EQUICONTINUOUS RANDOM FUNCTIONS

Jean SERRA

Centre de Morphologie Mathématique
Ecole des Mines de Paris

Upper Semi Continuous Functions (*G. M atheron, 1969)*

- The class F of the u.s.c. functions $f : E \rightarrow R$ is nothing but that of those closed sets C in $\mathcal{F}(E \otimes R)$ such that:
  - $C \supset E_{-\infty}$,
  - $\forall x \in E, \quad \forall t \in R : \qquad (x,t) \in C \quad \Rightarrow \quad \{x\} \otimes [\infty, t\,] \subset C.$
- This class is a compact family in $\mathcal{F}(E \otimes R)$. Hence, the open sets in F are the parts of F whose elements f satisfy the two conditions :

  $X^{+}_{f}\,(G) = \sup\,\{\, f(x)\,,\, x \in G \,\} > b$ and $\inf\,\{\, X^{+}_{f}\,(G)\,,\, G \supset K \,\} < a$,

  as G spans the open sets of E and K its compact sets.
- A sequence f_n converges towards f in F iff it satisfies the two conditions:

  i) for all $x \in E$, there exists a sequence $x_n \rightarrow x$ in E such that the sequence $f_n(x_n) \rightarrow f(x)$ in R ;

  ii) if a sequence x_{nk} converges towards x in E , then the sequence $f_{nk}(x_{nk})$ satisfies $\operatorname{Lim} f_{nk}(x_{nk}) \leq f(x)$.

U.S.C. Random Functions (*G. Matheron, 1969*)

- **Equip F with the σ - algebra generated by its topology, *i.e.* by the events**

$$X^{+}_{f}(G) = \sup \{ f(x) , x \in G \} > b .$$

- **A Random u.s.c. function f is then defined by providing the Measurable Space (F, σ) with a probability P. Such probabilities do exist because F is compact.**

- **Just as a random variable is characterized by its distribution function, a Random Function f ∈ (F, σ , P) is determined by the joint distributions**

$$\Pr \{ \sup\{ f(x) , x \in B_1 \} < \lambda_1 \; ; .... \sup\{ f(x) , x \in B_n \} < \lambda_n \}$$

for n finite, B_i compact sets , and λ_n real numbers
(Choquet- Matheron theorem, interpreted here for Random Functions).

Equicontinuous Functions (Reminder)

Modulus of Continuity φ

With any function $f \in R^E$, (*i.e.* $f : E \to R$), associate function $\varphi : R_+ \to R_+$ as follows :

$$\varphi(h) = \sup \{ |f(x) - f(y)| \quad x, y \in E, \quad d(x,y) \leq h \} \; .$$

Moreover, " f is uniformly continuous " ⇔ " $\lim_{\varphi \to 0} = 0$ " .

If so, then φ is called a *Modulus of Continuity* .

Equicontinuous Classes

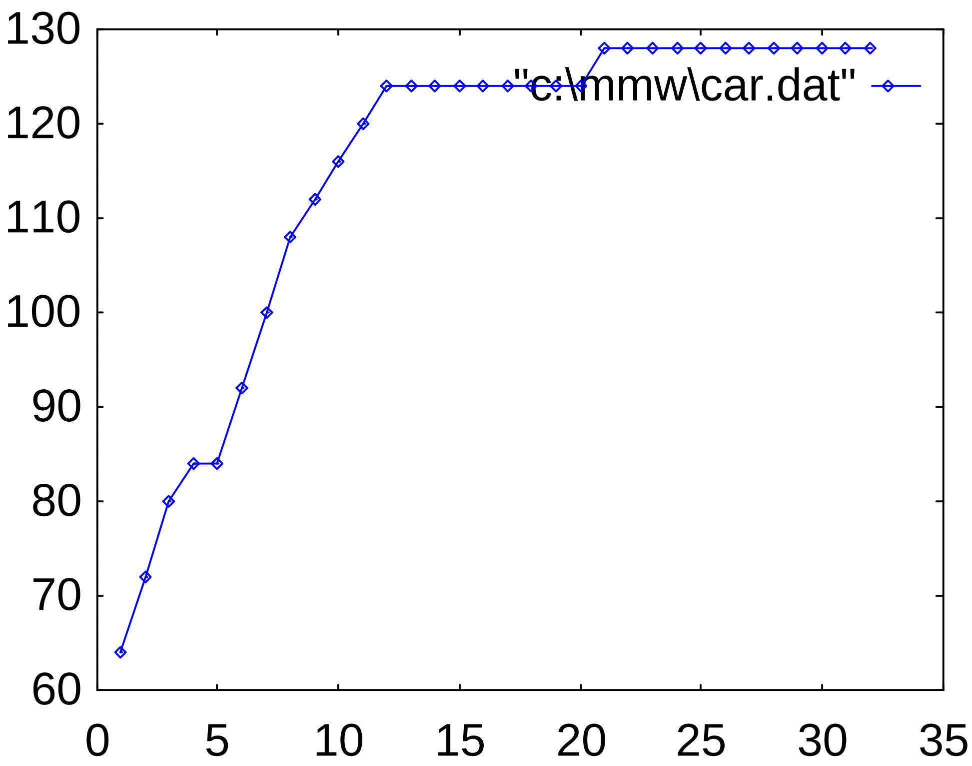
Given φ, a function f is said to be *φ - continuous* when

$$|f(x) - f(y)| \leq \varphi[d(x,y)] \qquad \text{for all} \quad x, y \in E \qquad (1)$$

The functions that satisfy Eq. (1) generate the so called *φ - continuous class*.

An Example of Modulus φ

Video Image



Corresponding Modulus φ
(square metrics)

Lattices L_φ of Equicontinuous Functions

- *Theorem* : *For every modulus φ, class L_φ is a complete sub-lattice of R^E*

  More generally, replace R by a Lattice T equipped with a topology that

  - makes T compact;

  - closes the ordering on T (i.e. $x_i \to x$, $y_i \to y$, $x_i \leq y_i \Rightarrow x \leq y$)

  Lattice T is said to be Compact and Close Ordered (*in brief* : C.C.O.)

- *Theorem* : *Let E be a metric space, T be a totally ordered CCO-lattice, and d_T be a distance on T such that*

  $$a \leq x \leq y \leq b \quad \text{in } T \quad \Rightarrow \quad d_T(x,y) \leq d_T(a,b)$$

  then the class L_φ of the φ - continuous functions $f: E \to T$ is a complete sub lattice of T^E.

- *Corollary* : *The theorem extends to any product $\Pi\{ T_i , i \in I \}$ of T type lattices.*

Dilations on R^E functions Lattices

- **In any lattice, the two basic operations are those which preserve either the supremum (namely the *dilations*) or the infimum (namely the *erosions*).**

The dilations δ that map the functions lattice R^E into itself admit a rather general form $(\delta f)(y) = \vee \{ g_y(z) + f(z) ,\ z \in E \}$

where each point $y \in E$ is associated with a *structuring function* g_y.

- **In order to describe the variation of the g_y over the space, introduce the following Hausdorff type metrics :**

Proposition **:** *Let $\mathcal{G}$ be a family of numerical functions over a metric space E,*
i/ which admit a common finite upper bound
ii/ whose cross sections $X_t(g) = \{ y : g(y) \geq t \}$ $\quad g \in \mathcal{G}$
are compact for all $t \in R \setminus \{-\infty\}$. If g_ρ stands for the dilate of g by a circular cylinder of radius ρ and height kρ, then the mapping $h: \mathcal{G} \otimes \mathcal{G} \rightarrow R_+$

$$h(g , g') = \inf \{ \rho : g \leq g'_\rho ,\ g' \leq g_\rho \}$$ *is a* **distance** *on* $\mathcal{G}$

Dilations on L_φ Lattices

We now wonder about the image $\delta(L_\varphi)$ of sub lattice L_φ under dilation δ .

- **Theorem** : ***let $\delta : R^E \rightarrow R^E$ be a dilation whose structuring functions admit a modulus of continuity φ' i.e.***

$$h(g_x, g_y) \leq \varphi' [\, d(x, y)\,] \qquad\qquad x, y \in E$$

Then δ maps L_φ into the sub lattice $L_{(\varphi + k)\circ \varphi'}$ of the $(\varphi + k)\circ\varphi'$- continuous functions.

- **Particular cases** :
  - **E is affine, and g_x is the translate of g_0 . Then $h(g_x , g_y) = d(x,y)$ and $(\varphi + k)\circ\varphi' = \varphi$. Dilation δ preserves each sub lattice L_φ ;**
  - **The g_x 's are flat of support K_x *i.e.***

$$g_x(y) = 0 \quad \text{when } y \in K_x , \qquad g_x(y) = 0 \quad \text{when not}$$

Then δ maps L_φ into $L_{\varphi\circ\varphi'}$. In particular, if $\varphi' \leq I$, then δ preserves L_φ . This latter case occurs, for example, when δ is the restriction of a translation invariant operator to a rectangular mask

Topologies on L_φ Lattices (I)

In the usual CCO lattices, when the mapping X→ ∨X from $\mathcal{F}$(T) into T is continuous, then X→ ∧X is u.s.c. only (*e.g.* closed sets in R , or u.s.c. functions R → R). The double continuity is thus an exceptionally strong property, and the following criterion a corner stone :

- ***Criterion (from G.Matheron)* : *An algebraic lattice admits a necessarily unique CCO topology such that ∨ and ∧ are both continuous iff for all s and all t in T, s ≰ t , one can find two elements s' and t' with***

$$s \notin M_{t'} \quad ; \quad t \notin M^{s'} \quad ; \quad M_{t'} \cup M^{s'} = T \ ,$$

where $M_{t'} = \{ z : z \in T ,\ z \leq t' \}$ *and* $M^{s'} = \{ z : z \in T ,\ z \geq s' \}$.

Remarkably, the criterion demands no topological prerequisit, and treats both questions of existence and of unicity.

Topologies on L_φ Lattices (II)

Owing to the above criterion , we may state

- ***theorem* :*Let L_φ be the lattice of the φ-continuous functions from E into R, (or more generally into a fully ordered CCO lattice T). Then, the unique topology that makes L_φ CCO, with continuous $\vee$ and $\wedge$, is the topology of the pointwise convergence.***

  ***Proof*: Let f,g $\in L_\varphi$, f $\neq$ g .There exists at leat one x$\in$ E with (*for ex*.) the strict inequalities g(x) < a < f(x) .Consider the two elements f_o and g_o of L_φ**

  $$f_o(y) = a - \varphi[d(x,y)] \quad \text{and} \quad g_o(y) = a + \varphi[d(x,y)] \quad \forall y \in E$$

  f is not a lower bound of g_0 since f(x) > a, hence f $\notin M^{g}{}_{o}$. Similarly, we have g $\notin M_{fo}$. Moreover, any function s $\in L_\varphi$ is either $\leq g_o$ (if s(x) $\leq$ a), or $\geq f_o$ (if s(x) $\leq$ a) .The criterion applies, and the topology is identified by observing that L_φ belongs to both classes of the u.s.c. and l.s.c. functions ■

- ***Corollary* : *the theorem remains true when T is replaced by any product $\Pi\{T_i, i \in I\}$ of T type lattices.***

Continuity of the Increasing Operators

The consequences of the theorem on increasing mappings are considerable. In the "flat " case, for example, we have :

- ***Theorem* : *Let δ be the dilation by the (variable) φ'- continuous Structuring Element K. Then, for each modulus φ, the mapping δ from L_φ into $L_{\varphi \circ \varphi'}$ is continuous. The continuity extends to all finite sup's, inf's, and composition products of such dilations.***

Similar results may be obtain for linear mappings. For example:

- ***Proposition*: *Let g(dh) be a measure such that $\int_E |g(dh)| \leq 1$. Then the convolution by g maps each L_φ into itself, and is continous.***

Consequently, all half residuals (*i.e.* the differences "identity minus mapping ") of the above increasing operators are continuous.

Random φ -Continuous Functions

Given modulus φ , the lattice L_φ is a compact sub-class of the family F of the u.s.c. functions from E into R.

- Therefore , the events

$$X^{+}_{f}(G) = \sup \{ f(x) , x \in G \} > b$$

that generate the σ-algebra on F admit a similar meaning in L_φ , and the compactness of L_φ ensures that there exist Probabilities on the Measurable Space (L_φ , σ) .

- Moreover, we draw from the above theorems that, as soons as they admit a modulus of continuity, dilations, erosions as well as their finite sup's , inf 's , and composition products do preserve φ -continuous Random Functions, with possible changes of moduli φ .

- Note that the expression of Choquet Functional is left unchanged .

Application to sampling

- **Given a digital image f ,**
  - **what is the minimum number of values of f which suffices for estimating f everywhere with an accuracy φ_o ,**
  - **and where must we locate the sampling points ?**

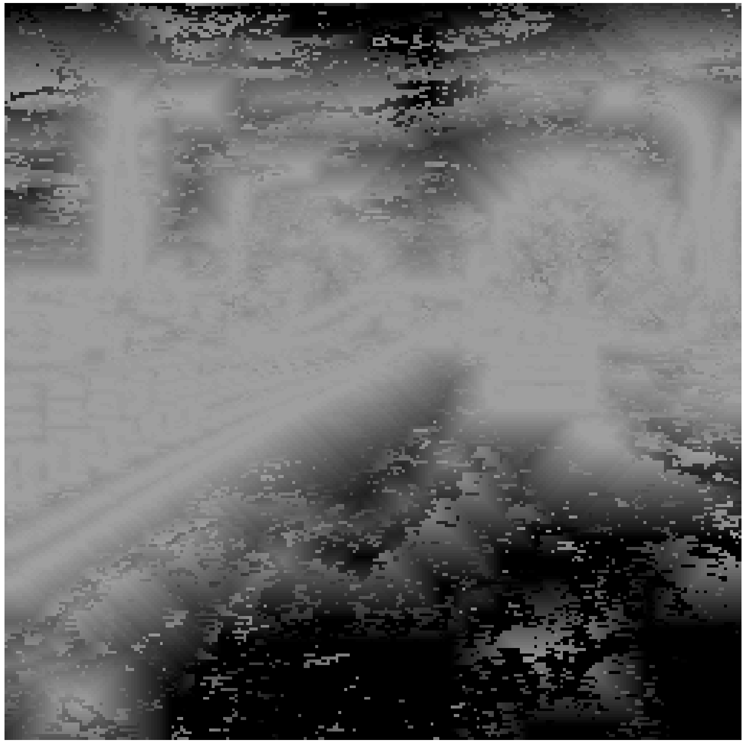
- **Consider f as a realization of a φ -continuous random function, and introduce the following local version of modulus φ :**

$$\varphi_x(h) = E\,[\,\sup\{\,|\,f(x) - f(y)\,|\quad y \in B_x\,(h)\,\}\,] \qquad (3)$$

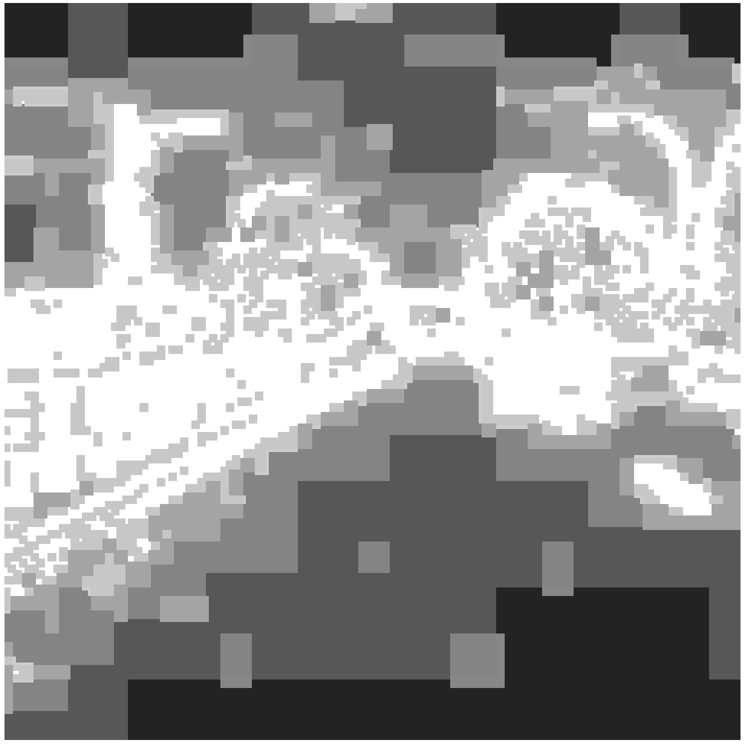
- **Let $h_x(\varphi)$ be the largest inverse of $\varphi_x(h)$, *i.e.* the value of the maximum disc centered at x and such that the variation, in the sense of Eq. (3) is $\leq \varphi$. Since $\varphi = \varphi_o$ is fixed, $h_x(\varphi) = h(x)$ becomes a function of x only.**

- **The goal comes back to construct a grid whose variable spacing fits with function h. We shall start from the four corners of the field.**

Example of Sampling (I)



(1) Inverse modulus
(car example)



Digitization of (1)
with a constant accuracy

Example of Sampling (II)

Initial Image
(65 536 pixels)

Sampled Image
(15 892 pixels)