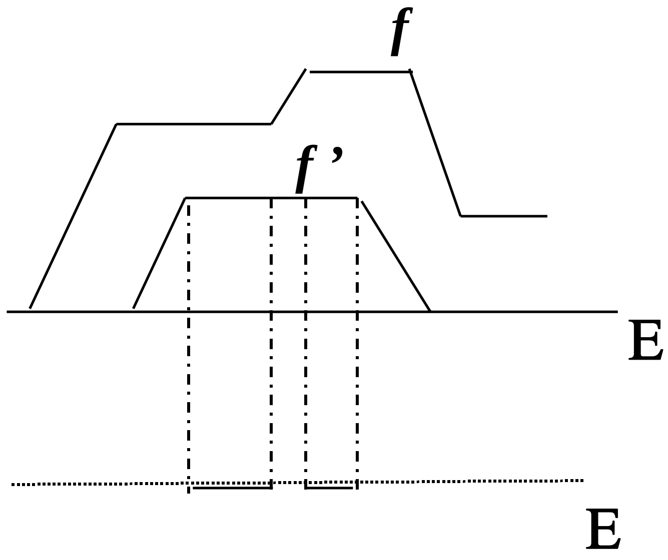
- ***Sup and Inf*** **:** In $\mathcal{L}$, the supremum $\curlyvee \Delta_i$ of family $\{\Delta_i\}$ admits $D = \vee D_i$ for partition. Each class D(x) of D, has for weight g the smaller φ-continuous function larger than $\vee (g_D)_i$ on D(x). The $\mathcal{L}$ infimum $\curlywedge \Delta_i$ is given, at each point x, by $\wedge g_{Di(x)}$ restricted to $\cap D_i(x)$).

- ***Extrema*** **:** Δ_{max} is the single class partition, weighted by m, and Δ_{min} is the partition into all points of E, each of them being weighted by 0.

An Example of Flat Weighted Partition

- ***Partitions* :** for $\varphi = 0$, given function f :
  - when $f(x) \neq 0$, every subset of the flat zone of f that contains point x can serve as a D(x), with weight f(x);
  - when $f(x) = 0$, class D(x) is reduced to {x}.

(*Note that f admits a largest flat partition Δ*)



Functions f and f'

Projection of their infimum partition

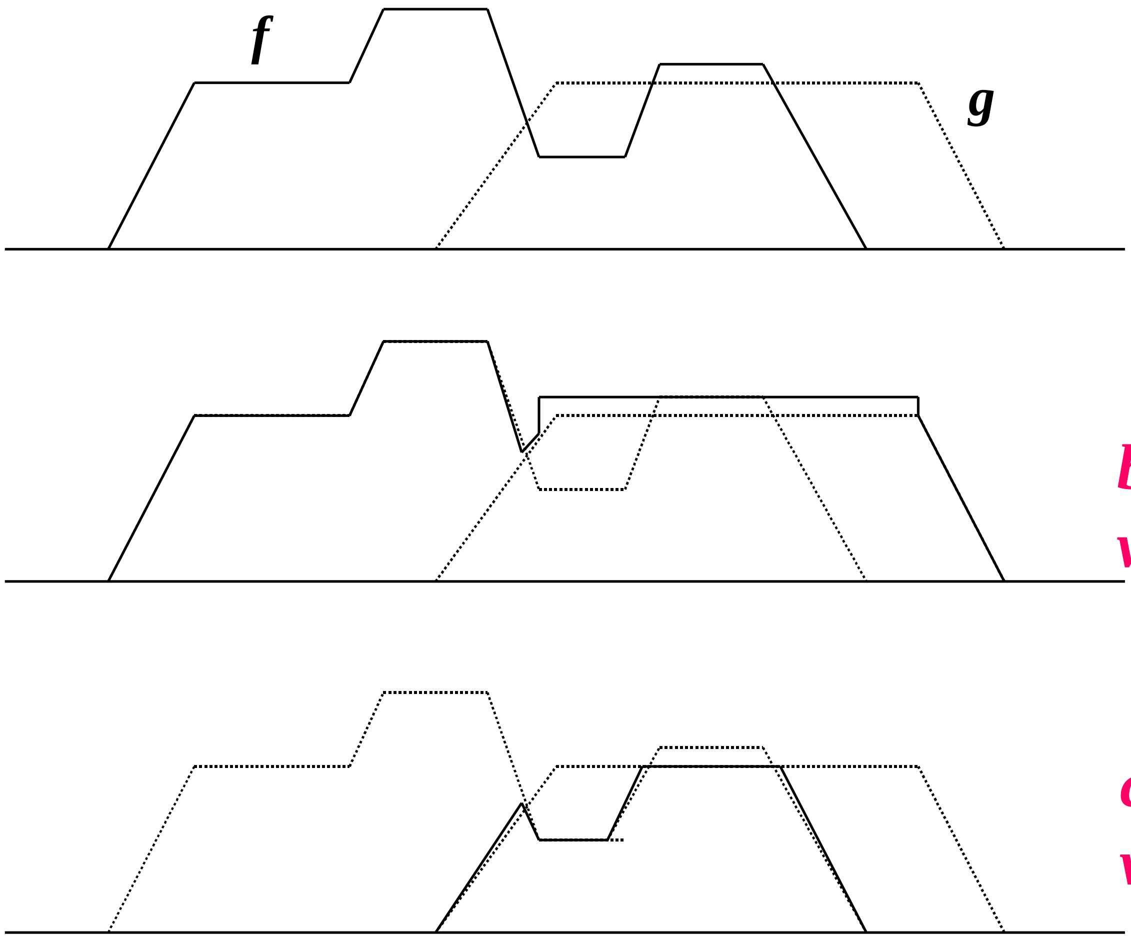
Δ ⋏ Δ'

- ***Ordering* :** the two largest flat partition Δ and Δ' generated from the flat zones of f and f ' are **not** comparable in $\mathcal{L}$, although f > f' (but in T^E !)

Their inf Δ ⋏ Δ' is given by two flat sub-zones of f ' and 0 elsewhere.

An Example of ⋎ and ⋏ in $\mathcal{L}$

Comment : ***Here the weights are taken constant in each flat zone of f and f', i.e. φ = 0 . This generates two weighted partitions Δ and Δ '.***



a) Non comparable weighted partitions Δ and Δ'

b) function associated with supremum Δ ⋎ Δ'

c) function associated with infimum Δ ⋏ Δ'

Cylinders in $\mathcal{L}$

- ***Cylinders* :** With any weighted set $g_A \in \mathcal{P}_\varphi(E)$, it is always possible to associate a weighted partition Δ_A as follows

$$x \to g_A \qquad \text{if } x \in A$$
$$x \to \{x\} \qquad \text{if } x \notin A \,.$$

Δ_A is composed of class g_A plus a **jumble of points**, all being weighted by 0. Such a partition is called a **cylinder, in $\mathcal{L}$, of base A**.

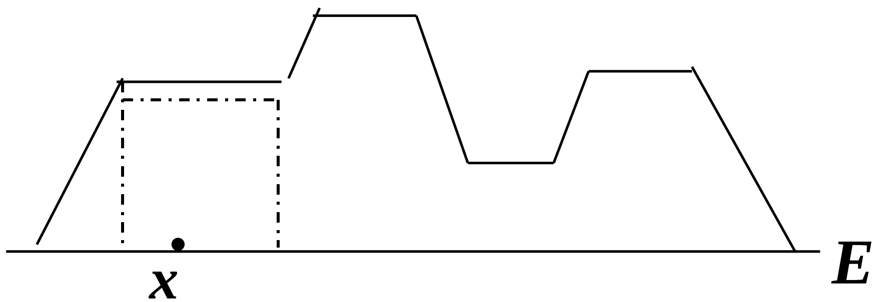
- ***Sup-generors* :** Every weighted partition Δ turns out to be the $\curlyvee$ of all cylinders Δ_{Dx} associated with each class $(g_D)_x$ of Δ . Hence the class of the cylinders is **sup-generating**.
- ***closure under* $\curlyvee$:** the supremum $\Delta_A = \curlyvee \Delta_{Ai}$ of family $\{\Delta_{Ai}\}$ of cylinders has for partition classes $\{\cup A_i$, plus all $\{x\} \subseteq [\cup A_i]^c \}$. Hence Δ_A is itself a cylinder.

Connections on Weighted Partitions

Suppose now that E is equipped with a connection $\mathcal{C}_0$. If the bases C_i's of cylinders Δ_{Ci} are connected and if $\cap C_i \neq \emptyset$, then $\curlyvee \Delta_{Ci}$ is a cylinder with a connected basis. Now, such cylinders are still sup-generating. Hence,

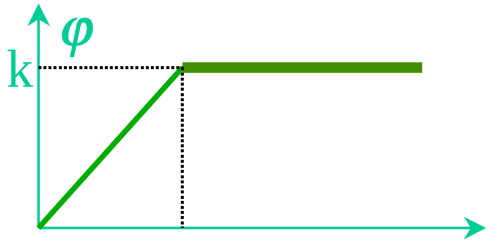
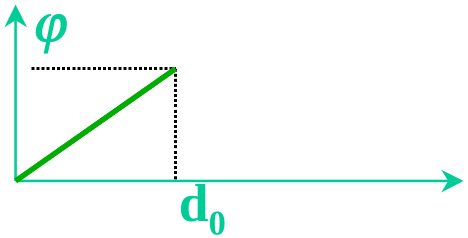
- ***Connection on $\mathcal{L}$:*** the cylinders Δ_C with a connected basis C in E, generate a **connection** $\mathcal{C}$ over $\mathcal{L}$.

- ***Associated opening :*** Given a weighted partition **$\Delta = (D, f)$**, the point opening **$\gamma_x(\Delta)$** of connection $\mathcal{C}$ extracts the **cylinder** whose base is the class D(x) of D covering point x, and weight the values of f inside D(x) .



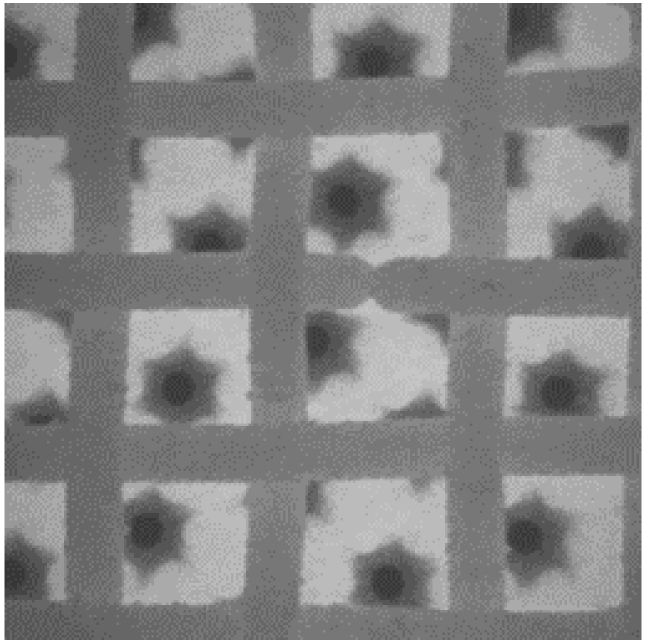
In $\mathcal{L}$, the connected opening at point x is a cylinder.

Typology for the Connections on Functions

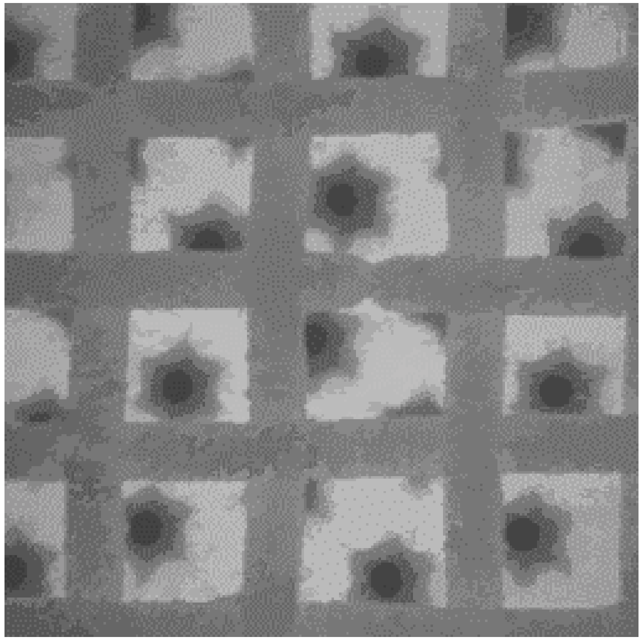
Module φ	Model for G_φ	Meaning for Function f
1) $\varphi = 0$	*Constant functions*	*Flat zones*
2) $\varphi(d) \leq k$ 	*Functions whose range of variation = k*	*Zones in which the variation of f is ≤ k , and jumps from one zone to another*
3) $d \leq d_0 \Rightarrow \varphi(d) = k \, . d^{\alpha}$ 	*Lipschitz geodesic functions*	*Zones in which the variation of f is smooth , but not from one zone to another*

An Example of Jump Connection in $\mathcal{L}$

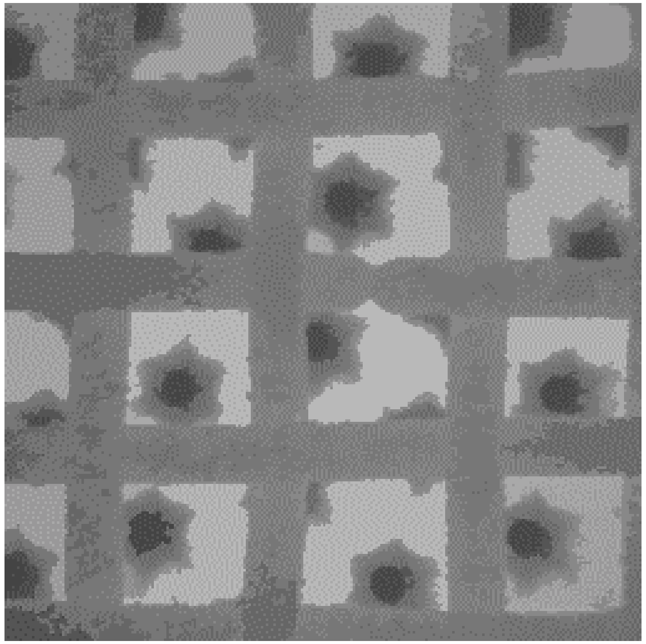
a)



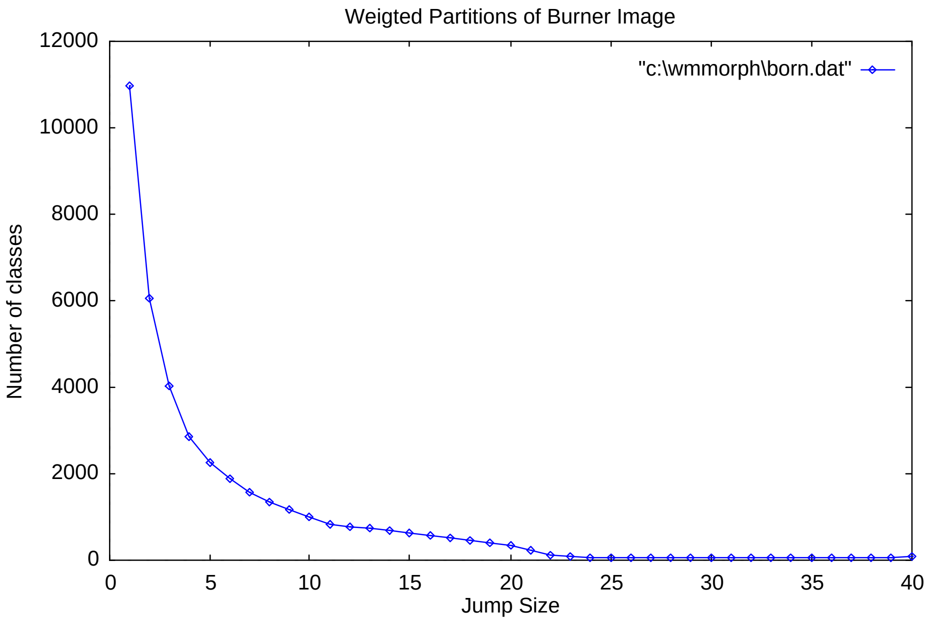
b)



c)



d)



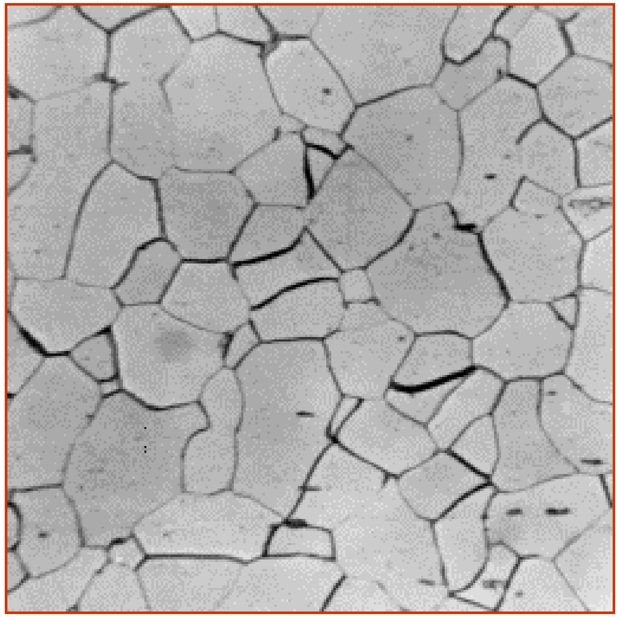
a) Initial image: gaz burner

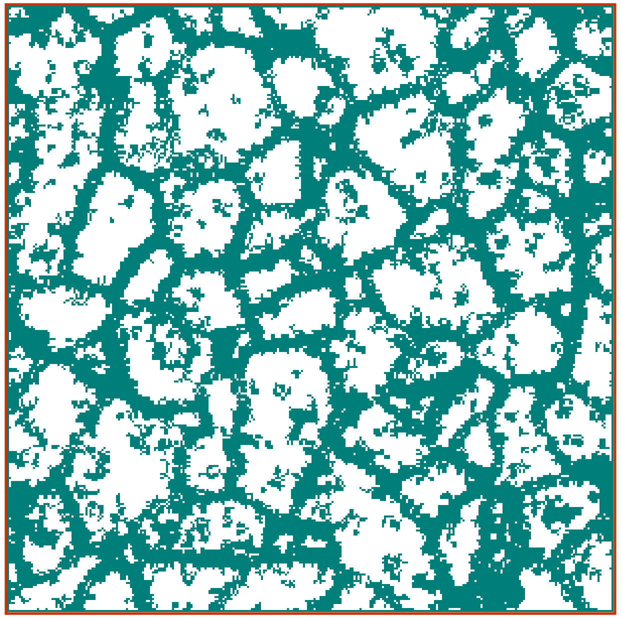
b) Jump of size 12 : 783 tiles

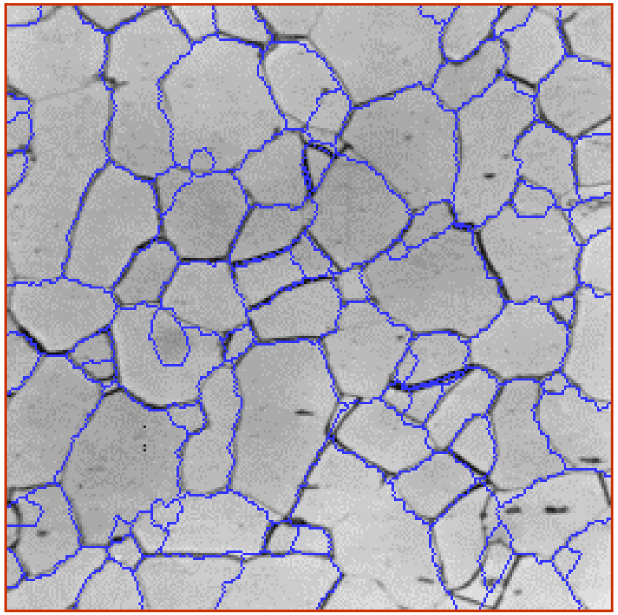
c) Jump of size 24 : 63 tiles

d) Number of tiles versus jump values

Other Example of Jump Connection in $\mathcal{L}$







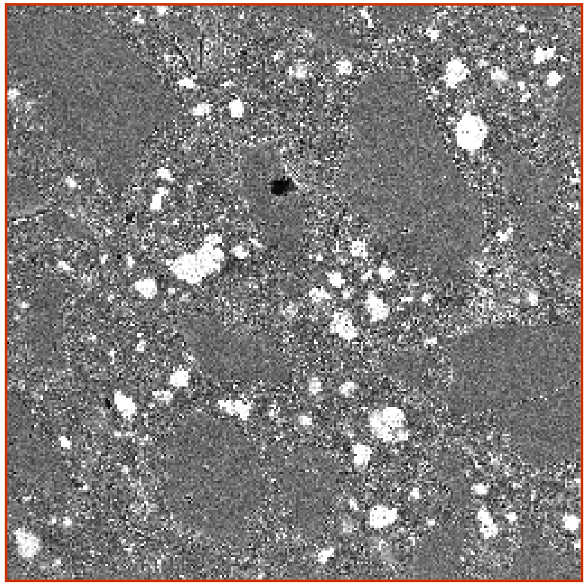
a) Initial image: polished section of alumine grains

b) Jump connection of size 12 :
- in dark, the point connected components
- in white, each particle is the base of a cylinder

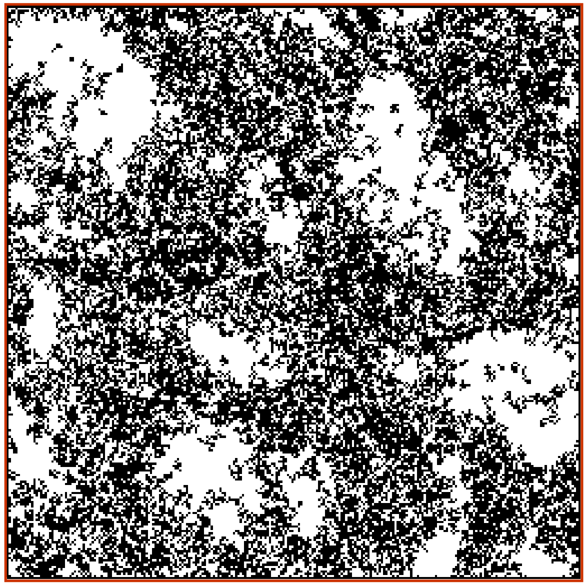
c) Skiz of the set of the dark points of image b)

An Example of Smooth Connection in $\mathcal{L}$ (I)

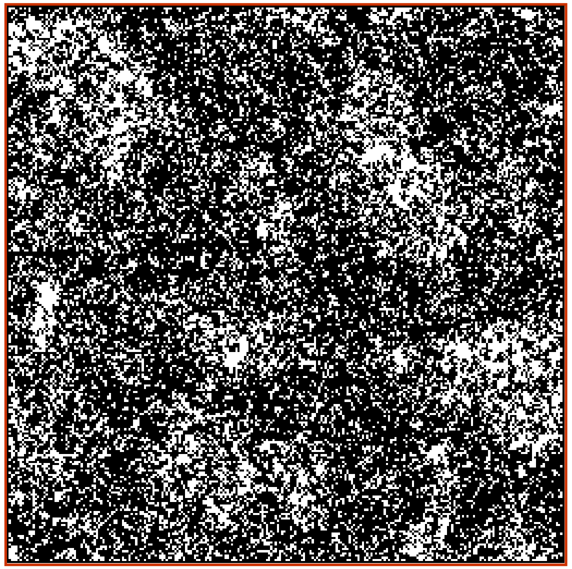
Comment : ***the two phases of the micrograph cannot be distinguished by means of jump connections.***



a) Initial image: rock electron micrograph



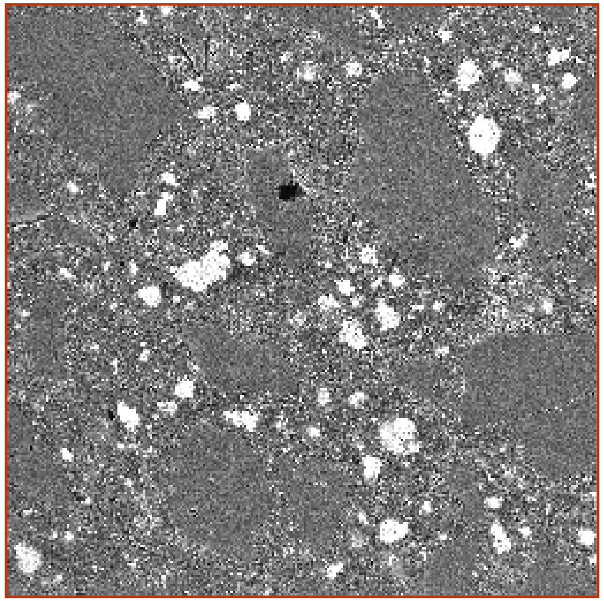
b) Jump connection of size 15 .



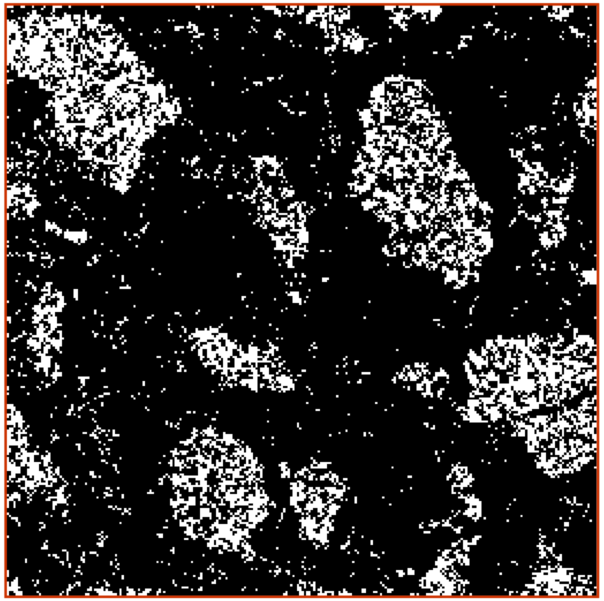
c) Jump connection of size 25 .

An Example of Smooth Connection in $\mathcal{L}$ (II)

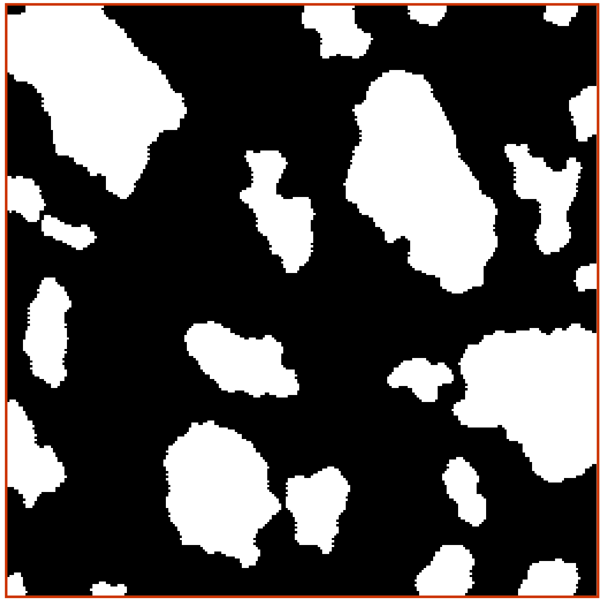
Comment : ***The smooth connection differentiates correctly the two phases according to their roughnesses.***



a) Initial image: rock electron micrograph .



d) smooth connection of slope 6 (in dark, union of all point connected components).



e) Filtering of Image d) which iyelds a correct segmentation of a) .

Jump Connection on a Color Image

Methodology:A jump connection of range 14 for the luminance yields 94 zones. The three color channels are averaged in each of the 94 regions.

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