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Paris (France), September 2-5 2024

CaLISTA Workshop Geometry-Informed Machine Learning

at Mines Paris (France)

Metripletic flow of Thermodynamics-Informed Neural Networks and Machine Learning on Lie Group: Symplectic Foliation Model from Carathéodory's seminal idea to Souriau's Lie Group Thermodynamics

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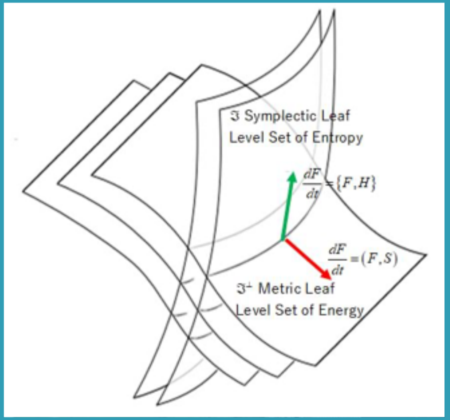
THALES

03/09/2024

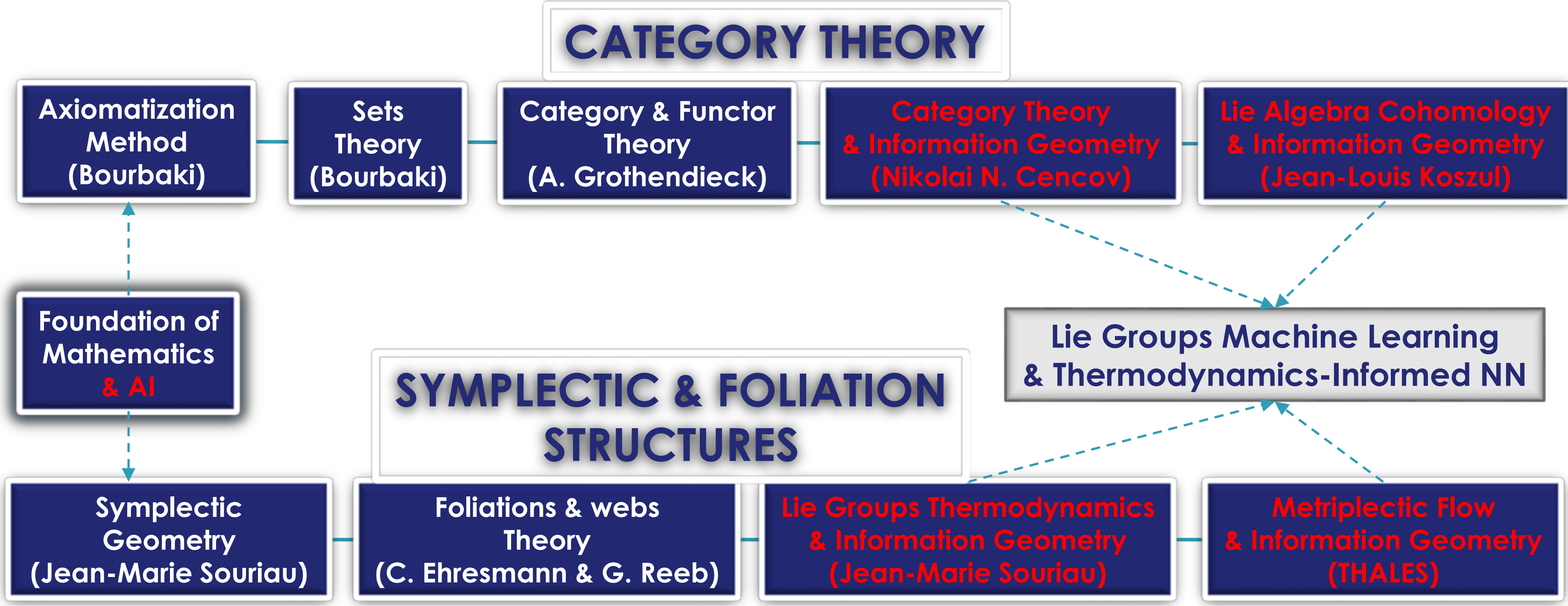
Mines ParisTech PSL, Paris

www.thalesgroup.com

« Every mathematician knows that it is impossible to understand an elementary course in thermodynamics. »
Vladimir Arnold



Mathematical Foundations of AI



AI Mathematical Foundations: Symplectic Foliations

Lie Groups & Lie Algebra Representation Theory

Lie Algebra Cohomology, Co-adjoint Orbits, Cocycle, Quivers

Symplectic Geometry & Poisson Geometry

Symplectic Manifold associated to Lie Group Co-adjoint Orbit by KKS 2 Form, Moment Map, Souriau's Maxwell Principle

Integrable System

complete integrability, Liouville integrability, Action-Angles Coordinates, Symplectic Integrators

Foliation Theory

Coisotropic Polar, Bifibration, Bifoliation, Transverse Poisson Structure, Symplectic Foliation, Riemannian Foliation, webs

Information Geometry

Statistical Manifold, Koszul-Fisher Metric, Natural Gradient, Natural Langevin Dynamics, Coordinate Free Neural Network, Hessian Structures, Flat connexion without torsion

Metriplectic Flow

GENERIC Flow, Poisson Bracket, Metric Flow Bracket, 1^{st} & 2^{nd} Thermodynamics Principles, Onsager-Casimir Relations, Dissipation, Transverse Foliations of Entropy and Energy levels sets

Calculus of Variations

Geometrization of Noether Theorem, Poincaré-Cartan Integral Invariant, Euler-Poincaré Equation

Souriau's Lie Groups Thermodynamics

Covariant Gibbs Density, Planck Temperature in Lie Algebra, Heat in dual Lie Algebra, Entropy as Casimir function, Souriau-Fisher Metric

GEOMETRIC DEFINITION OF ENTROPY

ENTROPY DEFINITION: Entropy is Invariant Casimir Function on Leaves of Symplectic Foliation associated to Coadjoint Orbits generated via Moment Map of Symmetry Group acting on the System.

$$\{S,H\}_{\tilde{\Theta}}(Q)=0$$

$$ad^{*}_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0$$

$$\beta\in Ker\tilde{\Theta}_{\beta}\Rightarrow\left\langle Q,\left[\frac{\partial S}{\partial Q},Z\right]\right\rangle+\tilde{\Theta}\left(\frac{\partial S}{\partial Q},Z\right)=0$$

Metriplectic Flow on Symplectic Foliation & Transverse Metric Foliation

Foliation Leaves = Level Sets of Energy

Foliation Leaves = Level Sets of Entropy

METRIC FOLIATION $\Im^{\perp} \perp \Im$ **SYMPLECTIC FOLIATION**

1st Principle of Thermodynamics Preservation of Energy

2nd Principle of Thermodynamics Entropy Production

METRIPLECTIC FLOW

$$\frac{dF}{dt} = \{F, H\} + (F, S)$$

Non-dissipative Entropy = Constant

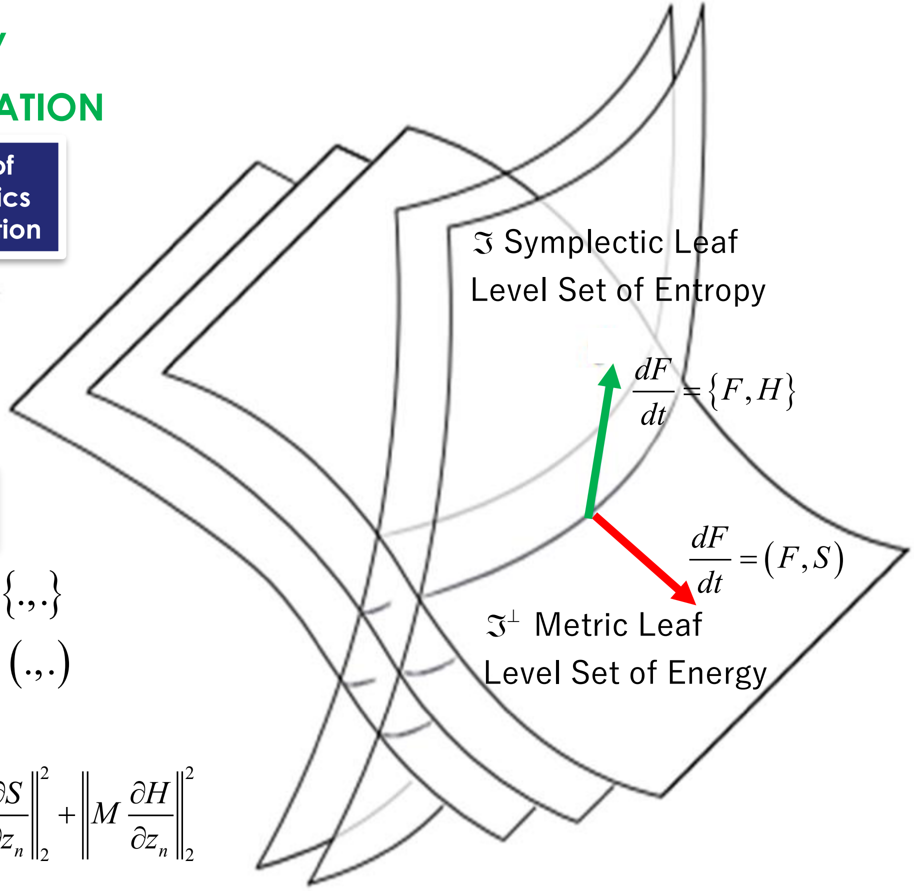
Dissipative Energy = Constant

$\{F, S\} = 0 \quad \forall F : S$ Entropy Casimir Function for $\{.,.\}$

$(F, H) = 0 \quad \forall F : H$ Energy Casimir Function for $(.,.)$

Thermodynamics-Informed Neural Networks

$$\Lambda = \frac{1}{N_{batch}} \sum_{n=0}^{N_{batch}} \left(\lambda \Lambda_n^{data} + \Lambda_n^{\deg} \right) \qquad \Lambda_n^{data} = \left\| \frac{dz^{GT}}{dt} - \frac{dz^{net}}{dt} \right\|_2^2 \qquad \Lambda_n^{\deg} = \left\| L \frac{\partial S}{\partial z_n} \right\|_2^2 + \left\| M \frac{\partial H}{\partial z_n} \right\|_2^2$$



Thermodynamics Triptych : symplectic & transverse metric foliatons

DUAL FISHER METRIC
HESSIAN OF ENTROPY

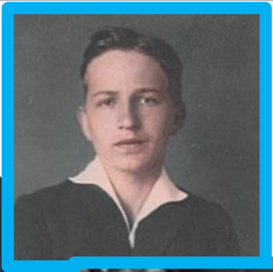
DISSIPATIVE

Transverse Riemanian Foliation

Metriplectic Flow & Onsager-Casimir Relations

Baptiste COQUINOT

Reeb's Foliation Theory: Symplectic Foliation, Transverse Metric Foliation



Charles EHRESMANN

Georges Henri REEB

Koszul Information Geometry

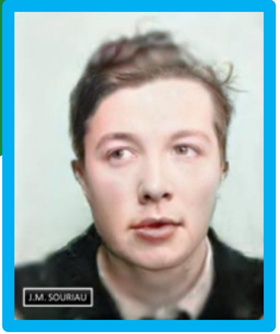
Symplectic Foliation

Souriau's Lie Groups Thermodynamics with Entropy as Casimir Function

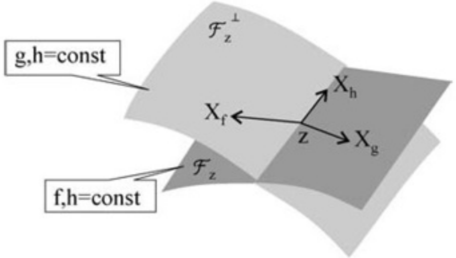
NON-DISSIPATIVE

FISHER METRIC
HESSIAN OF MASSIEU
KKS 2-FORM

Jean-Louis KOSZUL



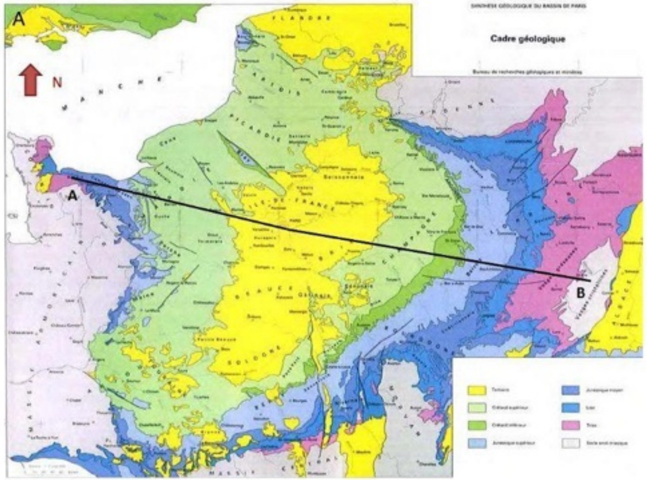
Jean-Marie SOURIAU

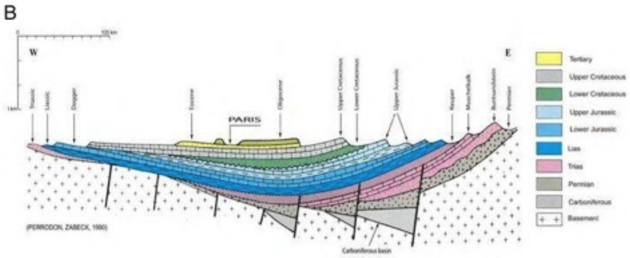


« Nord-Bassin Parisien » Seminar on « Geometric Structures of Dissipation »

« Nord-Bassin Parisien » Seminar 2020's

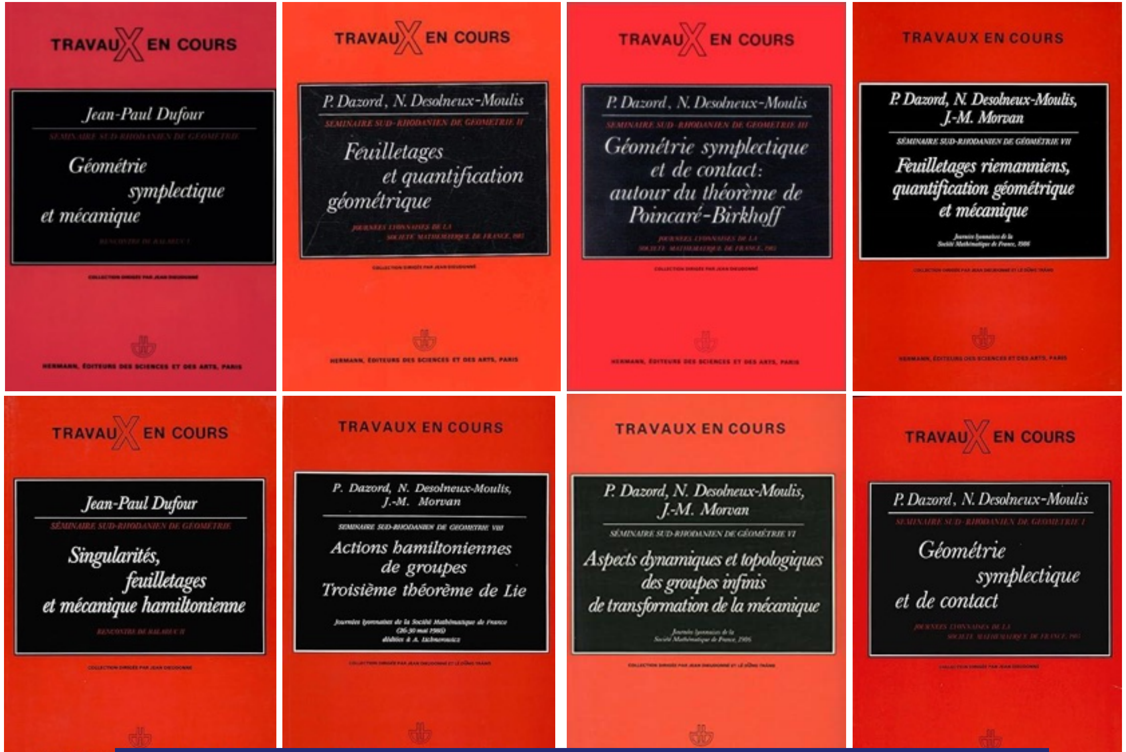
- Paris-Metz-Valenciennes-Lille-Louvain-Liège-Bruxelles





« Sud-Rhodanien » Seminar 1980's

- Lyon, Marseille, Montpellier



« L'oubli et la mémoire sont également inventifs »
J. L. Borges, Le Rapport de Brodie

Seminal Gibbs States from Moment Map of SE(3)

- As observed by Charles-Michel Marle, **Josiah Willard Gibbs** in chapter IV of his book "Elementary Principles in Statistical Mechanics, developed with Especial Reference to the Rational Foundation of Thermodynamics" published in 1902, considered **generalization of Gibbs states built with the moment map of the product of the one-dimensional group of translations in time and the three-dimensional group of rotations in space for the study of systems contained in a rotating vessel**, referring to a paper by Maxwell published in 1878. We can read in the book of Gibbs:

[…] The consideration of the above case of statistical equilibrium may be made the foundation of the theory of the thermodynamic equilibrium of rotating bodies, a subject which has been treated by Maxwell in his memoir On Boltzmann's theorem on the average distribution of energy in a system of material points (Gibbs 1902)

- **Jacques Hadamard** made a review of this book in 1906 (Hadamard 1906) and wrote:

[…] This book is not one of those that one analyzes hastily; but, on the other hand, the questions it deals with have been greatly agitated in recent times; the ideas defended by Gibbs have been the subject of much controversy; the reasoning with which he supported them has also been criticized. It seems interesting to me to study his work in the light of these controversies and by discussing these criticisms (Hadamard 1906)

Seminal Carathéodory’s axiomatization of Thermodynamics

- Constantin Carathéodory was born in Germany in Berlin in 1873 but was the son of a Turkish ambassador of Greek origin. From 1875 to 1895, Constantin Carathéodory lived in Bruxelles in Belgium, and studied at Belgian École Militaire. After a period in Greece, in London and Egypt, he returns to Berlin in 1900 to study mathematics.

- In 1904 in Göttingen, he defended his PhD on special Euler–Lagrange equations, with H. Minkowski,. He was successively professor in Göttingen, Bonn and Hannover. From 1905 to 1910. From Brussels, Carathéodory wrote to Born in 1907 about his definition of the concepts “amount of heat” and “reversibility” derived from the Carnot principle. He was back in Munich in 1924 to work on variational calculus applied to optics

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Pfaffian forms by Gaston Darboux

Sur le problème de Pfaff

Gaston Darboux (1822)

[…] The method that Pfaff made known in 1814, in the Memoirs of the Academy of Berlin, for the integration of a partial differential equation with any number of independent variables, was long neglected: the beautiful discoveries of Jacobi and de Cauchy have alone attracted the attention of geometers who deal with this theory. … In the first Part, I study the reduced forms, and I show that the integration of the first Pfaff system is sufficient and immediately gives the reduced form when it comes to the differential expression corresponding to a partial differential equation. In the second Part, I study the relations between reduced forms, and I demonstrate in particular three propositions which serve as a basis for Mr. Lie's group theory.

Johann Friedrich Pfaff

Pfaff problem and Pfaffian form attracted the attention of Clebsch (1862), Darboux (1882), Lie and Cartan (1899), among other as explained in the book of Edouard Goursat.

Gaston Darboux

Carathéodory's axiomatization of Thermodynamics

The main Axiom of Constantin Carathéodory of the 2nd law of Thermodynamics is written in 1909 paper:

[…] In jeder beliebigen Umgebung eins vorgeschriebenen Anfangszustandes gibt es Suztände, die durh adiabatische Zustandsänderungen nicht beliebig approximiert warden können [In the neighborhood of any equilibrium state of a system (of any number of thermodynamic coordinates), there exists states that are inaccessible by reversible adiabatic processes]. (Carathéodory 1909)

By analyzing the Pfaffian form, Carathéodory showed that if no heat exchange occurs, a system's adiabatic paths can be described mathematically, and implies the existence of an integrating factor related to temperature, leading to the definition of entropy. The integrating factor transforms the Pfaffian form into an exact differential, dS, which defines the change in entropy.

- **For the first law**, under any adiabatic change of state, the external work A by the change in energy is equal to zero and if we denote the initial ε and final $\bar{\varepsilon}$ energy:

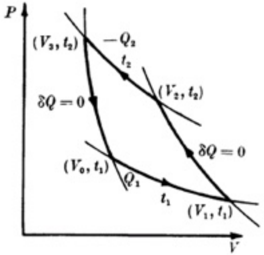
$$\bar{\varepsilon} - \varepsilon + A = 0$$

- **For the second law**, under all adiabatic changes of state that start from any given initial state, certain final states are not attainable and "unattainable" final states can be found in any neighborhood of the initial state.

Wolfgang Pauli Lectures on “Thermodynamics and the Kinetic Theory of Gases” – Rereading of Carathéodoty Axiomatization

- **Heat conduction** and **Internal Friction** are two main causes of **dissipation**
- […] If the second law were untrue, that is, if heat conduction were reversible, then work could be obtained from heat without compensating changes. Since it is only possible to convert work into heat without compensating changes, is the preferred direction in which actual processes take place. We can now also determine which of two temperature t1 and t2 is the higher, namely, that one from which heat can flow to the other without compensating changes.

(Pauli 1973)



Wolfgang Pauli Lectures on Physics, Volume 3 “Thermodynamics and the Kinetic Theory of Gases”, 2nd chapter dedicated to the 2nd Law:

[…] The second law distinguishes heat from the other forms of energy. It indicates a direction in time and makes apparent that heat is a disordered form of energy. (Pauli 1973)

Constantin Carathéodory axiomatization of thermodynamics interpreted by foliations by Theodore Frankel

- **Theodore Frankel (Frankel 2012)** was one of first to interpret Carathéory model with theory of foliations and Frobenius's theorem. In chapter entitled '**Heuristic Thermodynamics via Caratheodory (Can one go adiabatically from some state to any nearby state?)**' of his book "**The Geometry of Physics: An Introduction**", **Frankel identified entropy as a global adiabatic foliation of state space, guaranteed by the holonomy of entropy.**
- Theodore Frankel then rewrite Carathéodory results in a modern geometric framework

Caratheodory's Theorem: Let θ^1 be a continuously differentiable nonvanishing 1-form on an M^n, and suppose that $\theta = 0$ is not integrable; thus at some $x_0 \in M^n$ we have $\theta \wedge d\theta \neq 0$. Then there is a neighborhood U of x_0 such that any $y \in U$ can be joined to x_0 by a piecewise smooth path that is always tangent to the distribution.

- Franken concluded that the adiabatic distribution $Q^1 = 0$ is integrable and that there are locally defined a local entropy function S and τ on the state space M^{n+1} such that $Q^1 = \lambda dS$ that admit a local integrating factor τ .

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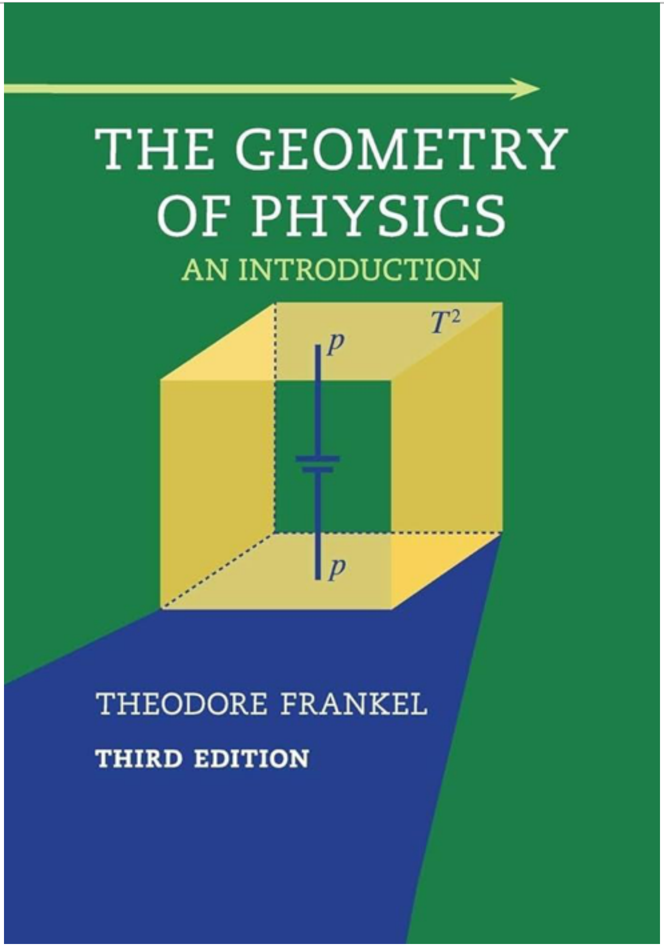
Constantin Carathéodory axiomatization of thermodynamics interpreted by foliations by Theodore Frankel



[…] If, for example, **the foliation defined by $Q = 0$ has leaves that wind densely (as in a torus) then there is no way that a global function S can exist, since such an S must be constant on each maximal leaf.** It is easy to see, however, that Kelvin's second law of thermodynamics rules out the possibility of not only dense adiabatic leaves, but even leaves that "double back"! For in the proof that "Kelvin implies Caratheodory," we saw that two states related by heating at constant volume cannot be joined by a quasi-static adiabatic. This says that no $\pi^{-1}(v)$ can meet a maximal adiabatic leaf twice. It might be thought that the space M^{n+1} and the adiabatic foliation must then be of a completely trivial nature. **(Frankel 1972, p.33)**

Talk outline

« There is nothing more in physical theories than symmetry groups except the mathematical construction which allows precisely to show that there is nothing more » -
Jean-Marie Souriau

1. Basic on Lie Groups
2. Information Geometry & Natural Gradient for Learning
3. Constantin Carathéodory: From the Pfaffian forms to Thermodynamics Axiomatization, & Adiabatic Foliation by Frankel
4. Jean-Marie Souriau: Symplectic Foliation via Moment Map and Koszul-Fisher Metric from Information Geometry for Lie Groups Thermodynamics
5. ENTROPY as Casimir Function in Coadjoint Representation Constant on Symplectic Leaves & Density of Maximum Entropy on Lie Groups
6. Lars Onsager & Baptiste Coquinot: Symplectic Foliation Model of METRIPLECTIC FLOW
7. Transverse Symplectic Foliation Structure
8. Liouville Complete Integrability and Information Geometry
9. Thermodynamics-Informed Neural Network and Symplectic Foliation Structure
10. Conclusion

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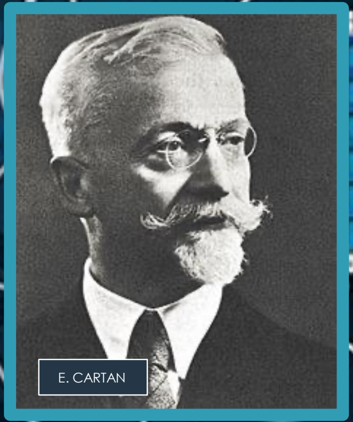
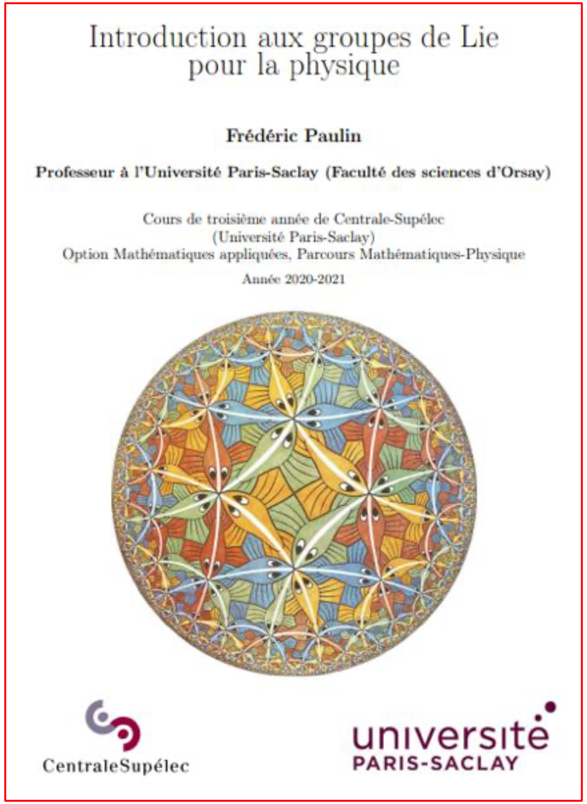
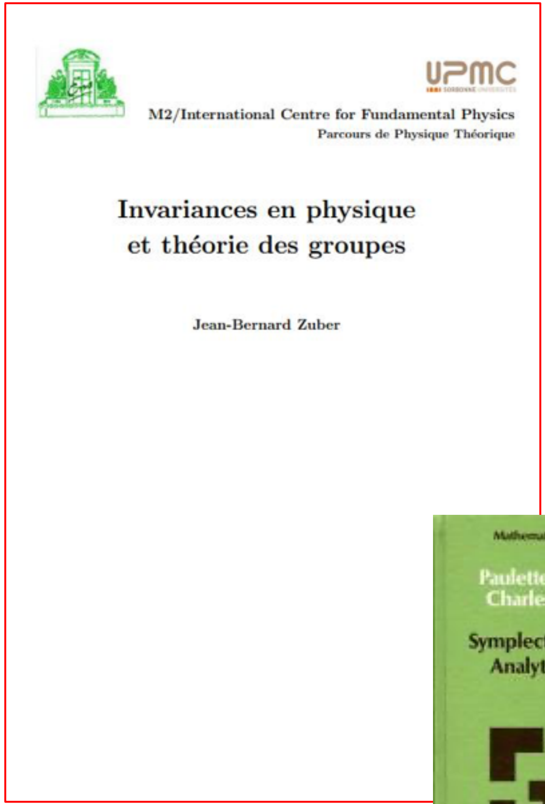
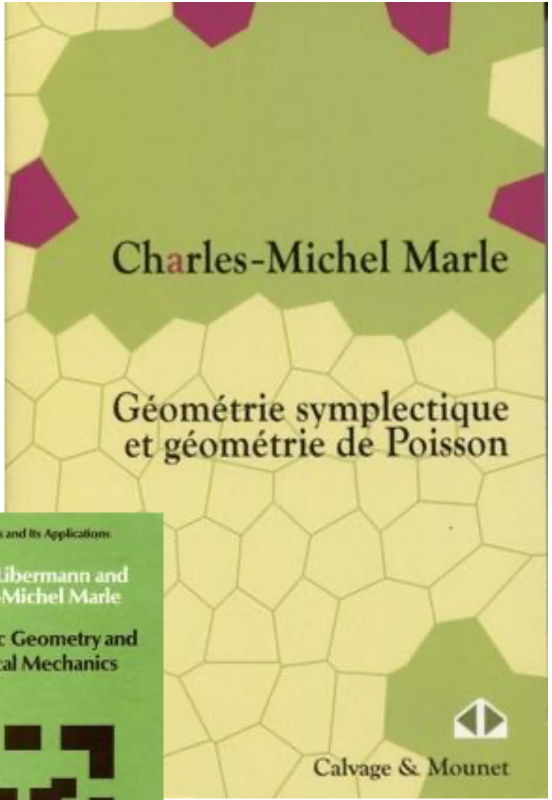
Basic on Lie Groups

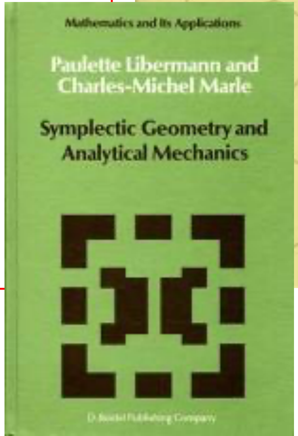


Introduction to Lie Groups Representation Theory , Symplectic and Poisson Geometries (Université Paris Saclay & Sorbonne Université Lectures)

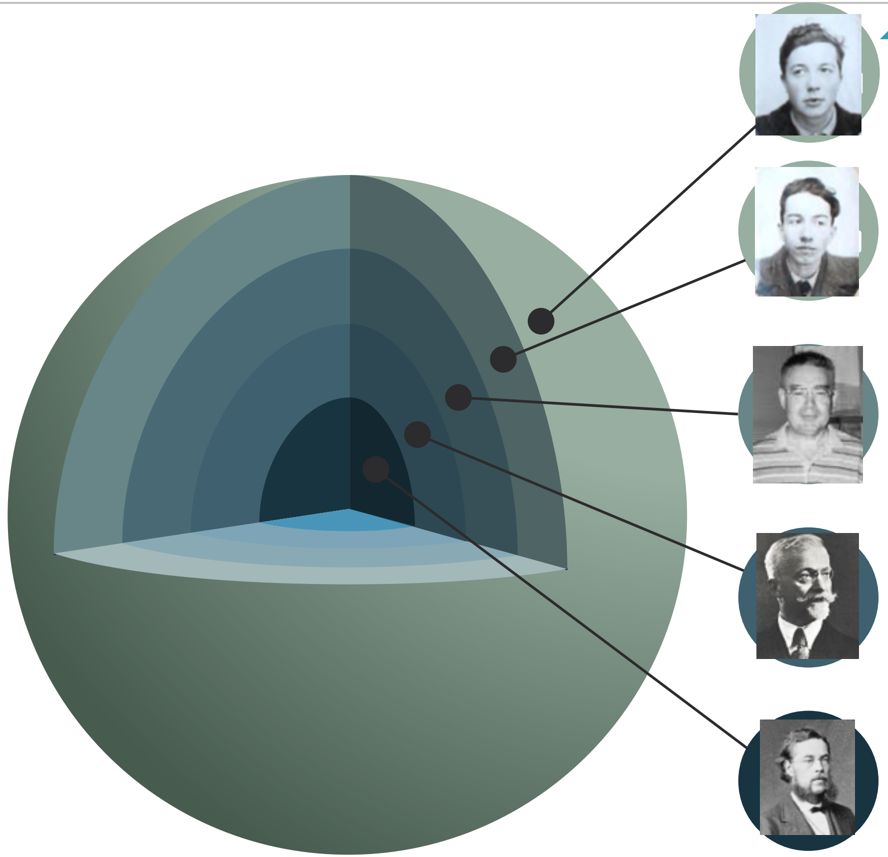








Lie Groups Tools Development: From Group to Co-adjoint Orbits



Lie Group & Statistical Physics

Jean-Michel Bismut – Random Mechanics
Jean-Marie Souriau – Lie Group Thermodynamics, Souriau Metric
Jean-Louis Koszul – Affine Lie Group & Algebra representation

Harmonic Analysis on Lie Group & Orbits Method

Pierre Torasso & Michèle Vergne – Poisson-Plancherel Formula
Michel Duflo – Extension of Orbits Method, Plancherel & Character
Alexandre Kirillov – Coadjoint Orbits, Kirillov Character
Jacques Dixmier – Unitary representation of nilpotent Group

Lie Group Representation

Bertram Kostant – KKS 2-form, Geometric Quantization
Alexandre Kirillov – Representation Theory, KKS 2-form
Jean-Marie Souriau – Moment Map, KKS 2-form, Souriau Cocycle
Valentine Bargmann – Unitary representation, Central extension

Lie Group Classification

Carl-Ludwig Siegel – Symplectic Group
Hermann Weyl – Conformal Geometry, Symplectic Group
Elie Cartan – Lie algebra classification, Symmetric Spaces
Willem Killing – Cartan-Killing form, Killing Vectors

Group/Lie Group Foundation

Henri Poincaré – Fuchsian Groups
Felix Klein – Erlangen Program (Homogeneous Manifold)
Sophus Lie – Lie Group
Evariste Galois/Louis Joseph Lagange – Substitution Group

Representation Theory & (Co-adjoint) Orbits Method: A. Kirillov

Grundlehren der mathematischen Wissenschaften 220

A Series of Comprehensive Studies in Mathematics

A. A. Kirillov

Elements of the Theory of Representations

Merits versus Demerits

Merits	Demerits
1. Universality: the method works for Lie groups of any type over any field.	1. The recipes are not accurately and precisely formulated.
2. The rules are visual, easy to memorize and illustrate by a picture.	2. Sometimes they are wrong and need corrections or modifications.
3. The method explains some facts which otherwise look mysterious.	3. It could be difficult to transform this explanation into a rigorous proof.
4. It provides a great number of symplectic manifolds and Poisson commuting families of functions.	4. Most completely integrable dynamical systems were discovered earlier by other methods.
5. The method introduces two new fundamental notions: coadjoint orbits and moment maps.	5. The description of coadjoint orbits and their structure is sometimes not an easy problem.

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§ 15. The Method of Orbits

At the basis of the method of orbits lies the following "experimental fact": the theory of infinite-dimensional representations of every Lie group is closely connected with a certain special finite-dimensional representation of this group. This representation acts in the dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of the group under study. We will call it a *co-adjoint* or briefly a *K-representation*.[1]

Orbits of a Lie group in the space of a K-representation are symplectic manifolds. They can be interpreted as phase spaces of a Hamiltonian mechanical system for which the given Lie group is the group of symmetries. In 15.2, we shall give a classification of all homogeneous symplectic manifolds with a given group of symmetries.

It turns out that unitary irreducible representations of the group G are connected with orbits of this group in the K-representation. The construction of the representation in an orbit is given in 15.3.

This is a generalization of the procedure of quantization that is used in quantum mechanics. This point of view is explained in more detail in 15.4.

The author sees the significance of the method of orbits not only in the specific theorems obtained by this method, but also in the great collection of simple and intuitive heuristic rules that give the solution of the basic questions of the theory of representations. With the passage of time, these rules will be elevated to the level of strict theorems, but already now their value is indisputable.

We shall show in 15.5 how the operations of restriction to a subgroup and induction from this subgroup can be described with the aid of the natural projection $p:\mathfrak{g}^*\to\mathfrak{h}^*$, where $\mathfrak{h}$ is the Lie algebra of the subgroup H.

As we shall see in 15.6, generalized characters of irreducible unitary representations admit a simple expression in the form of an integral over the corresponding orbit. In many cases, this allows us to write an explicit expression for the Plancherel measure.

Finally, in 15.7 we show that infinitesimal characters of irreducible unitary representations of a group G can be computed as values of G-invariant polynomials on the corresponding orbits.

15.1. The Co-Adjoint Representation of a Lie Group

Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{g}^*$ the dual space to $\mathfrak{g}$. The group G acts in $\mathfrak{g}$ with the aid of the adjoint representation Ad (see 6.3) and in $\mathfrak{g}^*$ with the aid of the co-adjoint representation, or, briefly, the K-representation. If the Lie algebra $\mathfrak{g}$ is realized in the form of the algebra of left-invariant vector-fields on G, then it is natural to realize $\mathfrak{g}^*$ in the form of the space of left-invariant differential forms of the first order on G. The K-representation of the group G acts in the space of 1-forms by right translations.

A. KIRILLOV

ÉLÉMENTS DE LA THÉORIE DES REPRÉSENTATIONS

ÉDITIONS MIR • MOSCOU

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Kirillov by Dixmier: « La thèse de Kirillov, parue en 1962, a suscité immédiatement beaucoup d'intérêt »

Brèves remarques sur l'œuvre de A. A. Kirillov

Jacques Dixmier

La thèse de Kirillov, parue en 1962, a suscité immédiatement beaucoup d'intérêt. Soit G un groupe de Lie nilpotent simplement connexe. Soient $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{g}^*$ l'espace vectoriel dual de $\mathfrak{g}$, dans lequel G opère par la représentation coadjointe. Soit $\mathfrak{g}^*/G$ l'ensemble des orbites coadjointes. Soit $\widehat{G}$ l'ensemble des représentations unitaires irréductibles de G (la plupart sont de dimension infinie). Alors Kirillov construit une bijection canonique $\mathfrak{g}^*/G \to \widehat{G}$; ainsi, les représentations unitaires irréductibles de G sont paramétrées par les orbites coadjointes. En outre, quantité de notions naturelles concernant les représentations s'interprètent géométriquement en termes d'orbites coadjointes: restriction à un sous-groupe, induction unitaire, produit tensoriel, mesure de Plancherel, topologie de $\widehat{G}$. Détaillons seulement un résultat, d'une élégance extraordinaire: soient $\pi \in \widehat{G}$ et Ω l'orbite associée ; soit μ la mesure G-invariante canonique sur Ω (c'est une distribution tempérée sur $\mathfrak{g}^*$) ; soit μ' sa transformée de Fourier, distribution tempérée sur $\mathfrak{g}$; transportant μ' sur G par l'application exponentielle, on obtient le caractère-distribution de π !

Kirillov s'est vite convaincu, et il a convaincu la communauté mathématique, que cette "méthode des orbites" devait être applicable à des groupes bien plus généraux que les groupes nilpotents.

Il n'a pas hésité à aborder le cas des groupes de Lie connexes quelconques. Evidemment, des difficultés considérables ont surgi immédiatement. Néanmoins, Kirillov a indiqué une voie d'accès, qui ensuite a été largement utilisée.

Depuis plus de trente ans, Kirillov applique la méthode des orbites aux groupes de Lie–Cartan de dimension infinie. Par exemple, soit M une variété C^∞ compacte. Soit G le groupe des difféomorphismes de M, ou l'un des sous-groupes obtenus en prenant les éléments de G qui conservent une forme volume, ou une structure symplectique, ou une structure de contact. On considère l'algèbre de Lie $\mathfrak{g}$ et son dual $\mathfrak{g}^*$, tous deux définis en tenant compte de la topologie. Aux orbites coadjointes sont associées des représentations unitaires irréductibles de G, du moins dans certains cas, par exemple si l'orbite est de dimension finie ou de codimension finie. Le groupe fondamental de ces orbites peut être non commutatif, par exemple être un groupe symétrique. Kirillov envisage parfois des groupes encore plus généraux (par exemple, le groupe $\mathrm{Diff}_H(E)$ où E est un fibré principal de groupe structural H), et parfois au contraire étudie de manière très approfondie des cas particuliers importants; par exemple, soient S le cercle, $\mathrm{Diff}_+(S)$ le groupe des difféomorphismes de S conservant l'orientation; alors les orbites coadjointes pour ce groupe sont liées aux fonctions holomorphes univalentes dans le disque unité, et à certaines équations différentielles linéaires; Kirillov définit sur ces orbites des structures complexes kählériennes invariantes, d'où des représentations de l'algèbre de Virasoro et du groupe de Virasoro–Bott.

Les démonstrations précédentes ont amené Kirillov à s'intéresser, dans plusieurs articles, à des questions de pure géométrie différentielle. Soient M une variété C^∞, E un fibré en droites sur M, $\Gamma(E)$ l'ensemble des sections C^∞ de E. Un crochet de Lie $[\ ,\ ]$ sur $\Gamma(E)$ est dit local si $[s_1, s_2]$ est continue en (s_1, s_2) et si $\mathrm{supp}[s_1, s_2] \subset \mathrm{supp}\, s_1 \bigcap \mathrm{supp}\, s_2$. Kirillov classifie ces crochets. Il détermine aussi, dans des cas généraux, quels sont les opérateurs multidifférentiels invariants par difféomorphismes (tels que la dérivation extérieure, ou le crochet ordinaire de deux champs de vecteurs).

Plus récemment, Kirillov a utilisé avec succès la méthode des orbites pour des groupes très différents, les $G_n(k)$ ($n = 1, 2, \ldots$; k corps commutatif). (On note $G_n(k)$ le groupe des matrices $n \times n$ unipotentes triangulaires supérieures à éléments dans k.) La théorie est particulièrement poussée lorsque k est un corps fini. Elle amène à introduire une suite remarquable des polynômes en une variable.

Voici un autre thème longuement étudié par Kirillov en collaboration avec Gelfand. Soient G un groupe algébrique complexe, $\mathfrak{g}$ son algèbre de Lie, U l'algèbre enveloppante de $\mathfrak{g}$, D le corps enveloppant de U. Soit $D_{n,k}$ le corps engendré par des indéterminées $x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_k$, où tous les crochets sont nuls sauf $[x_i, y_i] = 1$. Alors Gelfand et Kirillov conjecturent en 1966 que D est isomorphe à un corps $D_{n,k}$. Dans les années suivantes, ils démontrent des cas particuliers de plus en plus nombreux de cette conjecture. Bien après, la conjecture sera reconnue comme fausse, mais elle aura suscité un grand nombre de travaux intéressants. Surtout, dès l'article de 1966, Gelfand et Kirillov introduisent diverses notions de dimensions non commutatives; l'une d'entre elles, promise à un grand avenir, sera appelée dimension de Gelfand–Kirillov.

Il n'est pas surprenant que Kirillov ait étudié, pour elles-mêmes, les algèbres de Lie de dimension infinie. Dans un série d'articles, il a mis en évidence des cas où la "croissance" de ces algèbres est strictement intermédiaire entre la croissance polynomiale et la croissance exponentielle et il a construit des identités remarquables vérifiées dans l'algèbre de Lie des champs de vecteurs sur $\mathbb{R}$, ou des champs hamiltoniens sur $\mathbb{R}^2$.

La place me manque pour analyser les nombreux articles isolés abordant des thèmes sur lesquels Kirillov n'est pas revenu par la suite. Mentionnons tout de même 3 de ces articles: 1) il a complété sur un point très important l'étude (par Gelfand–Graev) des représentations unitaires irréductibles de $SL_2(k)$ (k corps localement compact non connexe). 2) Soient $\mathcal{H}$ un espace hilbertien, G le groupe des opérateurs unitaires u dans $\mathcal{H}$ tels que $1 - u$ soit compact. Alors Kirillov a déterminé les représentations unitaires irréductibles de G. 3) Kirillov a résolu un problème essentiel de géométrie intégrale posé par Gelfand. Ces études n'utilisent pas la méthode des orbites.

Kirillov a écrit plusieurs livres. Deux d'entre eux ont été traduits en français et en anglais: 1) *Eléments de la théorie des représentations*, 2) *Théorèmes et problèmes d'analyse fonctionnelle*. Il a d'autre part rédigé beaucoup d'articles d'exposition: chacun d'eux est en réalité un court livre. La clarté des exposés fait que ces écrits ont été largement lus, et notamment, avec profit, par des chercheurs débutants.

C'est l'occasion de signaler que Kirillov a eu de nombreux élèves de thèse. Les organisateurs de ce colloque ont eu la bonne idée de faire imprimer sur un T-shirt (offert à Kirillov) la liste de ces élèves: 57 noms, un record !

Kirillov a participé, en 1967, à la fondation d'un journal célèbre: *Functional Analysis and its Applications*. Il a, pendant 4 ans, été vice-président de la Société Mathématique de Moscou. Il est membre de comités éditoriaux de plusieurs journaux. Il a suscité la traduction en russe de nombreux livres et articles édités à l'ouest.

Dans sa conversation, Kirillov mentionne le soutien constant qu'il a reçu de son épouse (qui travaille en informatique). Il est fier de l'œuvre mathématique très connue de son fils.

Ce colloque honore donc un grand mathématicien: grand mathématicien par les voies qu'il a ouvertes, grand mathématicien par l'influence qu'il exerce depuis des années.

Jacques Dixmier
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F-75005 Paris, France

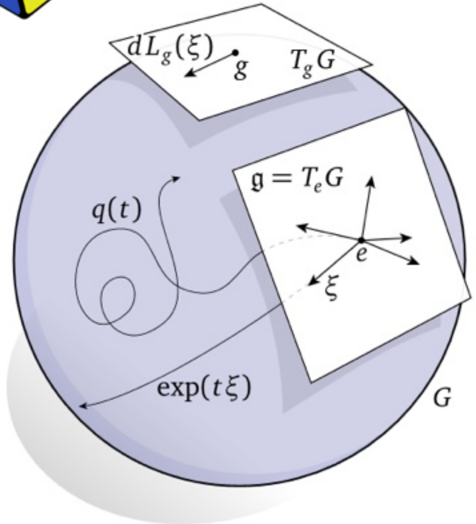
Lie Group

GROUP (Mathematics)

A set equipped with a binary operation with 4 axioms:

- Closure $\forall a,b \in G \text{ then } a \bullet b \in G$
- Associativity $\forall a,b,c \in G \text{ then } (a \bullet b) \bullet c = a \bullet (b \bullet c)$
- Identity $\exists e \in G \text{ such that } e \bullet a = a \bullet e = a$
- invertibility $\forall a \in G, \exists b \in G \text{ such that } b \bullet a = a \bullet b = e$

LIE GROUP



- A group that is a differentiable manifold, with the property that the group operations of multiplication and inversion are smooth maps:
  $\forall x,y \in G \text{ then } \phi : G \times G \to G \text{ then } \phi(x,y) = x^{-1}y \text{ is smooth}$
- A Lie algebra $\mathfrak{g} = T_e G$ is a vector space with a binary operation called the Lie bracket $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies axioms:

$$[ax+by,z] = a[x,z] + b[y,z] \;\; ; \;\; [x,x] = 0 \;\; ; \;\; [x,y] = -[y,x]$$

$$\text{Jacobi Identity: } [x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0$$

$$[x,y] = xy - yx \text{ for Matrix Lie Group}$$

Classical Lie Group

Lie Groups

$$
\begin{aligned}
\mathrm{SL}_n(\mathbb{R}) &= \{x \in \mathrm{GL}_n(\mathbb{R}) \ : \ \det \ x = 1\} \\
\mathrm{SL}_n(\mathbb{C}) &= \{x \in \mathrm{GL}_n(\mathbb{C}) \ : \ \det \ x = 1\} \\
\mathrm{O}(n) &= \{x \in \mathrm{GL}_n(\mathbb{R}) \ : \ {}^t x \, x = I_n\} \\
\mathrm{SO}(n) &= \mathrm{O}(n) \cap \mathrm{SL}_n(\mathbb{R}) \\
\mathrm{O}(p,q) &= \{x \in \mathrm{GL}_{p+q}(\mathbb{R}) \ : \ {}^t x \, I_{p,q} \, x = I_{p,q}\} \\
\mathrm{SO}(p,q) &= \mathrm{O}(p,q) \cap \mathrm{SL}_{p+q}(\mathbb{R}) \\
\mathrm{U}(n) &= \{x \in \mathrm{GL}_n(\mathbb{C}) \ : \ x^* \, x = I_n\} \\
\mathrm{SU}(n) &= \mathrm{U}(n) \cap \mathrm{SL}_n(\mathbb{C})
\end{aligned}
$$

Lie Algebra

$$
\begin{aligned}
\mathfrak{sl}_n(\mathbb{R}) &= \{X \in \mathfrak{gl}_n(\mathbb{R}) \ : \ \mathtt{tr} \ X = 0\} \\
\mathfrak{sl}_n(\mathbb{C}) &= \{X \in \mathfrak{gl}_n(\mathbb{C}) \ : \ \mathtt{tr} \ X = 0\} \\
\mathfrak{o}(n) = \mathfrak{so}(n) &= \{X \in \mathfrak{gl}_n(\mathbb{R}) \ : \ {}^t X = -X\} \\
\mathfrak{o}(p,q) = \mathfrak{so}(p,q) &= \{X \in \mathfrak{gl}_{p+q}(\mathbb{R}) \ : \ {}^t X \, I_{p,q} + I_{p,q} \, X = 0\} \\
\mathfrak{u}(n) &= \{X \in \mathfrak{gl}_n(\mathbb{C}) \ : \ X^* = -X\} \\
\mathfrak{su}(n) &= \{X \in \mathfrak{gl}_n(\mathbb{C}) \ : \ X^* = -X, \quad \mathtt{tr} \ X = 0\}
\end{aligned}
$$

Lie group $\mathcal{M}, \circ$		size	dim	$\mathcal{X} \in \mathcal{M}$	Constraint	$\boldsymbol{\tau}^\wedge \in \mathfrak{m}$	$\boldsymbol{\tau} \in \mathbb{R}^m$	$\mathrm{Exp}(\boldsymbol{\tau})$	Comp.	Action
n-D vector	$\mathbb{R}^n, +$	n	n	$\mathbf{v} \in \mathbb{R}^n$	$\mathbf{v} - \mathbf{v} = \mathbf{0}$	$\mathbf{v} \in \mathbb{R}^n$	$\mathbf{v} \in \mathbb{R}^n$	$\mathbf{v} = \exp(\mathbf{v})$	$\mathbf{v}_1 + \mathbf{v}_2$	$\mathbf{v} + \mathbf{x}$
circle	$S^1, \cdot$	2	1	$\mathbf{z} \in \mathbb{C}$	$\mathbf{z}^* \mathbf{z} = 1$	$i\theta \in i\mathbb{R}$	$\theta \in \mathbb{R}$	$\mathbf{z} = \exp(i\theta)$	$\mathbf{z}_1 \, \mathbf{z}_2$	$\mathbf{z} \, \mathbf{x}$
Rotation	$SO(2), \cdot$	4	1	$\mathbf{R}$	$\mathbf{R}^\top \mathbf{R} = \mathbf{I}$	$[\theta]_\times \in \mathfrak{so}(2)$	$\theta \in \mathbb{R}$	$\mathbf{R} = \exp([\theta]_\times)$	$\mathbf{R}_1 \, \mathbf{R}_2$	$\mathbf{R} \, \mathbf{x}$
Rigid motion	$SE(2), \cdot$	9	3	$\mathbf{M} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix}$	$\mathbf{R}^\top \mathbf{R} = \mathbf{I}$	$\begin{bmatrix} [\theta]_\times & \boldsymbol{\rho} \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(2)$	$\begin{bmatrix} \boldsymbol{\rho} \\ \theta \end{bmatrix} \in \mathbb{R}^3$	$\exp\left(\begin{bmatrix} [\theta]_\times & \boldsymbol{\rho} \\ 0 & 0 \end{bmatrix}\right)$	$\mathbf{M}_1 \, \mathbf{M}_2$	$\mathbf{R} \, \mathbf{x} + \mathbf{t}$
3-sphere	$S^3, \cdot$	4	3	$\mathbf{q} \in \mathbb{H}$	$\mathbf{q}^* \mathbf{q} = 1$	$\boldsymbol{\theta}/2 \in \mathbb{H}_p$	$\boldsymbol{\theta} \in \mathbb{R}^3$	$\mathbf{q} = \exp(\mathbf{u}\theta/2)$	$\mathbf{q}_1 \, \mathbf{q}_2$	$\mathbf{q} \, \mathbf{x} \, \mathbf{q}^*$
Rotation	$SO(3), \cdot$	9	3	$\mathbf{R}$	$\mathbf{R}^\top \mathbf{R} = \mathbf{I}$	$[\boldsymbol{\theta}]_\times \in \mathfrak{so}(3)$	$\boldsymbol{\theta} \in \mathbb{R}^3$	$\mathbf{R} = \exp([\boldsymbol{\theta}]_\times)$	$\mathbf{R}_1 \, \mathbf{R}_2$	$\mathbf{R} \, \mathbf{x}$
Rigid motion	$SE(3), \cdot$	16	6	$\mathbf{M} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix}$	$\mathbf{R}^\top \mathbf{R} = \mathbf{I}$	$\begin{bmatrix} [\boldsymbol{\theta}]_\times & \boldsymbol{\rho} \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(3)$	$\begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\theta} \end{bmatrix} \in \mathbb{R}^6$	$\exp\left(\begin{bmatrix} [\boldsymbol{\theta}]_\times & \boldsymbol{\rho} \\ 0 & 0 \end{bmatrix}\right)$	$\mathbf{M}_1 \, \mathbf{M}_2$	$\mathbf{R} \, \mathbf{x} + \mathbf{t}$

Example: SE(2) (Rigid Motion Group): Rotation and Translation in 2D

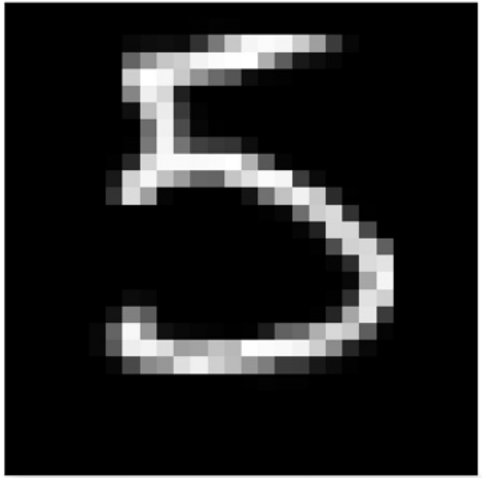
$$\mathbf{M} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \in SE(2) \subset \mathbb{R}^{3\times 3}$$

with $\mathbf{R} \in SO(2)$ a rotation and $\mathbf{t} \in \mathbb{R}^2$ a tran

$$\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{R}^\top & -\mathbf{R}^\top \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$$

$$\mathbf{M}_a \mathbf{M}_b = \begin{bmatrix} \mathbf{R}_a \mathbf{R}_b & \mathbf{t}_a + \mathbf{R}_a \mathbf{t}_b \\ \mathbf{0} & 1 \end{bmatrix}$$

$SE(2)$



Roto-translation=
Translation
+rotation

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Example: SE(2) (Rigid Motion Group)

Lie Group

$$\mathbf{M} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \in SE(2) \subset \mathbb{R}^{3\times 3}$$ with $\mathbf{R} \in SO(2)$ a rotation and $\mathbf{t} \in \mathbb{R}^2$ a translation.

Lie Algebra

$$\boldsymbol{\tau}^\wedge = \begin{bmatrix} [\theta]_\times & \boldsymbol{\rho} \\ \mathbf{0} & 0 \end{bmatrix} \in \mathfrak{se}(2) \quad , \qquad \boldsymbol{\tau} = \begin{bmatrix} \boldsymbol{\rho} \\ \theta \end{bmatrix} \in \mathbb{R}^3$$

$$\text{Hat}: \quad \mathbb{R}^m \to \mathfrak{m}; \qquad \boldsymbol{\tau} \mapsto \boldsymbol{\tau}^\wedge = \sum_{i=1}^{m} \tau_i E_i$$

$$\text{Vee}: \quad \mathfrak{m} \to \mathbb{R}^m; \quad \boldsymbol{\tau}^\wedge \mapsto (\boldsymbol{\tau}^\wedge)^\vee = \boldsymbol{\tau} = \sum_{i=1}^{m} \tau_i \mathbf{e}_i \ ,$$

with $\mathbf{e}_i$ the vectors of the base of $\mathbb{R}^m$ (we have $\mathbf{e}_i^\wedge = E_i$).

Inversion & Composition

$$\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{R}^\top & -\mathbf{R}^\top \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$$

$$\mathbf{M}_a \mathbf{M}_b = \begin{bmatrix} \mathbf{R}_a \mathbf{R}_b & \mathbf{t}_a + \mathbf{R}_a \mathbf{t}_b \\ \mathbf{0} & 1 \end{bmatrix}$$

Exp and Log maps

$$\exp: \quad \mathfrak{m} \to \mathcal{M} \quad ; \quad \boldsymbol{\tau}^\wedge \mapsto \mathcal{X} = \exp(\boldsymbol{\tau}^\wedge)$$

$$\log: \quad \mathcal{M} \to \mathfrak{m} \quad ; \quad \mathcal{X} \mapsto \boldsymbol{\tau}^\wedge = \log(\mathcal{X})$$

$$\mathbf{M} = \mathrm{Exp}(\boldsymbol{\tau}) \triangleq \begin{bmatrix} \mathrm{Exp}(\theta) & \mathbf{V}(\theta)\, \boldsymbol{\rho} \\ \mathbf{0} & 1 \end{bmatrix}$$

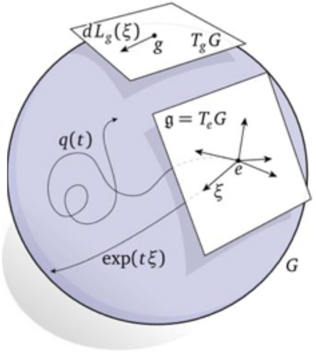
$$\boldsymbol{\tau} = \mathrm{Log}(\mathbf{M}) \triangleq \begin{bmatrix} \mathbf{V}^{-1}(\theta)\, \mathbf{t} \\ \mathrm{Log}(\mathbf{R}) \end{bmatrix} .$$

$$\exp: \mathfrak{gl}(n, \mathbb{R}) \to GL(n, \mathbb{R}),$$

$$X \mapsto \sum^{\infty} \frac{X^n}{n!}, \quad X \in \mathfrak{gl}(n, \mathbb{R}).$$

$$\log(A) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(A - I)^n}{n}$$

$$\mathbf{V}(\theta) = \frac{\sin\theta}{\theta} \mathbf{I} + \frac{1 - \cos\theta}{\theta} [1]_\times \ .$$



Adjoint Operator

$$\mathbf{Ad}_{\mathbf{M}} \boldsymbol{\tau} = (\mathbf{M} \boldsymbol{\tau}^\wedge \mathbf{M}^{-1})^\vee = \begin{bmatrix} \mathbf{R}\boldsymbol{\rho} - [\theta]_\times \mathbf{t} \\ \theta \end{bmatrix} = \mathbf{Ad}_{\mathbf{M}} \begin{bmatrix} \boldsymbol{\rho} \\ \theta \end{bmatrix}$$

leading to

$$\mathbf{Ad}_{\mathbf{M}} = \begin{bmatrix} \mathbf{R} & -[1]_\times \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} .$$

Examples

The rotation group $SO(3)$, its Lie algebra $\mathfrak{so}(3)$, and the vector space $\mathbb{R}^3$

In the rotation group $SO(3)$, of 3×3 rotation matrices $\mathbf{R}$, we have the orthogonality condition $\mathbf{R}^\top\mathbf{R}=\mathbf{I}$. The tangent space may be found by taking the time derivative of this constraint, that is $\mathbf{R}^\top\dot{\mathbf{R}}+\dot{\mathbf{R}}^\top\mathbf{R}=0$, which we rearrange as

$$\mathbf{R}^\top\dot{\mathbf{R}}=-(\mathbf{R}^\top\dot{\mathbf{R}})^\top .$$

This expression reveals that $\mathbf{R}^\top\dot{\mathbf{R}}$ is a skew-symmetric matrix (the negative of its transpose). Skew-symmetric matrices are often noted $[\boldsymbol{\omega}]_\times$ and have the form

$$[\boldsymbol{\omega}]_\times=\begin{bmatrix}0&-\omega_z&\omega_y\\ \omega_z&0&-\omega_x\\ -\omega_y&\omega_x&0\end{bmatrix}.$$

This gives $\mathbf{R}^\top\dot{\mathbf{R}}=[\boldsymbol{\omega}]_\times$. When $\mathbf{R}=\mathbf{I}$ we have

$$\dot{\mathbf{R}}=[\boldsymbol{\omega}]_\times ,$$

that is, $[\boldsymbol{\omega}]_\times$ is in the Lie algebra of $SO(3)$, which we name $\mathfrak{so}(3)$. Since $[\boldsymbol{\omega}]_\times\in\mathfrak{so}(3)$ has 3 DoF, the dimension of $SO(3)$ is $m=3$. The Lie algebra is a vector space whose elements can be decomposed into

$$[\boldsymbol{\omega}]_\times=\omega_x\mathbf{E}_x+\omega_y\mathbf{E}_y+\omega_z\mathbf{E}_z$$

with $\mathbf{E}_x=\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}$, $\mathbf{E}_y=\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix}$, $\mathbf{E}_z=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}$ the generators of $\mathfrak{so}(3)$, and where $\boldsymbol{\omega}=(\omega_x,\omega_y,\omega_z)\in\mathbb{R}^3$ is the vector of angular velocities. The one-to-one linear relation above allows us to identify $\mathfrak{so}(3)$ with $\mathbb{R}^3$ — we write $\mathfrak{so}(3)\cong\mathbb{R}^3$. We pass from $\mathfrak{so}(3)$ to $\mathbb{R}^3$ and viceversa using the linear operators *hat* and *vee*,

$$\text{Hat}:\quad \mathbb{R}^3\to\mathfrak{so}(3);\qquad \boldsymbol{\omega}\mapsto\boldsymbol{\omega}^\wedge=[\boldsymbol{\omega}]_\times$$
$$\text{Vee}:\quad \mathfrak{so}(3)\to\mathbb{R}^3;\qquad [\boldsymbol{\omega}]_\times\mapsto[\boldsymbol{\omega}]_\times^\vee=\boldsymbol{\omega} .$$

The exponential map of $SO(3)$

We have seen in Ex. 3 that $\dot{\mathbf{R}}=\mathbf{R}[\boldsymbol{\omega}]_\times\in T_{\mathbf{R}}SO(3)$. For $\boldsymbol{\omega}$ constant, this is an ordinary differential equation (ODE), whose solution is $\mathbf{R}(t)=\mathbf{R}_0\exp([\boldsymbol{\omega}]_\times t)$. At the origin $\mathbf{R}_0=\mathbf{I}$ we have the exponential map,

$$\mathbf{R}(t)=\exp([\boldsymbol{\omega}]_\times t)\qquad\in SO(3) .$$

We now define the vector $\boldsymbol{\theta}\triangleq\mathbf{u}\theta\triangleq\boldsymbol{\omega}t\in\mathbb{R}^3$ as the integrated rotation in angle-axis form, with angle θ and unit axis $\mathbf{u}$. Thus $[\boldsymbol{\theta}]_\times\in\mathfrak{so}(3)$ is the total rotation expressed in the Lie algebra. We substitute it above. Then write the exponential as a power series,

$$\mathbf{R}=\exp([\boldsymbol{\theta}]_\times)=\sum_k\frac{\theta^k}{k!}([\mathbf{u}]_\times)^k .$$

In order to find a closed-form expression, we write down a few powers of $[\mathbf{u}]_\times$,

$$[\mathbf{u}]_\times^0=\mathbf{I},\qquad [\mathbf{u}]_\times^1=[\mathbf{u}]_\times ,$$
$$[\mathbf{u}]_\times^2=\mathbf{u}\mathbf{u}^\top-\mathbf{I},\qquad [\mathbf{u}]_\times^3=-[\mathbf{u}]_\times ,$$
$$[\mathbf{u}]_\times^4=-[\mathbf{u}]_\times^2 ,\qquad \cdots$$

and realize that all can be expressed as multiples of $\mathbf{I}$, $[\mathbf{u}]_\times$ or $[\mathbf{u}]_\times^2$. We thus rewrite the series as,

$$\mathbf{R}=\mathbf{I}+[\mathbf{u}]_\times\left(\theta-\tfrac{1}{3!}\theta^3+\tfrac{1}{5!}\theta^5-\cdots\right)+[\mathbf{u}]_\times^2\left(\tfrac{1}{2}\theta^2-\tfrac{1}{4!}\theta^4+\tfrac{1}{6!}\theta^6-\cdots\right) ,$$

where we identify the series of $\sin\theta$ and $\cos\theta$, yielding the closed form,

$$\mathbf{R}=\exp([\mathbf{u}\theta]_\times)=\mathbf{I}+[\mathbf{u}]_\times\sin\theta+[\mathbf{u}]_\times^2(1-\cos\theta) .$$

This expression is the well known Rodrigues rotation formula. It can be used as the capitalized exponential just by doing $\mathbf{R}=\text{Exp}(\mathbf{u}\theta)=\exp([\mathbf{u}\theta]_\times)$.

The adjoint matrix of $SE(3)$

The $SE(3)$ group of rigid body motions (see App. D) has group, Lie algebra and vector elements,

$$\mathbf{M}=\begin{bmatrix}\mathbf{R}&\mathbf{t}\\ \mathbf{0}&1\end{bmatrix},\quad \boldsymbol{\tau}^\wedge=\begin{bmatrix}[\boldsymbol{\theta}]_\times&\boldsymbol{\rho}\\ \mathbf{0}&0\end{bmatrix},\quad \boldsymbol{\tau}=\begin{bmatrix}\boldsymbol{\rho}\\ \boldsymbol{\theta}\end{bmatrix} .$$

The adjoint matrix is identified by developing (31) as

$$\begin{aligned}\mathbf{Ad}_{\mathbf{M}}\boldsymbol{\tau}&=(\mathbf{M}\boldsymbol{\tau}^\wedge\mathbf{M}^{-1})^\vee=\cdots=\\ &=\left(\begin{bmatrix}\mathbf{R}[\boldsymbol{\theta}]_\times\mathbf{R}^\top & -\mathbf{R}[\boldsymbol{\theta}]_\times\mathbf{R}^\top\mathbf{t}+\mathbf{R}\boldsymbol{\rho}\\ \mathbf{0}&0\end{bmatrix}\right)^\vee\\ &=\left(\begin{bmatrix}[\mathbf{R}\boldsymbol{\theta}]_\times & [\mathbf{t}]_\times\mathbf{R}\boldsymbol{\theta}+\mathbf{R}\boldsymbol{\rho}\\ \mathbf{0}&0\end{bmatrix}\right)^\vee\\ &=\begin{bmatrix}[\mathbf{t}]_\times\mathbf{R}\boldsymbol{\theta}+\mathbf{R}\boldsymbol{\rho}\\ \mathbf{R}\boldsymbol{\theta}\end{bmatrix}=\begin{bmatrix}\mathbf{R}&[\mathbf{t}]_\times\mathbf{R}\\ \mathbf{0}&\mathbf{R}\end{bmatrix}\begin{bmatrix}\boldsymbol{\rho}\\ \boldsymbol{\theta}\end{bmatrix}\end{aligned}$$

where we used $[\mathbf{R}\boldsymbol{\theta}]_\times=\mathbf{R}[\boldsymbol{\theta}]_\times\mathbf{R}^\top$ and $[\mathbf{a}]_\times\mathbf{b}=-[\mathbf{b}]_\times\mathbf{a}$. So the adjoint matrix is

$$\mathbf{Ad}_{\mathbf{M}}=\begin{bmatrix}\mathbf{R}&[\mathbf{t}]_\times\mathbf{R}\\ \mathbf{0}&\mathbf{R}\end{bmatrix}\quad\in\mathbb{R}^{6\times6} .$$

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Maurice Fréchet, papier de 1943 et cours de l'IHP de l'Hiver 1939

M. Fréchet, Sur l'extension de certaines evaluations statistiques au cas de petits échantillons, Revue de l'Institut International de Statistique, Vol. 11, No. 3/4 (1943), pp. 182-205

SUR L'EXTENSION DE CERTAINES EVALUATIONS STATISTIQUES AU CAS DE PETITS ECHANTILLONS

par Maurice Fréchet.

Introduction.

Ce mémoire [1]) est consacré à l'extension au cas de petits échantillons de la méthode de détermination empirique d'un paramètre basée sur le principe de la moindre dispersion et à sa comparaison avec les méthodes basées sur le principe de la valeur dominante et sur celui de la plus grande plausibilité.

Si nous nous en étions tenus aux démonstrations, nous aurions pu abréger sensiblement ce mémoire. Mais il nous a paru nécessaire d'entrer dans plus de détails qu'on ne le fait généralement, afin de séparer plus nettement des déductions mathématiques, les hypothèses et les conventions sur lesquelles elles reposent et dont le choix, aussi plausible que possible, n'a cependant rien de nécessaire.

Notations. — Soient $X_1, X_2, \ldots. X_n$, n valeurs prises par une variable aléatoire X au cours de n épreuves indépendantes. On se limitera, dans la suite, au cas où la loi de répartition de X peut s'exprimer par une probabilité élémentaire $\delta\, dx$ et où, de plus, la densité de probabilité δ en un point x est une fonction d'une forme connue $f(x, \theta)$, dépendant d'un paramètre dont la valeur vraie θ_0 est inconnue.

On se propose d'évaluer θ_0 connaissant d'une part, la forme $f(x, \theta)$ de δ et, d'autre part, les résultats de n épreuves qui ont donné les valeurs numériques $x_1, x_2, \ldots. x_n$ à $X_1, X_2, \ldots. X_n$. Sous cette forme stricte, le problème *ne peut être résolu* par de simples déductions mathématiques.

Il s'agit donc de fixer *certaines conventions plausibles* qui assigneront à tout „échantillon" de n valeurs $x_1, x_2, \ldots. x_n$ de X une valeur déterminée t, laquelle sera prise comme valeur empirique de la valeur vraie θ_0. t est donc une certaine fonction convenablement choisie de $x_1, x_2, \ldots. x_n$

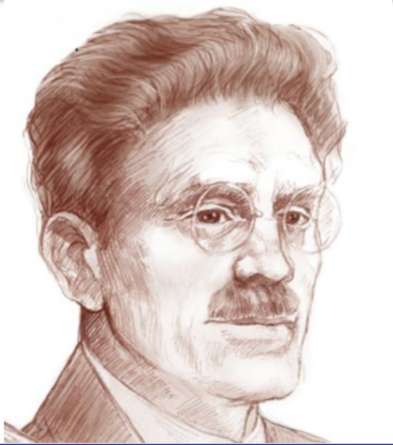
(1) $$t = H_n(x_1, \ldots. x_n).$$

On voit que chaque échantillon détermine t, de sorte que

(2) $$T_n = H_n(X_1, \ldots. X_n)$$

est une variable aléatoire dont chaque échantillon détermine une valeur. (Quand, dans nos raisonnements, n sera fixe, nous écrirons, pour simplifier, H au lieu de H_n et T au lieu de T_n).

[1]) Le contenu de ce mémoire a formé une partie de notre cours de statistique mathématique à l'Institut Henri Poincaré pendant l'hiver 1939—1940. Il constitue l'un des chapitres du deuxième cahier (en préparation) de nos „Leçons de Statistique Mathématique", dont le premier cahier „Introduction: Exposé préliminaire de Calcul des Probabilités" (119 pages in-quarto, dactylographiées) vient de paraitre au „Centre de Documentation Universitaire", Tournois et Constans, Paris.



Manuscrit perdu du cours de statistique mathématique à l'Institut Henri Poincaré pendant l'Hiver 1939-1940 !

[1]) Le contenu de ce mémoire a formé une partie de notre cours de statistique mathématique à l'Institut Henri Poincaré pendant l'hiver 1939—1940. Il constitue l'un des chapitres du deuxième cahier (en préparation) de nos „Leçons de Statistique Mathématique", dont le premier cahier „Introduction: Exposé préliminaire de Calcul des Probabilités" (119 pages in-quarto, dactylographiées) vient de paraitre au „Centre de Documentation Universitaire", Tournois et Constans, Paris.

https://www.jstor.org/stable/1401114

THALES

Maurice Fréchet 1943 Seminal Paper (Clairaut Equation)

Fréchet, M. Sur l'extension de certaines évaluations statistiques au cas de petits échantillons. Revue de l'Institut International de Statistique 1943, 11, 182–205.

Etude des densités distinguées. Appelons (provisoirement, dans ce mémoire) *densité distinguée*, toute densité de probabilité $f(x, \theta)$ telle que la fonction

(46) $$\theta + \frac{\dfrac{\partial L f(x,\theta)}{\partial \theta}}{\displaystyle\int_{-\infty}^{+\infty} \left[\frac{\partial}{\partial \theta} f(x,\theta)\right]^2 \frac{dx}{f(x,\theta)}}$$

soit indépendante de θ.

Pour ces densités distinguées, on va pouvoir déterminer la fonction minimisante $H'(X_1, \ldots. X_n)$ et étendre au cas des petits échantillons la comparaison des méthodes d'estimation faites par divers auteurs dans le cas des grands échantillons. Il vaut donc la peine de chercher la forme générale de $f(x, \theta)$ pour cette catégorie de variables.

de θ. En appelant $h(x)$ cette fonction, on voit qu'on a l'identité de la forme

(47) $$\lambda(\theta) \frac{\partial}{\partial \theta} L f(x,\theta) = h(x) - \theta$$

où $\lambda(\theta) > 0$. On peut considérer $\frac{1}{\lambda(\theta)}$ comme la dérivée seconde d'une fonction $\mu(\theta)$; d'où $\frac{\partial}{\partial \theta} L f(x,\theta) = \mu_\Theta''(\theta)\ [h(x) - \theta]$.

Par suite $L f(x,\theta) - \mu'_\Theta\ [h(x) - \theta] - \mu(\theta)$ est une quantité indépendante de θ que nous pouvons représenter par $l(x)$.

Ainsi toute densité distinguée, $f(x, \theta)$, est de la forme

(48) $$f(x,\theta) = e^{\mu'_\Theta [h(x) - \Theta] + \mu(\Theta) + l(x)}$$

(52bis) $$\lambda \mu'' = 1.$$

Incidemment, puisque, d'après (52), $\lambda(\theta)$ est positif, il en résulte aussi que $\mu''\left(= \frac{1}{\lambda(\theta)}\right)$ est aussi positif. **Métrique de Fisher**

On peut d'ailleurs préciser d'une manière plus directe que par (50), le choix des fonctions $\mu(\theta)$, $h(x)$, $l(x)$: *on peut prendre arbitrairement $h(x)$ et $l(x)$* [1]) *et alors $\mu(\theta)$ est déterminé par* (50) ou même mieux *par une formule explicite.* En effet, (50) peut s'écrire

$$e^{\Theta\mu' - \mu} = \int_{-\infty}^{+\infty} e^{\mu'_\Theta h(x) + l(x)}\, dx.$$

Donnons-nous alors arbitrairement $h(x)$ et $l(x)$ et soit s une variable arbitraire : la fonction

$$\int_{-\infty}^{+\infty} e^{s h(x) + l(x)}\, dx \qquad ^{1})$$

sera une fonction positive connue que nous pourrons représenter par $e^{\Psi(s)}$. On voit alors que $\mu(\theta)$ sera défini par

$$\theta \mu' - \mu = \psi(\mu') \quad \text{ou}$$

(55) $$\mu = \theta \mu' - \psi(\mu')$$ **Legendre-Clairaut**

c'est-à-dire une équation de Clairaut. La solution μ' = constante réduirait $f(x, \theta)$, d'après (48) à une fonction indépendante de θ, cas où le problème n'aurait plus de sens. μ est donc donné par la solution singulière de (55), qui est unique et s'obtient en éliminant s entre $\mu = \theta s - \psi(s)$ et $\theta = \psi'(s)$ ou encore entre

$$e^{\Theta s - \mu} = \int_{-\infty}^{+\infty} e^{s h(x) + l(x)}\, dx \text{ et}$$

(55bis) $$\int_{-\infty}^{+\infty} e^{s h(x) + l(x)} [h(x) - \theta]\, dx = 0.$$

Si l'on veut, $\mu(\theta)$ est donné par la relation

$$e^{-\mu} = e^{-\Theta s} \int_{-\infty}^{+\infty} e^{s h(x) + l(x)}\, dx$$

où s est donné en fonction de θ par la relation implicite (55bis).

Calyampudi Radhakrishna Rao, papier de 1945 (2 ans après le papier et 6 ans après le cours de Maurice Fréchet)

C.R. Rao, “Information and accuracy attainable in the estimation of statistical parameters”, Bulletin of the Calcutta Mathematical Society, Vol.37, No.3, pp.81–91, 1945

Information and the Accuracy Attainable in the Estimation of Statistical Parameters

C Radhakrishna Rao

(Communicated by Mr. R C Bose–Received August 23, 1945

Introduction

The earliest method of estimation of statistical parameters is the method of least squares due to Markoff. A set of observations whose expectations are linear functions of a number of unknown parameters being given, the problem which Markoff posed for solution is to find out a linear function of observations whose expectation is an assigned linear function of the unknown parameters and whose variance is a minimum. There is no assumption about the distribution of the observations except that each has a finite variance.

A significant advance in the theory of estimation is due to Fisher (1921) who introduced the concepts of *consistency, efficiency and sufficiency* of estimating functions and advocated the use of the maximum likelihood method. The principle accepts as the estimate of an unknown parameter θ, in a probability function $\phi(\theta)$ of an assigned type, that function $t(x_1, \ldots, x_n)$ of the sampled observations which makes the probability density a maximum. The validity of this principle arises from the fact that out of a large class of unbiased estimating functions following the normal distribution the function given by maximising the probability density has the least variance. Even when the distribution of t is not normal the property of minimum variance tends to hold as the size of the sample is increased.

Rao et Fréchet au Colloque de 1948

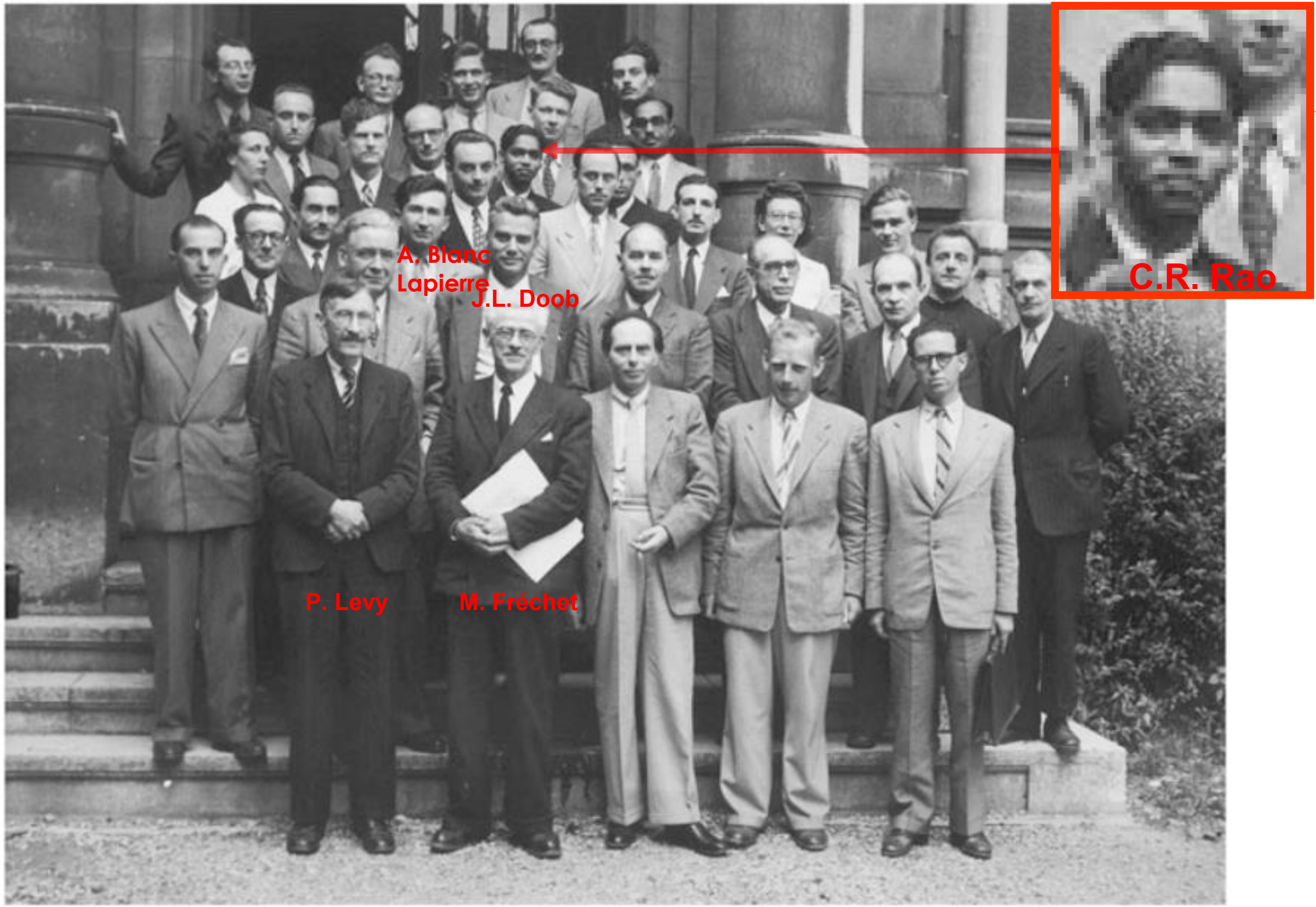


Fig. 9 Colloque International sur le Calcul des Probabilités, Lyon 1948. First row: Paul Lévy and Maurice Fréchet. On the picture one can find among others J. Doob, R. Fortet, D. Van Dantzig, E. Mourier, J. Kampé de Fériet, A. Blanc-Lapierre. . . . (Photo: © Private collection F. Lederer)

Axiomatization of Information Geometry via Category Theory
Nikolai Nikolaevich Chentsov in 60's

Math. Operationsforsch. Statist., Ser. Statistics, Vol. 9 (1978) No. 2, 267–276

Algebraic Foundation of Mathematical Statistics[2]

N. N. Čencov[1].

It results from the following sentence which is a non-symmetric analogy to Pyphagor's theorem for the information deviation [10], [6].

Theorem 6. *Let the probability measures R be dominated by* $\{P_s\}$*, which form the exponential family. If there is such distribution* $P_\sigma \in \{P_s\}$ *that*

$$\int_\Omega \left[\ln \frac{dP_{s'}}{dP_{s''}}(\omega)\right](R - P_\sigma)\,(d\omega) = 0\,, \qquad (21)$$

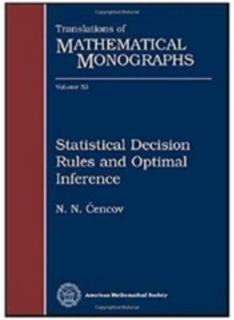
whatever $P_{s'}, P_{s''} \in \{P_s\}$ *are, then*

$$I(R: P_s) = I(R: P_\sigma) + I(P_\sigma: P_s), \quad \forall P_s\,. \qquad (22)$$

Corollary. *The measure* P_σ *is the I-nearest to R point of the exponential family.*

Recently a number of proposals for the arising non-symmetric Pyphagoric information geometry has been recovered by Csiszar [11].

Nikolai Nikolaevich Chentsov (1930-1992)



Fisher Metric and Fréchet-Darmois (Cramer-Rao) Bound

- Cramer-Rao –Fréchet-Darmois Bound has been introduced by Fréchet in 1939 and by Rao in 1945 as inverse of the Fisher Information Matrix: $I(\theta)$

$$R_{\hat{\theta}} = E\left[\left(\theta - \hat{\theta}\right)\left(\theta - \hat{\theta}\right)^{+}\right] \geq I(\theta)^{-1} \qquad \left[I(\theta)\right]_{i,j} = -E\left[\frac{\partial^2 \log p_\theta(z)}{\partial \theta_i \partial \theta_j^*}\right]$$

- Rao has proposed to introduced an invariant metric in parameter space of density of probabilities (axiomatised by N. Chentsov):

$$ds_\theta^2 = Kullback\_Divergence\left(p_\theta(z), p_{\theta+d\theta}(z)\right)$$

$$ds_\theta^2 = -\int p_\theta(z) \log \frac{p_{\theta+d\theta}(z)}{p_\theta(z)} dz$$

$$ds_\theta^2 \underset{Taylor}{\approx} \sum_{i,j} g_{ij} d\theta_i d\theta_j^* = \sum_{i,j} \left[I(\theta)\right]_{i,j} d\theta_i d\theta_j^* = d\theta^+ . I(\theta) . d\theta$$

$$w = W(\theta) \Rightarrow ds_w^2 = ds_\theta^2$$

Distance Between Gaussian Density with Fisher Metric

Fisher Matrix for Gaussian Densities:

$$I(\theta) = \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{2}{\sigma^2} \end{bmatrix} \quad \text{avec} \quad E\left[\left(\theta - \hat{\theta}\right)\left(\theta - \hat{\theta}\right)^T\right] \geq I(\theta)^{-1} \quad \text{et} \quad \theta = \begin{pmatrix} m \\ \sigma \end{pmatrix}$$

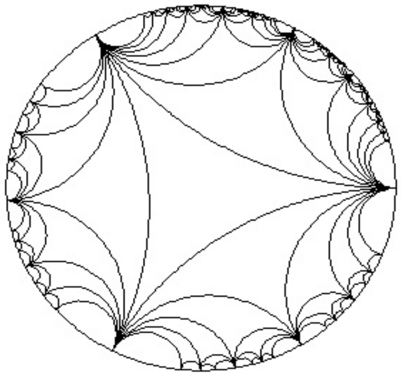
- Fisher matrix induced the following differential metric :

$$ds^2 = d\theta^T . I(\theta) . d\theta = \frac{dm^2}{\sigma^2} + 2.\frac{d\sigma^2}{\sigma^2} = \frac{2}{\sigma^2}\left[\left(\frac{dm}{\sqrt{2}}\right)^2 + (d\sigma)^2\right]$$

- Poincaré Model of upper half-plane and unit disk

$$z = \frac{m}{\sqrt{2}} + i.\sigma \qquad \omega = \frac{z - i}{z + i} \quad \left(|\omega| < 1\right)$$

$$\Rightarrow ds^2 = 8.\frac{|d\omega|^2}{\left(1 - |\omega|^2\right)^2}$$



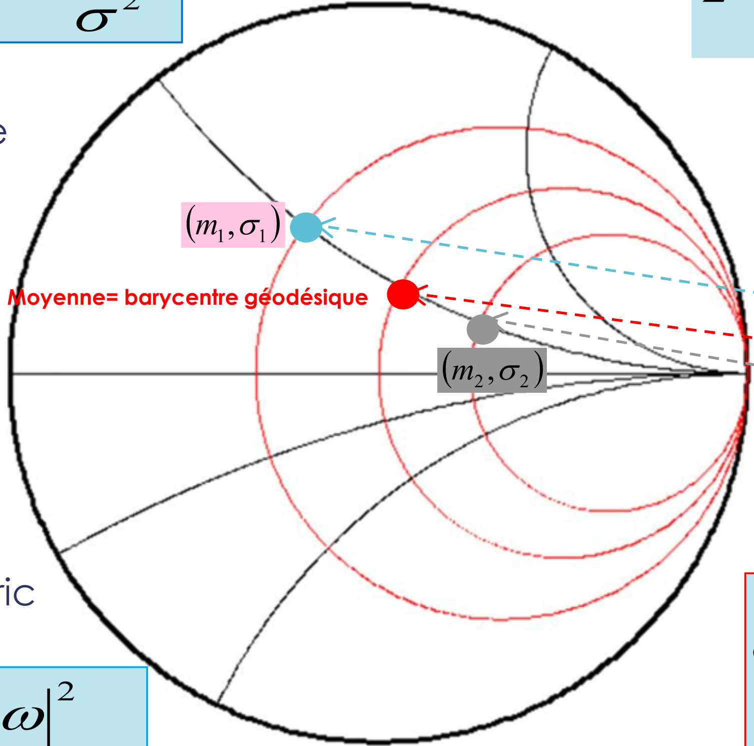
1 monovariate gaussian = 1 point in Poincaré unit disk

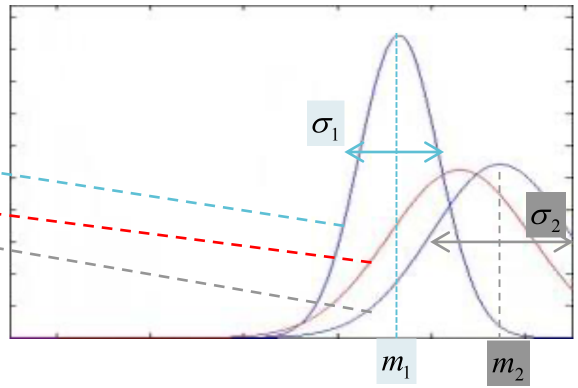
$$ds^2 = \frac{dm^2}{\sigma^2} + 2.\frac{d\sigma^2}{\sigma^2}$$

Fisher Metric in
Poincaré Half-Plane

$$z = \frac{m}{\sqrt{2}} + i.\sigma$$

$$\omega = \frac{z-i}{z+i} \quad \left(|\omega| < 1\right)$$





Poincaré-Fisher metric
In Unit Disk

$$ds^2 = 8.\frac{|d\omega|^2}{\left(1-|\omega|^2\right)^2}$$

$$d^2\left(\{m_1,\sigma_1\},\{m_2,\sigma_2\}\right) = 2.\left(\log \frac{1+\delta(\omega^{(1)},\omega^{(2)})}{1-\delta(\omega^{(1)},\omega^{(2)})}\right)^2$$

$$\text{with} \quad \delta(\omega^{(1)},\omega^{(2)}) = \left|\frac{\omega^{(1)}-\omega^{(2)}}{1-\omega^{(1)}\omega^{(2)*}}\right|$$

Information Geometry & Natural Gradient

- This simple gradient descent has a first drawback of using the same non-adaptive learning rate for all parameter components, and a second drawback of non invariance with respect to parameter re-encoding inducing different learning rates. **S.I. Amari** has introduced the **natural gradient** to preserve this invariance to be insensitive to the characteristic scale of each parameter direction. The gradient descent could be corrected by $I(\theta)^{-1}$ where I is the **Fisher information matrix** with respect to parameter θ, given by:

$$\theta_t \leftarrow \theta_{t-1} - \eta_t I(\theta_{t-1})^{-1} \frac{\partial l_t(y_t)^T}{\partial \theta}$$

$$I(\theta) = \left[g_{ij} \right]$$

$$\text{with } g_{ij} = \left[-E_{y \approx p(y/\theta)} \left[\frac{\partial^2 \log p(y/\theta)}{\partial \theta_i \partial \theta_j} \right] \right]_{ij} = \left[E_{y \approx p(y/\theta)} \left[\frac{\partial \log p(y/\theta)}{\partial \theta_i} \frac{\partial \log p(y/\theta)}{\partial \theta_j} \right] \right]_{ij}$$

Natural Gradient & Stochastic Gradient: Natural Langevin Dynamics

Natural Langevin Dynamics: Natural Gradient with Langevin Stochastics descent

- To regularize solution and avoid over-fitting, Stochastic gradient is used, as Langevin Stochastic Gradients
- **Yann Ollivier** (FACEBOOK FAIR, previously CNRS LRI Orsay) and **Gaëtan Marceau-Caron** (MILA, previously CNRS LRI Orsay and THALES LAS/ATM & TRT PhD) have proposed to coupled **Natural Gradient** with **Langevin Dynamics**: **Natural Langevin Dynamics (Best SMF/SEE GSI'17 paper)**

$$\theta_t \leftarrow \theta_{t-1} - \eta_t I(\theta_{t-1})^{-1} \frac{\partial\left(l_t(y_t)^T - \frac{1}{N}\log\alpha(\theta_{t-1})\right)}{\partial\theta} + \sqrt{\frac{2\eta_t}{N}} I(\theta_{t-1})^{-1/2} N(0, I_d)$$

- The resulting natural Langevin dynamics combines the advantages of Amari's natural gradient descent and Fisher-preconditioned Langevin dynamics for large neural networks

Natural Gradient of Information Geometry for LLM (Large Language Model) by GOOGLE LLC

FAdam: Adam is a natural gradient optimizer using diagonal empirical Fisher information

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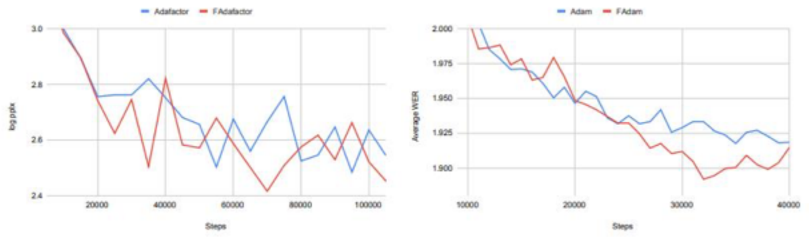
Abstract

This paper establishes a mathematical foundation for the Adam optimizer, elucidating its connection to natural gradient descent through Riemannian and information geometry. We rigorously analyze the diagonal empirical Fisher information matrix (FIM) in Adam, clarifying all detailed approximations and advocating for the use of log probability functions as loss, which should be based on discrete distributions, due to the limitations of empirical FIM. Our analysis uncovers flaws in the original Adam algorithm, leading to proposed corrections such as enhanced momentum calculations, adjusted bias corrections, adaptive epsilon, and gradient clipping. We refine the weight decay term based on our theoretical framework. Our modified algorithm, **Fisher Adam (FAdam)**, demonstrates superior performance across diverse domains including LLM, ASR, and VQ-VAE, achieving state-of-the-art results in ASR.

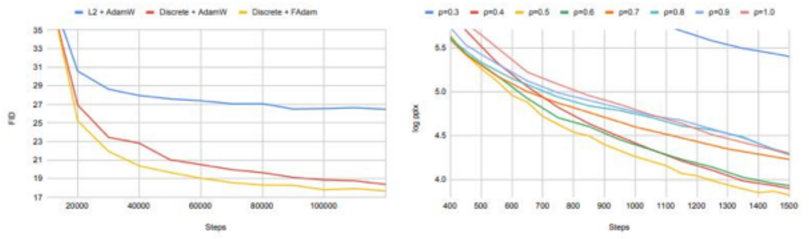
$$\theta_{t+1} = \theta_t - \eta F^{-1} \nabla \mathcal{L}(\theta)$$

LibriSpeech WERs	dev	test	dev-other	test-other	avg
Adam (w2v-BERT paper [44])	1.30	2.60	1.40	2.70	2.00
Adam	1.30	2.54	1.33	2.59	1.93
FAdam	1.29	2.49	1.34	2.49	**1.90**

Table 1: LibriSpeech WERs



(a) LLM 1B (text domain) (b) ASR 600M (speech domain)



(c) VQ-VAE 100M (image domain) (d) Exponent sweep

Figure 1: Comparison of FAdam and Adam performance. (a) Eval loss (log pplx) on 1B LLMs, presenting FAdafactor outperforms Adafactor. (b) Average WER on LibriSpeech using 600M Conformer models, presenting FAdam outperforms Adam. (c) FID of ImageNet generation using 100M VQ-VAE models, presenting FAdam outperforms AdamW. (d) Comparison of FIM exponents on a 1B LLM, showing 0.5 (square root) as the optimal choice.

THALES

Building a future we can all trust

Jean-Marie Souriau: Symplectic Foliation via Moment Map and Koszul-Fisher Metric from Information Geometry for Lie Groups Thermodynamics

SOURIAU 2019

SOURIAU 2019

- Internet website : http://souriau2019.fr
- In 1969, 50 years ago, Jean-Marie Souriau published the book "**Structure des système dynamiques**", in which using the ideas of J.L. Lagrange, he formalized the "**Geometric Mechanics**" in its modern form based on **Symplectic Geometry**
- Chapter IV was dedicated to "Thermodynamics of Lie groups" (ref André Blanc-Lapierre)
- Testimony of **Jean-Pierre Bourguignon** at Souriau'19 (IHES director of the European ERC)

https://www.youtube.com/watch?v=93hFolIBo0Q&t=3s

SOURIAU 2019

Conference May 27-31 2019, Paris-Diderot University

https://www.youtube.com/watch?v=beM2pUK1H7o

Jean-Marie Souriau @ ENS Ulm 1942

Jean-Marie Souriau

- Structure des systèmes dynamiques

Jacques Dixmier

- Algèbres enveloppantes

René Deheuvels

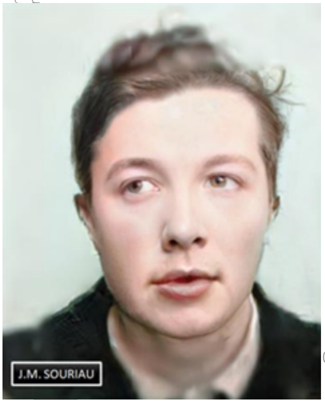
- Formes quadratiques et groupes classiques
- Tenseurs et spineurs

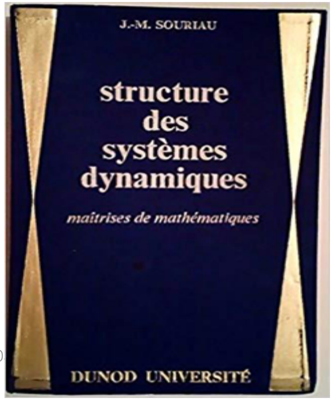
Machine Learning on Lie Groups

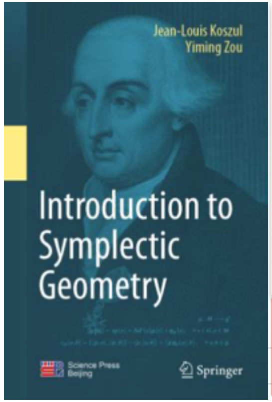
Lie Groups Machine Learning based on Souriau Lie-Groups Thermodynamics

- We will introduce "**Lie Groups Machine Learning**" [1] that extends statistics and machine learning on Lie Groups based on representation theory and Lie algebra cohomology.
- From Jean-Marie Souriau "**Lie Groups Thermodynamics**" [3] initiated in the framework of symplectic model of statistical mechanics, new geometric statistical tools have been developed to define:

- maximum Entropy densities of probability on Lie Groups for supervised methods
- Fisher metric extension from Information Geometry for unsupervised methods.







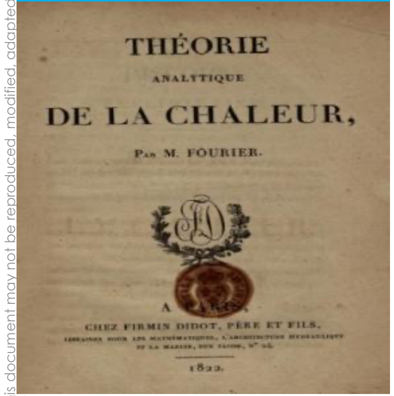
« Il n'y a rien de plus dans les théories physiques que les groupes de symétrie si ce n'est la construction mathématique qui permet précisément de montrer qu'il n'y a rien de plus » - **Jean-Marie Souriau**

Souriau's Moment Map = Geometrization of Noether Theorem

Souriau Lie Groups Thermodynamics: General Equations

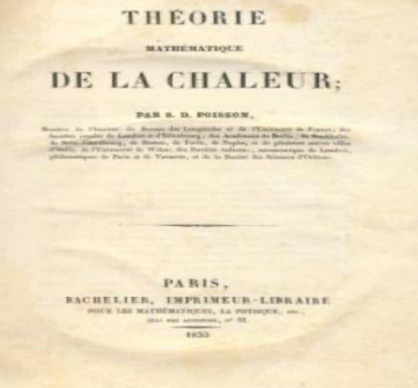
Ces formules sont universelles, en ce sens qu'elles ne mettent pas en jeu la variété symplectique U - mais seulement le groupe G, son cocycle symplectique f et les couples Θ, Q. Peut-être cette "thermodynamique des groupes de Lie" a-t-elle un intérêt mathématique.

FOURIER
Analytical Theory of Heat

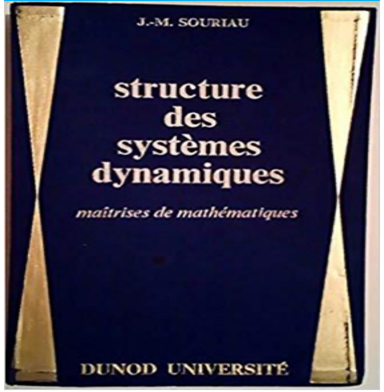


CLAUSIUS
Mechanical Theory of Heat

POISSON
Mathematical Theory of Heat



SOURIAU
Geometric Theory of Heat



SYMPLECTIC THEORY OF HEAT

Souriau SSD Chapter IV: Gibbs Equilibrium is not covariant with respect to Dynamic Groups of Physics

MÉCANIQUE STATISTIQUE COVARIANTE

Le groupe des translations dans le temps (7.9) est un sous-groupe du groupe de Galilée ; mais *ce n'est pas un sous-groupe invariant*, ainsi que le montre un calcul trivial. Si un système dynamique est *conservatif* dans un repère d'inertie, il en résulte qu'il peut *ne plus être conservatif dans un autre*. La formulation (17.24) du principe de Gibbs doit donc être élargie, pour devenir compatible avec la relativité galiléenne.

Nous proposons donc le principe suivant :

(17.77) Si un système dynamique est invariant par un sous-groupe de Lie G' du groupe de Galilée, les équilibres naturels du système constituent l'ensemble de Gibbs du groupe dynamique G'.

Soit $\mathcal{G}'$ l'algèbre de Lie G' ; on sait que $\mathcal{G}'$ est une sous-algèbre de Lie de celle de G, notée $\mathcal{G}$; un équilibre du système sera caractérisé par un élément Z de $\mathcal{G}'$, donc de $\mathcal{G}$; on pourra écrire

(17.78)
$$Z = \begin{bmatrix} j(\omega) & \beta & \gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{bmatrix}$$

en utilisant les notations (13.4) ; Z parcourt l'ensemble Ω défini en (16.219) ; à chaque valeur de Z est associé un élément M du dual $\mathcal{G}'^*$ de $\mathcal{G}'$, valeur moyenne du moment μ ; on peut appliquer les formules (16.219), (16.220), qui généralisent les relations thermodynamiques (17.26), (17.27), (17.28).

(17.79) On voit que c'est Z (17.78) qui généralise la « température » ; le théorème d'isothermie (17.32) s'étend immédiatement : l'équilibre d'un système composé de plusieurs parties sans interactions s'obtient en attribuant à chaque composante un équilibre *correspondant à la même valeur de* Z ; l'entropie s, le potentiel de Planck z et le moment moyen M sont *additifs*. W

J.M. Souriau, Structure des systèmes dynamiques, **Chapitre IV « Mécanique Statistique »**

Trompette de Souriau

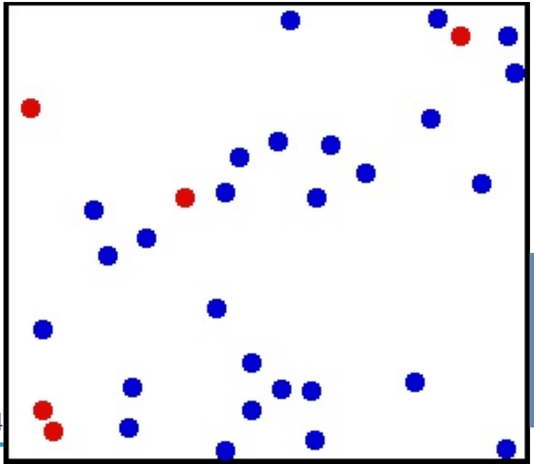
Lorsque le fait qu'on rencontre est en opposition avec une théorie régnante, il faut accepter le fait et abandonner la théorie, alors même que celle-ci, soutenue par de grands noms, est généralement adoptée
- Claude Bernard "Introduction à l'Étude de la Médecine Expérimentale"

Symplectic Model of Souriau Lie Groups Thermodynamics

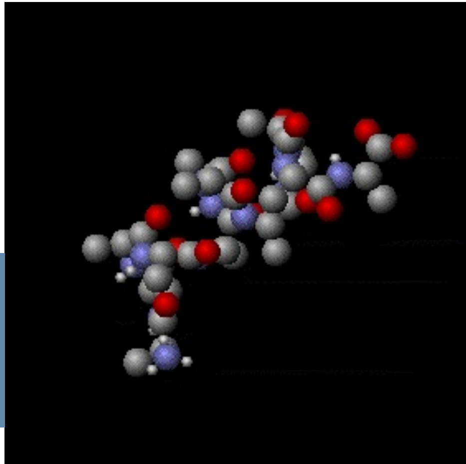
- Souriau Geometric (Planck) Temperature is **an element of Lie Algebra** of Dynamical Group (Galileo/Poincaré groups) acting on the system

- Generalized Entropy is **Legendre Transform of minus logarithm of Laplace Transform** and is **Casimir Function on Symplectic leaves (obtained by coadjoint orbit via moment map)**

- Fisher(-Souriau) Metric is a **Geometric Calorific Capacity (hessian of Massieu Potential)**



Souriau formalism is fully **covariant**, with no special coordinates (**covariance of Gibbs density wrt Dynamical Groups**)



Jean-Marie Souriau Seminal Paper - 1974

Statistical Mechanics, Lie Group and Cosmology - 1st part: Symplectic Model of Statistical Mechanics

Jean-Marie Souriau

Abstract: The classical notion of Gibbs' canonical ensemble is extended to the case of a symplectic manifold on which a Lie group has a symplectic action ("dynamic group"). The rigorous definition given here makes it possible to extend a certain number of classical thermodynamic properties (temperature is here an element of the Lie algebra of the group, heat an element of its dual), notably inequalities of convexity. In the case of non-commutative groups, particular properties appear: the symmetry is spontaneously broken, certain relations of cohomological type are verified in the Lie algebra of the group. Various applications are considered (rotating bodies, covariant or relativistic statistical Mechanics). [These results specify and complement a study published in an earlier work (*), which will be designated by the initials SSD].

(*) Souriau, J.-M., Structure des systèmes dynamique. Dunod, collection Dunod Université, Paris 1969. http://www.jmsouriau.com/structure_des_systemes_dynamiques.htm

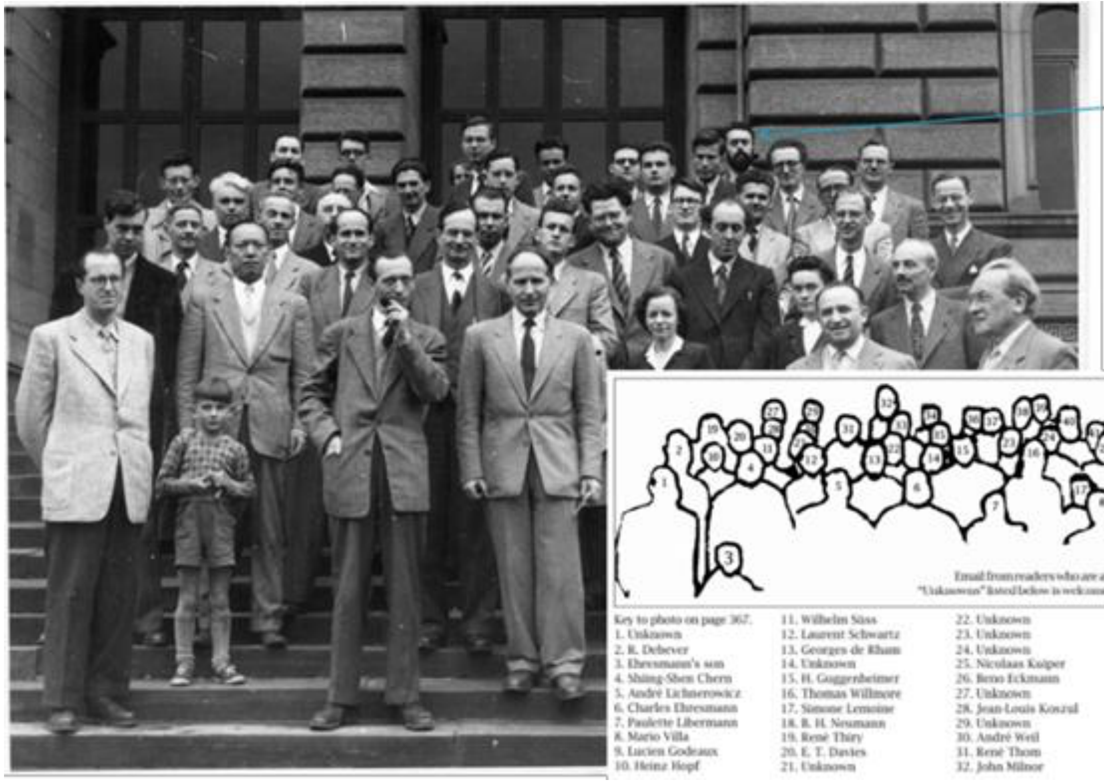
Souriau, J-M., Mécanique statistique, groupes de Lie et cosmologie, Colloques Internationaux C.N.R.S., n°237 – Géométrie symplectique et physique mathématique, pp.59-113, 1974

English translation by F. Barbaresco:
https://www.academia.edu/42630654/Statistical_Mechanics_Lie_Group_and_Cosmology_1_st_part_Symplectic_Model_of_Statistical_Mechanics

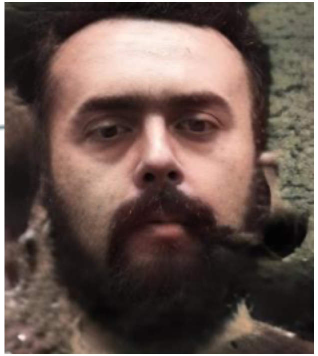
Symplectic Geometry & Foliations Theory Epic

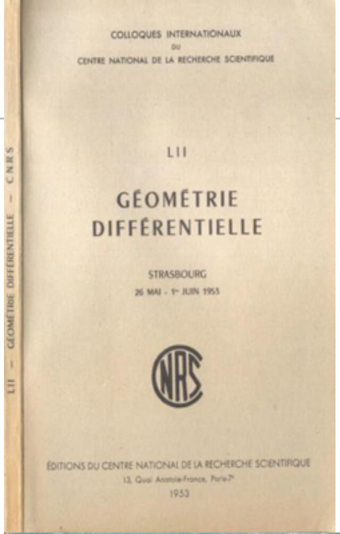
F. Barbaresco, *Esthétique structurelle des feuilletages symplectiques du mouvement, de la chaleur et de l'information : la quête romantique de Jean-Marie Souriau de Carthage à Massilia ou le triptyque de la Nature des choses (De Rerum Natura) « Esthétisme – Structure – Mouvement »*, ENS/IRCAM MAMUPHI seminar, to be published by Spartacus Editor for MAMUPHI anniversary https://spartacus-idh.com/
MAMUPHI video: https://www.youtube.com/watch?v=dWyWXubGfXA
MAMUPHI Slides: http://www.entretemps.asso.fr/2022-2023/Barbaresco.pdf
https://www.academia.edu/112471996/Structural_aesthetics_of_the_symplectic_foliations_of_movement_heat_and_information_the_romantic_quest_of_Jean_Marie_Souriau_from_Carthage_to_Massilia_or_the_triptych_of_the_Nature_of_things_De_Rerum_Natura_Aesthetics_Structure_Movement_

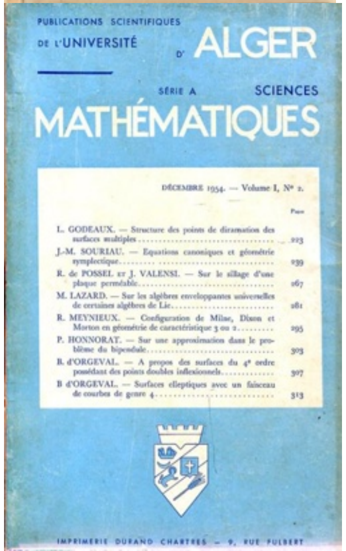
Strasbourg 1953


Email from readers who are able to identify any of the "Unknowns" listed below is welcome at out.i.caffans.org.

Key to photo on page 367.
1. Unknown
2. R. Debever
3. Ehresmann's son
4. Shiing-Shen Chern
5. André Lichnerowicz
6. Charles Ehresmann
7. Paulette Libermann
8. Mario Villa
9. Lucien Godeaux
10. Heinz Hopf
11. Wilhelm Süss
12. Laurent Schwartz
13. Georges de Rham
14. Unknown
15. H. Guggenheimer
16. Thomas Willmore
17. Simone Lemoine
18. B. H. Neumann
19. René Thiry
20. E. T. Davies
21. Unknown
22. Unknown
23. Unknown
24. Unknown
25. Nicolaas Kuiper
26. Beno Eckmann
27. Unknown
28. Jean-Louis Koszul
29. Unknown
30. André Weil
31. René Thom
32. John Milnor
33. Marcel Berger
34. Unknown
35. Bernard Malgrange
36. Daniel Bernard
37. André Aragnol
38. G. Legrand
39. Jean-Marie Souriau
40. Unknown
41. Georges Reeb
42. Unknown






PUBLICATIONS SCIENTIFIQUES DE L'UNIVERSITÉ D'ALGER

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DÉCEMBRE 1954. — Volume I, N° 2.

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IMPRIMERIE DURAND CHARTRES — 9, RUE FULBERT

Lagrange 2-form rediscovered by Jean-Marie Souriau

- Rewriting equations of classical mechanics in phase space $\begin{pmatrix} r \\ v \end{pmatrix}$

$$m\frac{d^2r}{dt^2}=F \quad \Longrightarrow \quad m\frac{dv}{dt}=F \quad \text{et} \quad v=\frac{dr}{dt}$$

- Souriau rediscovered that Lagrange had considered the evolution space: $y=\begin{pmatrix} t \\ r \\ v \end{pmatrix}\in V$

$$\begin{cases} m\delta v - F\delta t = 0 \\ \delta r - v\delta t = 0 \end{cases}$$

- **A dynamic system is represented by a foliation**. This foliation is determined by an antisymmetric covariant 2nd order tensor σ, called the Lagrange (-Souriau) form, a bilinear operator on the tangent vectors of V.

$$\sigma(\delta y)(\delta' y)=\langle m\delta v - F\delta t, \delta' r - v\delta' t\rangle - \langle m\delta' v - F\delta' t, \delta r - v\delta t\rangle \quad \delta y=\begin{pmatrix} \delta t \\ \delta r \\ \delta v \end{pmatrix} \quad \text{et} \quad \delta' y=\begin{pmatrix} \delta' t \\ \delta' r \\ \delta' v \end{pmatrix}$$

- *In the Lagrange-Souriau model, σ is a 2-form on the evolution space V, and the differential equation of motion implies:* $\delta y \in \varepsilon$

$$\sigma(\delta y)(\delta' y)=0 \ , \ \forall \delta' y \qquad \sigma(\delta y)=0 \ \text{ ou } \ \delta y\in\ker(\sigma)$$

Souriau Work Roots: François Gallissot Theorem

- **Gallissot Theorem**: There are 3 types of differential forms generating the equations of a material point motion, **invariant by the action of the Galileo group**

$$A:\begin{cases} s=\dfrac{1}{2m}\displaystyle\sum_{i=1}^{3}\left(mdv_i-F_idt\right)^2 \\ e=\dfrac{m}{2}\displaystyle\sum_{j=1}^{3}\left(dx_j-v_jdt\right)^2 \end{cases}$$

F. Gallissot, Les formes extérieures en Mécanique (*Thèse*), Durand, Chartres, 1954.

$$B: f=\sum_{1}^{3}\delta_{ij}\left(dx_i-v_idt\right)\left(mdv_j-F_jdt\right)\ \text{with}\ \delta_{ij}\ \text{kröneckersymbol}$$

$$C:\ \omega=\sum_{1}^{3}\delta_{ij}\left(mdv_i-F_idt\right)\wedge\left(dx_j-v_jdt\right)$$

- $d\omega=0$ constrained the Pfaff form $\delta_{ij}F_idx_j$ to be closed and to be reduced to the differential of U : $C\Rightarrow\omega=m\delta_{ij}dv_i\wedge dx_j-dH\wedge dt$ with $H=T-U$ and $T=1/2\sum_{i=1}^{3}m\left(v_i\right)^2$
- It proves that ω has an exterior differential $d\omega$ generating **Poincaré-Cartan Integral invariant:**

$$d\omega=\sum_{i=1}^{3}mv_idx_j-Hdt$$

François Gallissot Work in 1952 based on Elie and Henri Cartan works

1952

LES FORMES EXTÉRIEURES EN MÉCANIQUE

par F. GALLISSOT.

INTRODUCTION

La mécanique des systèmes paramétriques développée traditionnellement d'après les idées de Lagrange s'est toujours heurtée à des difficultés notables lorsqu'elle a désiré aborder les questions de frottement entre solides (impossibilité et indétermination) ou la notion générale de liaison (asservissement de M. Béghin), d'autre part la forme lagrangienne des équations du mouvement ne nous donne aucune indication sur la nature du problème de l'intégration.

Dans ces célèbres leçons sur les invariants intégraux Élie Cartan a montré que toutes les propriétés des équations différentielles de la dynamique des sytèmes holonomes résultaient de l'existence de l'invariant intégral $\int \omega$, $\omega = p_i dq^i - H dt$. Ainsi à tout système holonome dont les forces dérivent d'une fonction de forces est associé une forme ω, les équations du mouvement étant les caractéristiques de la forme extérieure $d\omega$. Au cours de ces dix dernières années, sous l'influence des topologistes s'est édifiée sur des bases qui semblent définitives la théorie des formes extérieures sur les variétés différentiables. Il est alors naturel de se demander si la mécanique classique ne peut pas bénéficier largement de ce courant d'idées, si elle ne peut pas être construite en plaçant à sa base une forme extérieure de degré deux, si grâce à la notion de variétés, la notion de liaison ne peut pas être envisagée sous un angle plus intelligible, si les indéterminations et impossibilités qui paraissent paradoxales dans le cadre lagrangien n'ont pas une explication naturelle, enfin s'il n'est pas possible de considérer sous un jour nouveau le problème de l'intégration des équations du mouvement, ces dernières étant engendrées par une forme Ω de degré deux.

S'affranchir de la servitude des coordonnées

$$i(E)\Omega = 0$$

Pour atteindre ces divers objectifs il m'a semblé utile de reprendre dans le chapitre I l'étude des bases logiques sur lesquelles est édifiée la mécanique galiléenne. Je montre ainsi dans le § 1 que lorsqu'on se propose de trouver des formes génératrices des équations du mouvement d'un point matériel invariantes dans les transformations du groupe galiléen, la forme la plus intéressante est une forme extérieure de degré deux définie sur une variété $V_7 = E_3 \otimes E_3 \otimes T$ (E_3 espace euclidien, T droite numérique temporelle)(1). Dans le § 11 on montre qu'à tout système paramétrique holonome à n degrés de liberté est associé une forme Ω de degré deux de rang $2n$ définie sur une variété différentiable dont les caractéristiques sont les équations du mouvement (2). Cette forme s'exprime si l'on veut au moyen de $2n$ formes de Pfaff et de dt, la forme hamiltonienne n'étant qu'un cas particulier simple. Dans le § 3 j'indique sommairement comment on peut s'affranchir de la servitude des coordonnées dans l'étude des systèmes dynamiques et le rôle important joué par l'opérateur $i(\ )$ antidérivation de M. H. Cartan (3), le champ caractéristique E de la forme Ω étant défini par la relation $i(E)\Omega = 0$.

(1) M. Kravtchenko a présenté cette conception au VIIIe Congrès de Mécanique.

(2) Dès 1946 M. Lichnerowicz au *Bulletin des Sciences Mathématiques* tome LXX, p. 90 a déjà introduit les formes extérieures pour la formation des équations des systèmes holonomes et linéairement non holonomes.

(3) M. H. Cartan, Colloque de Topologie, Bruxelles, 1950. Masson, Paris, 1951.

F. Gallissot, Les formes extérieures en Mécanique (*Thèse*), Durand, Chartres, 1954.

Interior/Exterior Products and Lie derivative

- $i_V\omega$ is the (p-1)-form on X obtained by inserting $V(x)$ as the first argument of ω :

Interior product : $i_V\omega\left(v_2,\cdots v_p\right)=\omega\left(V(x),v_2,\cdots,v_p\right)$

- $\theta\wedge\omega$ is the (p + 1)-form on X where ω is a p-form and θ is a 1-form on X:

Exterior product : $\theta\wedge\omega\left(v_0,\cdots,v_p\right)=\sum_{i=0}^{p}(-1)^i\,\theta(v_i)\omega\left(v_0,\cdots,\hat{v}_i,\cdots,v_p\right)$

(where the hat indicates a term to be omitted).

- $L_V\omega$ is a p-form on X ,and $L_V\omega=0$ if the flow of V consists of symmetries of ω:

Lie derivative : $L_V\omega\left(v_1,\cdots,v_p\right)=\left.\frac{d}{dt}e^{tV*}\omega\left(v_1,\cdots,v_p\right)\right|_{t=0}$

Exterior derivative and E.Cartan, H. Cartan & S. Lie formulas

- $d\omega$ is the (p+1)-form on X defined by taking the ordinary derivative of ω and then antisymmetrizing:

Exterior derivative : $d\omega\left(v_0,\cdots,v_p\right)=\sum_{i=0}^{p}\left(-1\right)^i\frac{\partial\omega}{\partial x}(v_i)(v_0,\cdots,\hat{v}_i,\cdots,v_p)$

$$p=0,\left[d\omega\right]_i=\partial_i\omega\ \ ;\ \ p=1,\left[d\omega\right]_{ij}=\partial_i\omega_j-\partial_j\omega_i\ \ ;\ \ p=2,\left[d\omega\right]_{ijk}=\partial_i\omega_{jk}+\partial_j\omega_{ki}+\partial_k\omega_{ij}$$

- The properties of the exterior and Lie Derivative are the following:

$L_V\omega=di_V\omega+i_Vd\omega$ (E. Cartan)

$i_{[U,V]}\omega=i_VL_U\omega-L_Ui_V\omega$ (H.Cartan)

$L_{[U,V]}\omega=L_VL_U\omega-L_UL_V\omega$ (S. Lie)

Souriau Moment Map (1/2)

- Let (X,σ) be a connected symplectic manifold.
- A vector field η on X is called symplectic if its flow preserves the 2-form :

$$L_\eta \sigma = 0$$

- If we use Elie Cartan's formula, we can deduce that :

$$L_\eta \sigma = di_\eta \sigma + i_\eta d\sigma = 0$$

- but as $d\sigma = 0$ then $di_\eta \sigma = 0$. We observe that the 1-form $i_\eta \sigma$ is closed.
- When this 1-form is exact, there is a smooth function $x \mapsto H$ on X with:

$$i_\eta \sigma = -dH$$

- This vector field η is called Hamiltonian and could be define as s symplectic gradient :

$$\eta = \nabla_{Symp} H$$

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THALES

Souriau Moment Map (2/2)

$$di_\eta \sigma = 0 \qquad i_\eta \sigma = -dH$$

- We define the Poisson bracket of two functions H , H' by :

$$\{H, H'\} = \sigma(\eta, \eta') = \sigma\left(\nabla_{Symp} H', \nabla_{Symp} H\right)$$

with $i_\eta \sigma = -dH$ and $i_{\eta'} \sigma = -dH'$

- Let a Lie group G that acts on X and that also preserve σ .

- A moment map exists if these infinitesimal generators are actually hamiltonian, so that a map exists:

$\Phi : X \to \mathfrak{g}^*$ with $i_{Z_X} \sigma = -dH_Z$ where $H_Z = \langle \Phi(x), Z \rangle$

Notation !

- I use notation as used by Koszul and Souriau which is not the most classical one

$$Ad_g^* = \left(Ad_{g^{-1}} \right)^*$$

with

$$\left\langle Ad_g^* F, Y \right\rangle = \left\langle F, Ad_{g^{-1}} Y \right\rangle, \forall g \in G, Y \in \mathfrak{g}, F \in \mathfrak{g}^*$$

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Coadjoint operator and Coadjoint Orbits (Kirillov Representation)

Lie Group Adjoint Representation

- the adjoint representation of a Lie group Ad_g is a way of representing its elements as linear transformations of the Lie algebra, considered as a vector space

$$Ad_g = \left(d\Psi_g\right)_e : \mathfrak{g} \to \mathfrak{g}$$

$$X \mapsto Ad_g(X) = gXg^{-1}$$

$$\mathrm{Ad}_g(X) = \frac{\mathrm{d}}{\mathrm{d}t} g e^{tX} g^{-1}\Big|_{t=0} \in T_e G$$

$$\Psi : G \to Aut(G)$$

$$g \mapsto \Psi_g(h) = ghg^{-1}$$

$$ad = T_e Ad : T_e G \to End\left(T_e G\right)$$

$$X, Y \in T_e G \mapsto ad_X(Y) = [X, Y]$$

Lie Group Co-Adjoint Representation

- the coadjoint representation of a Lie group Ad_g^*, is the dual of the adjoint representation ($\mathfrak{g}^*$ denotes the dual space to $\mathfrak{g}$):

$$\forall g \in G, Y \in \mathfrak{g}, F \in \mathfrak{g}^*, \text{ then } \left\langle Ad_g^* F, Y \right\rangle = \left\langle F, Ad_{g^{-1}} Y \right\rangle$$

$$K = Ad_g^* = \left(Ad_{g^{-1}}\right)^* \quad \text{and} \quad K_*(X) = -\left(ad_X\right)^*$$

THALES
Building a future we can all trust

Coadjoint operator and Coadjoint Orbits (Kirillov Representation)

Co-adjoint Orbits as Homogeneous Symplectic Manifold by KKS 2-form

- A coadjoint orbit: $O_F = \left\{ Ad_g^* F, g \in G \right\}$ subset of $\mathfrak{g}^*, F \in \mathfrak{g}^*$

  carry a natural homogeneous symplectic structure by a closed G-invariant 2-form:

$$\sigma_\Omega \left(K_{*X} F, K_{*Y} F \right) = B_F \left(X, Y \right) = \left\langle F, \left[X, Y \right] \right\rangle, X, Y \in \mathfrak{g}$$

- The coadjoint action on O_F is a Hamiltonian G-action with moment map $\Omega \to \mathfrak{g}^*$

Souriau Foundamental Theorem « *Every symplectic manifold is a coadjoint orbit* » is based on classification of symplectic homogeneous Lie group actions by Souriau, Kostant and Kirillov

$g \in G$ → $O_F = \left\{ Ad_g^* F, g \in G, F \in \mathfrak{g}^* \right\}$ → $\sigma_\Omega \left(ad_F X, ad_F Y \right) = \left\langle F, \left[X, Y \right] \right\rangle$, $X, Y \in \mathfrak{g}, F \in \mathfrak{g}^*$

Lie Group

Coadjoint Orbit
(action of Lie Group on dual Lie algebra)

Homogeneous Symplectic Manifold
(a smooth manifold with a closed differential 2-form σ, such that dσ=0, where the Lie Group acts transitively)

To make statistics on Lie Groups, migrate on the symplectic manifold generated by coadjoint orbits to capture symmetries

Bedrock of Information Geometry

Jean-Marie Souriau (ENS 1942)

Jean-Louis Koszul (ENS 1940)

Information Geometry & Machine Learning : Legendre structure

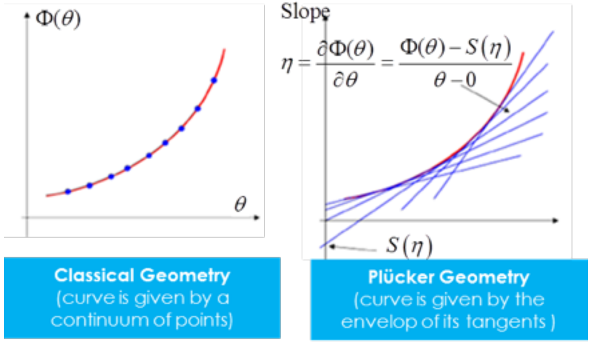
Legendre Transform, Dual Potentials & Fisher Metric

- **S.I. Amari** has proved that the Riemannian metric in an exponential family is the **Fisher information matrix** defined by:

$$g_{ij} = -\left[\frac{\partial^2 \Phi}{\partial \theta_i \partial \theta_j}\right]_{ij} \quad \text{with} \quad \Phi(\theta) = -\log \int_R e^{-\langle \theta, y \rangle} d\lambda$$

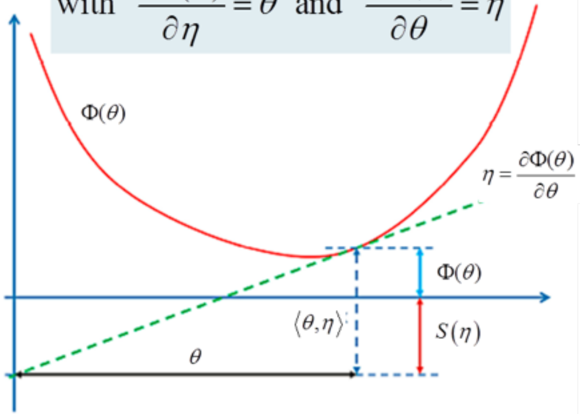
- and the dual potential, the **Shannon entropy**, is given by the **Legendre transform**:

$$S(\eta) = \langle \theta, \eta \rangle - \Phi(\theta) \quad \text{with} \quad \eta_i = \frac{\partial \Phi(\theta)}{\partial \theta_i} \quad \text{and} \quad \theta_i = \frac{\partial S(\eta)}{\partial \eta_i}$$



$$S(\eta) = \langle \theta, \eta \rangle - \Phi(\theta)$$

$$\text{with} \quad \frac{\partial S(\eta)}{\partial \eta} = \theta \quad \text{and} \quad \frac{\partial \Phi(\theta)}{\partial \theta} = \eta$$



Fisher Metric and Koszul 2 form on sharp convex cones

Koszul-Vinberg Characteristic Function, Koszul Forms

- **J.L. Koszul** and **E. Vinberg** have introduced an affinely invariant Hessian metric on a sharp convex cone through its **characteristic function**

$$\Phi_\Omega(\theta) = -\log \int_{\Omega^*} e^{-\langle \theta, y \rangle} dy = -\log \psi_\Omega(\theta) \quad \text{with } \theta \in \Omega \text{ sharp convex cone}$$

$$\psi_\Omega(\theta) = \int_{\Omega^*} e^{-\langle \theta, y \rangle} dy \quad \text{with Koszul-Vinberg Characteristic function}$$

- **1st Koszul form α**: $\alpha = d\Phi_\Omega(\theta) = -d\log\psi_\Omega(\theta)$
- **2nd Koszul form γ**: $\gamma = D\alpha = Dd\log\psi_\Omega(\theta)$

$$\left(Dd\log\psi_\Omega(x)\right)(u) = \frac{1}{\psi_\Omega(u)^2}\left[\int_{\Omega^*} F(\xi)^2 d\xi . \int_{\Omega^*} G(\xi)^2 d\xi - \left(\int_{\Omega^*} F(\xi).G(\xi) d\xi\right)^2\right] > 0 \text{ with } F(\xi) = e^{-\frac{1}{2}\langle x, \xi \rangle} \text{ and } G(\xi) = e^{-\frac{1}{2}\langle x, \xi \rangle}\langle u, \xi \rangle$$

- Diffeomorphism: $\eta = \alpha = -d\log\psi_\Omega(\theta) = \int_{\Omega^*} \xi p_\theta(\xi) d\xi \text{ with } p_\theta(\xi) = \dfrac{e^{-\langle \xi, \theta \rangle}}{\int_{\Omega^*} e^{-\langle \xi, \theta \rangle} d\xi}$
- Legendre transform: $S_\Omega(\eta) = \langle \theta, \eta \rangle - \Phi_\Omega(\theta) \text{ with } \eta = d\Phi_\Omega(\theta) \text{ and } \theta = dS_\Omega(\eta)$

Koszul-Vinberg Characteristic Function

Koszul-Vinberg Characteristic function

- The name "characteristic function" come from the following link:
  Let Ω be a cone in U and Ω^* its dual, for any $\lambda > 0, H_\lambda(x) = \left\{ y \in U / \left\langle x, y \right\rangle = \lambda \right\}$

  and let $d^{(\lambda)}y$ denote the Lebesgue measure on $H_\lambda(x)$:

$$\psi_\Omega(x) = \int_{\Omega^*} e^{-\langle x,y\rangle} dy = \frac{(m-1)!}{\lambda^{m-1}} \int_{\Omega^* \cap H_\lambda(x)} d^{(\lambda)}y$$

- There exist a bijection $x \in \Omega \mapsto x^* \in \Omega^*$, satisfying the relation $(gx)^* = {}^t g^{-1} x^*$ for all $g \in G(\Omega) = \left\{ g \in GL(U) / g\Omega = \Omega \right\}$ the linear automorphism group of Ω and x^* is:

$$x^* = \int_{\Omega^* \cap H_\lambda(x)} y d^{(\lambda)}y \Big/ \int_{\Omega^* \cap H_\lambda(x)} d^{(\lambda)}y$$

- We can observe that x^* is the center of gravity of $\Omega^* \cap H_\lambda(x)$. We have the property that $\psi_\Omega(gx) = \left| \det(g) \right|^{-1} \psi_\Omega(x)$ for all $x \in \Omega, g \in G(\Omega)$ and then that $\psi_\Omega(x)dx$ is an invariant measure on Ω.

Koszul-Vinberg Characteristic Function

Koszul-Vinberg Characteristic function

- We can observe that x^* is the center of gravity of $\Omega^* \cap H_\lambda(x)$. We have the property that $\psi_\Omega(gx) = \left|\det(g)\right|^{-1} \psi_\Omega(x)$ for all $x \in \Omega, g \in G(\Omega)$ and then that $\psi_\Omega(x)dx$ is an invariant measure on Ω.
- Writing $\partial_a = \sum_{i=1}^{m} a^i \frac{\partial}{\partial x^i}$, one can write:

$$\partial_a \Phi_\Omega(x) = \partial_a\left(-\log \psi_\Omega(x)\right) \quad , \quad a \in U, x \in \Omega$$

$$\partial_a \Phi_\Omega(x) == \psi_\Omega(x)^{-1} \int_{\Omega^*} \langle a, y \rangle e^{-\langle x, y \rangle} dy = \left\langle a, x^* \right\rangle$$

- Then, the tangent space to the hypersurface $\{y \in U / \psi_\Omega(y) = \psi_\Omega(x)\}$ at $x \in \Omega$ is given by $\left\{y \in U / \left\langle x^*, y \right\rangle = m\right\}$. For $x \in \Omega, a, b \in U$, the bilinear form $\partial_a \partial_b \log \psi_\Omega(x)$ is symmetric and positive definite, so that it defines an invariant Riemannian metric on Ω.

Fisher Metric and Souriau 2-form: Lie Groups Thermodyamics

Statistical Mechanics, Dual Potentials & Fisher Metric

- In geometric statistical mechanics, **J.M. Souriau** has developed a "**Lie groups thermodynamics**" of dynamical systems where the (maximum entropy) **Gibbs density is covariant** with respect to the action of the Lie group. In the Souriau model, previous structures of information geometry are preserved:

$$I(\beta) = -\frac{\partial^2 \Phi}{\partial \beta^2} \text{ with } \Phi(\beta) = -\log \int_M e^{-\langle \beta, U(\xi) \rangle} d\lambda \qquad U : M \to \mathfrak{g}^*$$

$$S(Q) = \langle \beta, Q \rangle - \Phi(\beta) \text{ with } Q = \frac{\partial \Phi(\beta)}{\partial \beta} \in \mathfrak{g}^* \text{ and } \beta = \frac{\partial S(Q)}{\partial Q} \in \mathfrak{g}$$

Jean-Marie Souriau

- In the Souriau **Lie groups thermodynamics** model, β is a "geometric" (Planck) temperature, element of Lie algebra $\mathfrak{g}$ of the group, and Q is a "geometric" heat, element of dual Lie algebra $\mathfrak{g}^*$ of the group.

Fisher-Souriau Metric as a non-null Cohomology extension of KKS 2 form (Kirillov-Kostant-Souriau 2 form)

Souriau definition of Fisher Metric is related to the extension of KKS 2-form (Kostant-Kirillov-Souriau) in case of non-null Cohomogy:

Souriau-Fisher Metric

$$I(\beta)=\left[g_\beta\right] \text{ with } g_\beta\left([\beta,Z_1],[\beta,Z_2]\right)=\tilde{\Theta}_\beta\left(Z_1,[\beta,Z_2]\right)$$

$$\text{with } \tilde{\Theta}_\beta\left(Z_1,Z_2\right)=\tilde{\Theta}\left(Z_1,Z_2\right)+\left\langle Q,[Z_1,Z_2]\right\rangle$$

Non-null cohomology: aditional term from Souriau Cocycle

Equivariant KKS 2 form

$$\tilde{\Theta}(X,Y)=J_{[X,Y]}-\left\{J_X,J_Y\right\} \text{ with } J(x):M\rightarrow\mathfrak{g}^* \text{ such that } J_X(x)=\left\langle J(x),X\right\rangle, X\in\mathfrak{g}$$

$$\tilde{\Theta}\left(X,Y\right):\mathfrak{g}\times\mathfrak{g}\rightarrow\Re$$

$$X,Y\mapsto\left\langle\Theta(X),Y\right\rangle$$

$$\text{with } \Theta(X)=T_e\theta\left(X(e)\right)$$

$$\tilde{\Theta}\left(\beta,Z\right)+\left\langle Q,[\beta,Z]\right\rangle=0$$

$$\beta\in Ker\,\tilde{\Theta}_\beta$$

Souriau Fundamental Equation of Lie Group Thermodynamics

$$Q\left(Ad_g(\beta)\right)=Ad_g^*(Q)+\theta(g)$$

Non-equivariance of Coadjoint operator

- Non-equivariance of Coadjoint operator:

$$Q\left(Ad_g(\beta)\right) = Ad_g^*(Q) + \theta\left(g\right)$$

- This is the action of Lie Group on Moment map:

$$J\left(\Phi_g(x)\right) = a(g, J(x)) = Ad_g^*\left(J(x)\right) + \theta(g)$$

- By noting the action of the group on the dual space of the Lie algebra:

$$G \times \mathfrak{g}^* \to \mathfrak{g}^*, (s,\xi) \mapsto s\xi = Ad_s^*\xi + \theta(s)$$

- Associativity is given by:

$$(s_1 s_2)\xi = Ad_{s_1 s_2}^*\xi + \theta(s_1 s_2) = Ad_{s_1}^* Ad_{s_2}^*\xi + \theta(s_1) + Ad_{s_1}^*\theta(s_2)$$

$$(s_1 s_2)\xi = Ad_{s_1}^*\left(Ad_{s_2}^*\xi + \theta(s_2)\right) + \theta(s_1) = s_1\left(s_2\xi\right) \quad , \quad \forall s_1, s_2 \in G, \xi \in \mathfrak{g}^*$$

Souriau Cocycle

- $\theta(g) \in \mathfrak{g}^*$ is called nonequivariance one-cocycle, and it is a measure of the lack of equivariance of the moment map.

$$\theta(st) = J((st).x) - Ad_{st}^* J(x)$$

$$\theta(st) = \left[J(s.(t.x)) - Ad_s^* J(t.x) \right] + \left[Ad_s^* J(t.x) - Ad_s^* Ad_t^* J(x) \right]$$

$$\theta(st) = \theta(s) + Ad_s^* \left[J(t.x) - Ad_t^* J(x) \right]$$

$$\theta(st) = \theta(s) + Ad_s^* \theta(t)$$

Souriau one-cocycle and compute 2-cocycle

$$\tilde{\Theta}(X,Y):\mathfrak{g}\times\mathfrak{g}\to\Re \qquad \text{with } \Theta(X)=T_e\theta(X(e))$$

$$X,Y \mapsto \langle\Theta(X),Y\rangle$$

- We can also compute tangent of one-cocycle θ at neutral element, to compute 2-cocycle Θ :

$$\zeta\in\mathfrak{g}\ ,\quad \theta_\zeta(s)=\langle\theta(s),\zeta\rangle=\langle J(s.x),\zeta\rangle-\langle Ad_s^* J(x),\zeta\rangle$$

$$\theta_\zeta(s)=\langle J(s.x),\zeta\rangle-\langle J(x),Ad_{s^{-1}}\zeta\rangle$$

$$T_e\theta_\zeta(\xi)=\langle T_xJ.\xi_p(x),\zeta\rangle+\langle J(x),ad_\xi\zeta\rangle \quad \text{with } \xi_p=X_{\langle J,\xi\rangle}$$

$$T_e\theta_\zeta(\xi)=X_{\langle J(x),\xi\rangle}\left[\langle J(x),\zeta\rangle\right]+\langle J(x),[\xi,\zeta]\rangle$$

$$T_e\theta_\zeta(\xi)=-\{\langle J,\xi\rangle,\langle J,\zeta\rangle\}+\langle J(x),[\xi,\zeta]\rangle=\Theta(\xi)$$

Souriau Tensor

$$\tilde{\Theta}(X,Y) = J_{[X,Y]} - \{J_X, J_Y\} = -\langle d\theta(X), Y\rangle \quad , \quad X,Y \in \mathfrak{g}$$

$$\tilde{\Theta}([X,Y],Z) + \tilde{\Theta}([Y,Z],X) + \tilde{\Theta}([Z,X],Y) = 0 \quad , \quad X,Y,Z \in \mathrm{g}$$

- By differentiating the equation on affine action, we have:

$$T_x J\left(\xi_p(x)\right) = -ad_\xi^* J(x) + \Theta(\xi,.)$$

$$dJ(Xx) = ad_X J(x) + d\theta(X) \quad , \quad x \in M, X \in \mathfrak{g}$$

$$\langle dJ(Xx), Y\rangle = \langle ad_X J(x), Y\rangle + \langle \mathrm{d}\theta(X), Y\rangle, \quad x \in M, X, Y \in \mathfrak{g}$$

$$\langle dJ(Xx), Y\rangle = \langle J(x), [X,Y]\rangle + \langle \mathrm{d}\theta(X), Y\rangle \quad = \{\langle J, X\rangle, \langle J, Y\rangle\}(x)$$

$$\langle J(x), [X,Y]\rangle - \{\langle J, X\rangle, \langle J, Y\rangle\}(x) = -\langle \mathrm{d}\theta(X), Y\rangle$$

Souriau Riemannan Metric and Fisher Metric

- For $\beta \in \Omega$, let g_β be the Hessian form on $T_\beta \Omega \equiv \mathfrak{g}$ with the potential $\Phi(\beta) = -\log \psi_\Omega(\beta)$. For $X, Y \in \mathfrak{g}$, we define:

$$g_\beta(X,Y) = -\frac{\partial^2 \Phi}{\partial \beta^2}(X,Y) = \left(\frac{\partial^2}{\partial s \partial t}\right)_{s=t=0} \log \psi_\Omega(\beta + sX + tY)$$

- The positive definitiveness is given by Cauchy-Schwarz inequality:

$$g_\beta(X,Y) = \frac{1}{\psi_\Omega(\beta)^2} \left\{ \begin{array}{l} \int_M e^{-\langle U(\xi),\beta\rangle} d\lambda(\xi) . \int_M \langle U(\xi), X\rangle^2 e^{-\langle U(\xi),\beta\rangle} d\lambda(\xi) \\ -\left(\int_M \langle U(\xi), X\rangle e^{-\langle U(\xi),\beta\rangle} d\lambda(\xi) \right)^2 \end{array} \right\}$$

$$= \frac{1}{\psi_\Omega(\beta)^2} \left\{ \begin{array}{l} \int_M \left(e^{-\langle U(\xi),\beta\rangle/2} \right)^2 d\lambda(\xi) . \int_M \left(\langle U(\xi), X\rangle e^{-\langle U(\xi),\beta\rangle/2} \right)^2 d\lambda(\xi) \\ -\left(\int_M e^{-\langle U(\xi),\beta\rangle/2} . \langle U(\xi), X\rangle e^{-\langle U(\xi),\beta\rangle/2} d\lambda(\xi) \right)^2 \end{array} \right\} \geq 0$$

Souriau Riemannan Metric and Fisher Metric

$$g_\beta(X,Y)=\left\langle -\frac{\partial Q}{\partial \beta}(X),Y\right\rangle \text{ for } X,Y\in\mathfrak{g}$$

- we have for any $\beta\in\Omega, g\in G$ and $Y\in\mathfrak{g}$:

$$\left\langle Q\left(Ad_g\beta\right),Y\right\rangle=\left\langle Q(\beta),Ad_{g^{-1}}Y\right\rangle+\left\langle\theta(g),Y\right\rangle$$

- Let us differentiate the above expression with respect to g. Namely, we substitute $g=\exp\left(tZ_1\right),t\in R$ and differentiate at t = 0. Then the left-hand side becomes:

$$\left(\frac{d}{dt}\right)_{t=0}\left\langle Q\left(\beta+t\left[Z_1,\beta\right]+o\left(t^2\right)\right),Y\right\rangle=\left\langle\frac{\partial Q}{\partial\beta}([Z_1,\beta]),Y\right\rangle$$

- and the right-hand side of is calculated as:

$$\left(\frac{d}{dt}\right)_{t=0}\left\langle Q(\beta),Y-t\left[Z_1,Y\right]+o\left(t^2\right)\right\rangle+\left\langle\theta\left(I+tZ_1+o\left(t^2\right)\right),Y\right\rangle$$

$$=-\left\langle Q(\beta),[Z_1,Y]\right\rangle+\left\langle d\theta\left(Z_1\right),Y\right\rangle$$

Souriau Riemannan Metric and Fisher Metric

- Therefore,

$$\left\langle \frac{\partial Q}{\partial \beta}([Z_1,\beta]),Y \right\rangle = \langle d\theta(Z_1),Y\rangle - \langle Q(\beta),[Z_1,Y]\rangle$$

- Substituting $Y=-[\beta,Z_2]$ to the above expression:

$$g_\beta([\beta,Z_1],[\beta,Z_2]) = \left\langle -\frac{\partial Q}{\partial \beta}([Z_1,\beta]),[\beta,Z_2] \right\rangle$$

$$g_\beta([\beta,Z_1],[\beta,Z_2]) = \langle d\theta(Z_1),[\beta,Z_2]\rangle + \langle Q(\beta),[Z_1,[\beta,Z_2]]\rangle$$

- We define then symplectic 2-cocycle and the tensor:

$$\Theta(Z_1) = d\theta(Z_1)$$

$$\tilde{\Theta}(Z_1,Z_2) = \langle \Theta(Z_1),Z_2\rangle = J_{[Z_1,Z_2]} - \{J_{Z_1},J_{Z_2}\}$$

$$\tilde{\Theta}_\beta(Z_1,Z_2) = \langle Q(\beta),[Z_1,Z_2]\rangle + \tilde{\Theta}(Z_1,Z_2) \Rightarrow g_\beta([\beta,Z_1],[\beta,Z_2]) = \tilde{\Theta}_\beta(Z_1,[\beta,Z_2])$$

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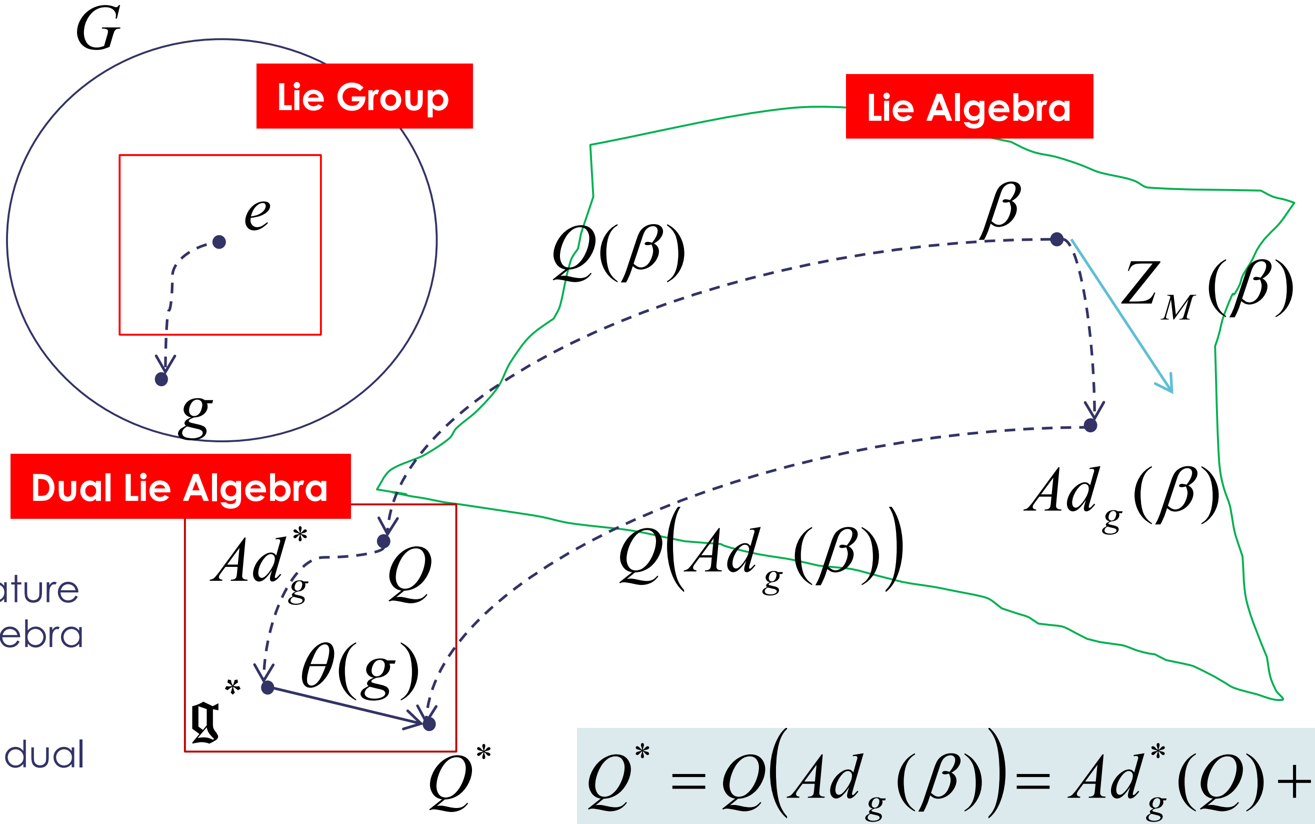
Fundamental Souriau Theorem

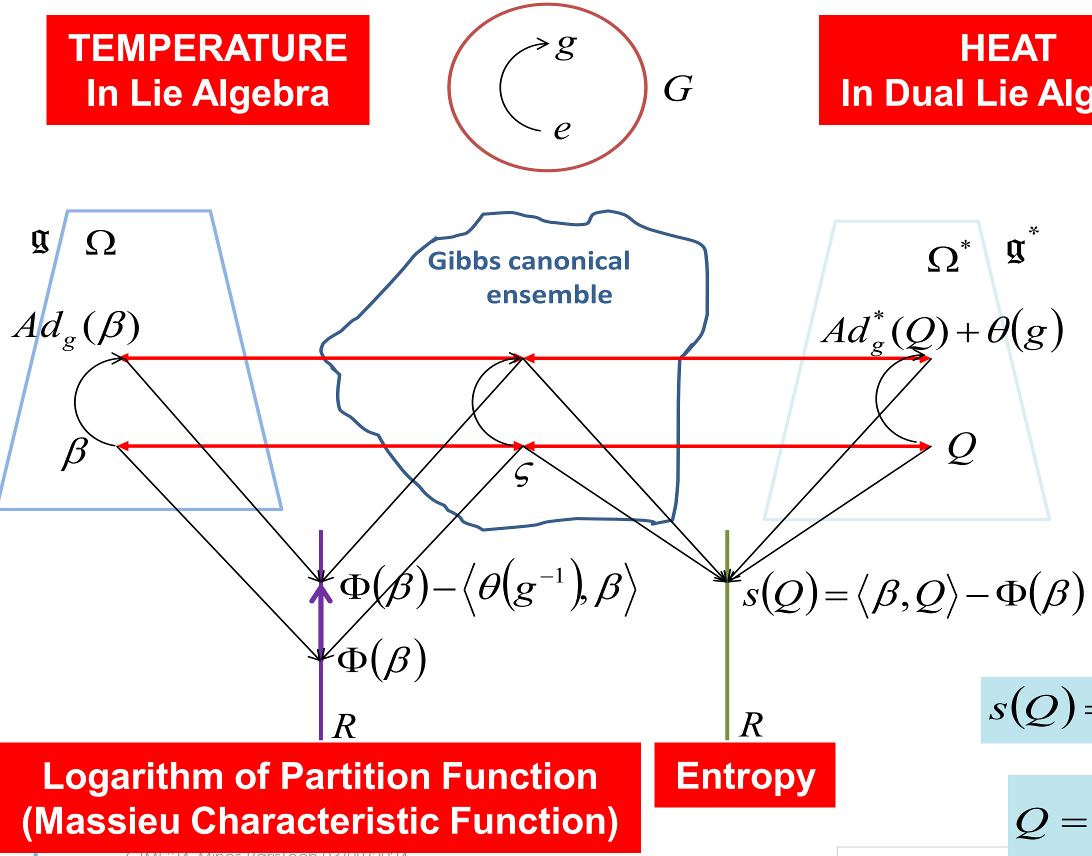


β : (Planck) température element of Lie algebra

Q : Heat, element of dual Lie Algebra

$$Q^* = Q\left(Ad_g(\beta)\right) = Ad_g^*(Q) + \theta(g)$$

Souriau-Fisher Metric & Souriau Lie Groups Thermodynamics: Bedrock for Lie Group Machine Learning



Fisher Metric

$$g_{\beta}([\beta,Z_1],[\beta,Z_2])=\tilde{\Theta}_{\beta}(Z_1,[\beta,Z_2])\geq 0$$

$$I(\beta)=I(Ad_g(\beta))=-\frac{\partial^2\Phi}{\partial\beta^2}=\frac{\partial^2\log\int_M e^{-\langle\beta,U(\xi)\rangle}d\lambda}{\partial\beta^2}$$

Entropy invariant under the action of the group

Legendre — Clairaut

$$s(Q)=\langle\beta,Q\rangle-\Phi(\beta)=\langle\Theta^{-1}(Q),Q\rangle-\Phi(\Theta^{-1}(Q))$$

$$Q=\Theta(\beta)=\frac{\partial\Phi}{\partial\beta}\in\mathfrak{g}^*$$

$$\beta=\Theta^{-1}(Q)\in\mathfrak{g}$$

Gallileo Group & Algebra & V. Bargman Central extensions

- Symplectic cocycles of the Galilean group: V. Bargmann (Ann. Math. 59, 1954, pp 1–46) has proven that the symplectic cohomology space of the Galilean group is one-dimensional.

- Gallileo Lie Group & Algebra

$$\begin{cases} \vec{x}' = R.\vec{x} + \vec{u}.t + \vec{w} \\ t' = t + e \end{cases}$$

$$\begin{bmatrix} \vec{x}' \\ t' \\ 1 \end{bmatrix} = \begin{bmatrix} R & \vec{u} & \vec{w} \\ 0 & 1 & e \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{x} \\ t \\ 1 \end{bmatrix}$$

$\vec{x}, \vec{u}$ and $\vec{w} \in R^3, e \in R^+$

$R \in SO(3)$

$$\begin{bmatrix} \vec{\omega} & \vec{\eta} & \vec{\gamma} \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{bmatrix}, \begin{cases} \vec{\eta} \text{ and } \vec{\gamma} \in R^3, \varepsilon \in R^+ \\ \vec{\omega} \in so(3) : \vec{x} \mapsto \vec{\omega} \times \vec{x} \end{cases}$$

- Bargmann Central extension:

$$\begin{bmatrix} R & \vec{u} & 0 & \vec{w} \\ 0 & 1 & 0 & e \\ -\vec{u}^t R & -\frac{\|\vec{u}\|^2}{2} & 1 & f \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Souriau Gibbs states for Hamiltonian actions of subgroups of the Galilean group

Souriau Gibbs states for one-parameter subgroups of the Galilean group

- **Souriau Result:** Action of the full Galilean group on the space of motions of an isolated mechanical system is not related to any Equilibrium Gibbs state (the open subset of the Lie algebra, associated to this Gibbs state, is empty)
- The **1-parameter subgroup of the Galilean group** generated by β element of Lie Algebra, is the set of matrices

$$\exp(\tau\beta) = \begin{pmatrix} A(\tau) & \vec{b}(\tau) & \vec{d}(\tau) \\ 0 & 1 & \tau\varepsilon \\ 0 & 0 & 1 \end{pmatrix} \text{ with } \begin{cases} A(\tau) = \exp\left(\tau j(\vec{\omega})\right) \text{ and } \vec{b}(\tau) = \left(\sum_{i=1}^{\infty} \frac{\tau^i}{i!} \left(j(\vec{\omega})\right)^{i-1} \right) \vec{\alpha} \\ \vec{d}(\tau) = \left(\sum_{i=1}^{\infty} \frac{\tau^i}{i!} \left(j(\vec{\omega})\right)^{i-1} \right) \vec{\delta} + \varepsilon \left(\sum_{i=2}^{\infty} \frac{\tau^i}{i!} \left(j(\vec{\omega})\right)^{i-2} \right) \vec{\alpha} \end{cases}$$

$$\beta = \begin{pmatrix} j(\vec{\omega}) & \vec{\alpha} & \vec{\delta} \\ 0 & 1 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}$$

Souriau Thermodynamics of butter churn (device used to convert cream into butter) or “La Thermodynamique de la crémière”

If we consider the case of the centrifuge

$$\vec{\omega} = \omega \vec{e}_z \,,\, \vec{\alpha} = 0 \text{ and } \vec{\delta} = 0$$

$$Rotation\ speed : \frac{\omega}{\varepsilon}$$

$$f_i\left(\vec{r}_{i0}\right) = -\frac{\omega^2}{2\varepsilon^2}\left\|\vec{e}_z \times \vec{r}_{i0}\right\|^2$$

$$\text{with } \Delta = \left\|\vec{e}_z \times \vec{r}_{i0}\right\| \text{ distance to axis z}$$

> “The angular momentum is imparted to the gas when the molecules collide with the rotating walls, which changes the Maxwell distribution at every point, shifting its origin. The walls play the role of an angular momentum reservoir. Their motion is characterized by a certain angular velocity, and the angular velocities of the fluid and of the walls become equal at equilibrium, exactly like the equalization of the temperature through energy exchanges”. – Roger Balian

$$\rho_i(\beta) = \frac{1}{P_i(\beta)}\exp\left(-\langle J_i, \beta\rangle\right) = cst.\exp\left(-\frac{1}{2m_i\kappa T}\left\|\vec{p}_{i0}\right\|^2 + \frac{m_i}{2\kappa T}\left(\frac{\omega}{\varepsilon}\right)^2\Delta^2\right)$$

- the behaviour of a gas made of point particles of various masses in a centrifuge rotating at a constant angular velocity (the heavier particles concentrate farther from the rotation axis than the lighter ones)

$$\frac{\omega}{\varepsilon}$$

Link with Classical Thermodynamics

We have the reciprocal formula:

$Q = \dfrac{\partial \Phi}{\partial \beta}$	$\beta = \dfrac{\partial S}{\partial Q}$
$s(Q) = \left\langle \dfrac{\partial \Phi}{\partial \beta}, \beta \right\rangle - \Phi$	$\Phi(\beta) = \left\langle Q, \dfrac{\partial S}{\partial Q} \right\rangle - S$

For Classical Thermodynamics (Time translation only), we recover the definition of Boltzmann-Clausius Entropy:

$$\begin{cases} \beta = \dfrac{\partial S}{\partial Q} \\ \beta = \dfrac{1}{T} \end{cases} \Rightarrow dS = \frac{dQ}{T}$$

Souriau Model of Covariant Gibbs Density

Covariant Souriau-Gibbs density

- Souriau has then defined a Gibbs density that is covariant under the action of the group:

$$p_{Gibbs}(\xi) = e^{\Phi(\beta) - \langle U(\xi), \beta \rangle} = \frac{e^{-\langle U(\xi), \beta \rangle}}{\int_M e^{-\langle U(\xi), \beta \rangle} d\lambda_\omega}$$

$$\text{with } \Phi(\beta) = -\log \int_M e^{-\langle U(\xi), \beta \rangle} d\lambda_\omega$$

$$Q = \frac{\partial \Phi(\beta)}{\partial \beta} = \frac{\int_M U(\xi) e^{-\langle U(\xi), \beta \rangle} d\lambda_\omega}{\int_M e^{-\langle U(\xi), \beta \rangle} d\lambda_\omega} = \int_M U(\xi) p(\xi) d\lambda_\omega$$

- We can express the Gibbs density with respect to Q by inverting the relation $Q = \frac{\partial \Phi(\beta)}{\partial \beta} = \Theta(\beta)$. Then $p_{Gibbs,Q}(\xi) = e^{\Phi(\beta) - \langle U(\xi), \Theta^{-1}(Q) \rangle}$ with $\beta = \Theta^{-1}(Q)$

SOURIAU'S MOMENT MAP
By VLADIMIR ARNOLD

A Poisson action of a group G on a symplectic manifold M defines a mapping of M into the dual space of the Lie algebra of the group

$$P: M \to \mathfrak{g}^*.$$

That is, we fix a point x in M and consider the function on the Lie algebra which associates to an element a of the Lie algebra the value of the Hamiltonian H_a at the fixed point x:

$$p_x(a) = H_a(x).$$

This p_x is a linear function on the Lie algebra and is the element of the dual space to the algebra associated to x:

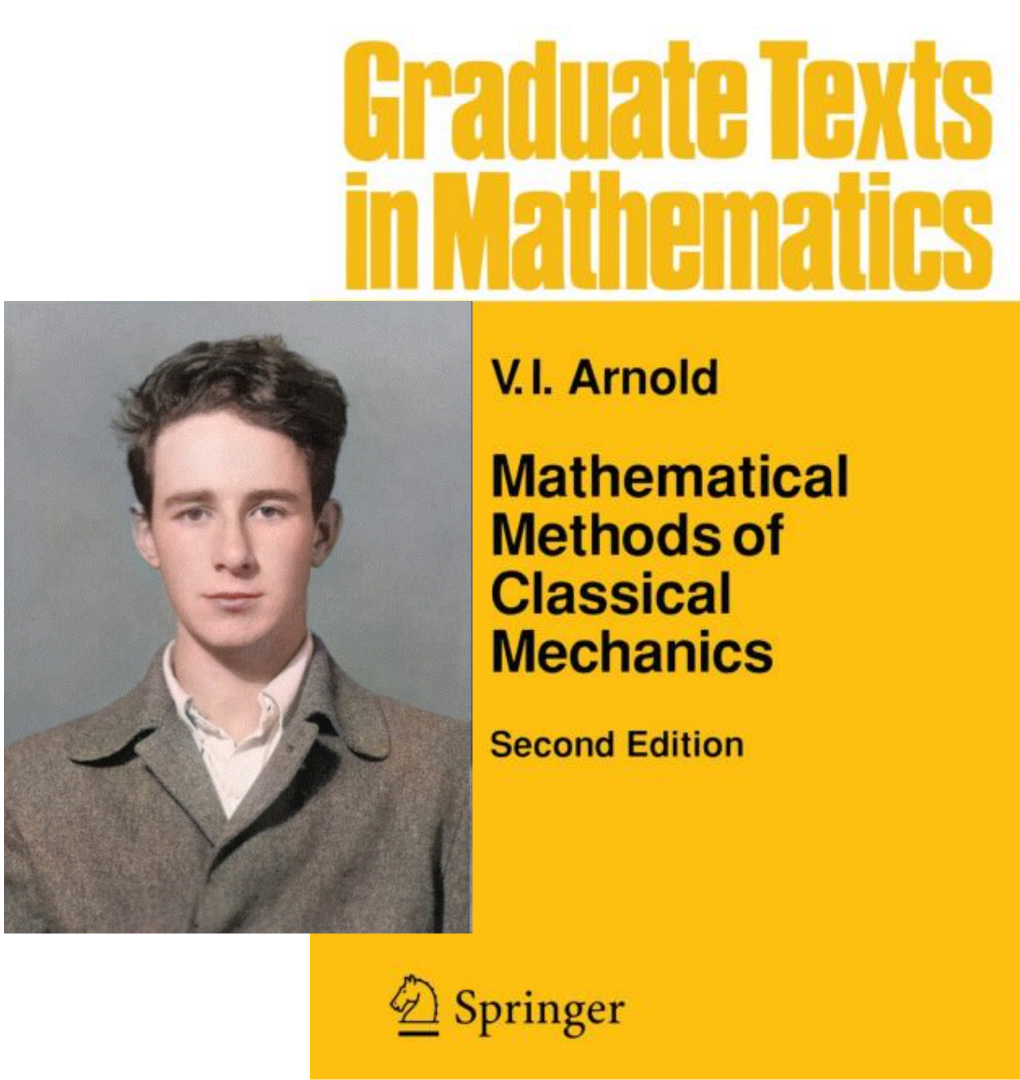
$$P(x) = p_x.$$

Following Souriau (*Structure des systèmes dynamiques*, Dunod, 1970), we will call the mapping P the *momentum*. Note that the value of the momentum is always a vector in the space $\mathfrak{g}^*$.

Theorem. *Under the momentum mapping P, a Poisson action of a connected Lie group G is taken to the co-adjoint action of G on the dual space $\mathfrak{g}^*$ of its Lie algebra (cf. Appendix 2), i.e., the following diagram commutes:*

$$\begin{array}{ccc} M & \xrightarrow{g} & M \\ \downarrow P & & \downarrow P \\ \mathfrak{g}^* & \xrightarrow{\mathrm{Ad}^*_{g^{-1}}} & \mathfrak{g}^* \end{array}$$

Corollary. *Suppose that a hamiltonian function $H: M \to \mathbb{R}$ is invariant under the Poisson action of a group G on M. Then the momentum is a first integral of the system with hamiltonian function H.*



SOURIAU'S MOMENT MAP = GEOMETRIZATION OF NOETHER THEOREM

3.5 Noether type theorems

Consider now the "levels" of the moment map[13] $\mu : W \to \mathfrak{g}^\star$ of a hamiltonian G-action on the symplectic manifold (W, ω). The most classical form of E. Noether's theorem seems to be stated nowadays as:

[13] I shall call *level* ξ the inverse image $\mu^{-1}(\xi)$ even if ξ is not a number.

Theorem 3.5.1 *Let H be a function on W which is invariant by the G-action. Then μ is constant on the trajectories of the hamiltonian vector field X_H.*

Proof. Indeed, if $\gamma(t)$ is a trajectory of X_H, one can write, for any $X \in \mathfrak{g}$:

$$\begin{aligned} \tfrac{d}{dt} < \mu \circ \gamma(t), X > &= < T_{\gamma(t)}\mu(X_H(\gamma(t))), X > \\ &= < X_H(\gamma(t)), {}^tT_{\gamma(t)}\mu(\underline{X}) > \\ &= < X_H(\gamma(t)), (i_X\omega)_{\gamma(t)} > \\ &= \omega(\underline{X}, X_H)_{\gamma(t)} \\ &= -dH_{\gamma(t)}(\underline{X}) \end{aligned}$$

But H is invariant and $\underline{X}$ a fundamental vector field of the action, and thus:

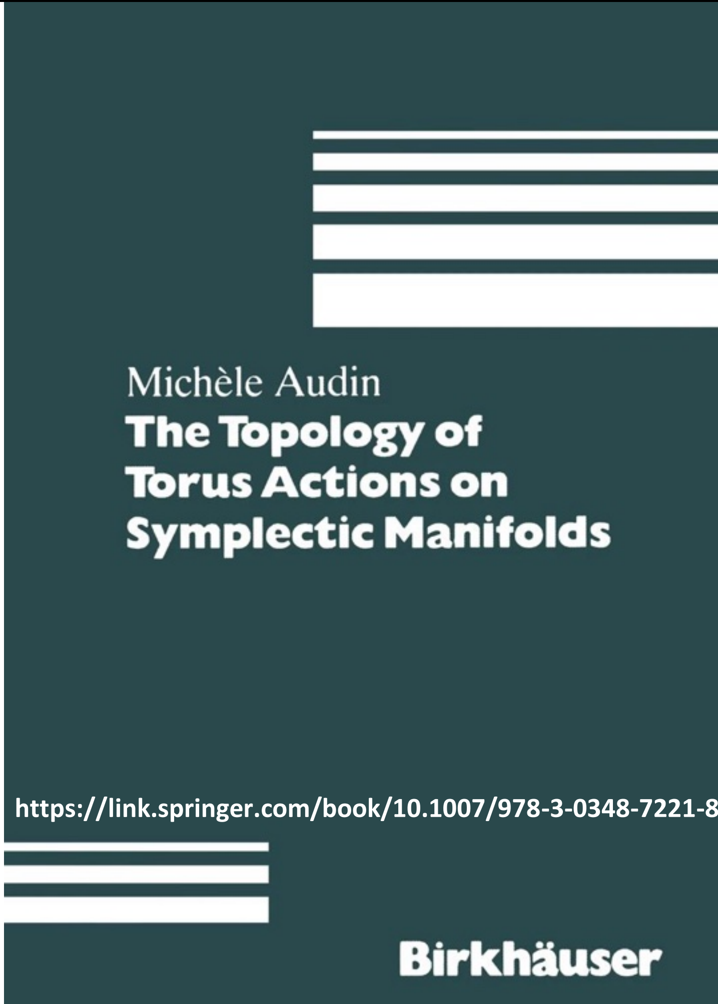
$$H(\exp(sX) \cdot \gamma(t)) = H(\gamma(t))$$

which, when differentiated at $s = 0$, is:

$$dH(\underline{X}(\gamma(t))) = 0.$$

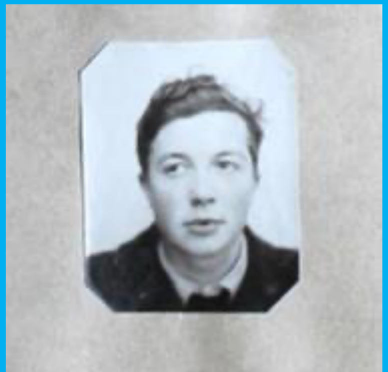
□

The field X_H is thus tangent to the levels $\mu^{-1}(\xi)$.



https://link.springer.com/book/10.1007/978-3-0348-7221-8

ENTROPY as Casimir Function in Coadjoint Representation Constant on Symplectic Leaves & Density of Maximum Entropy on Lie Groups



Jean-Marie Souriau

Souriau Entropy Invariance

Casimir Invariant Function in coadjoint representation

- We observe that Souriau Entropy $S(Q)$ defined on coadjoint orbit of the group has a property of invariance :

$$S\left(Ad_g^{\#}(Q)\right) = S(Q)$$

- with respect to Souriau affine definition of coadjoint action:

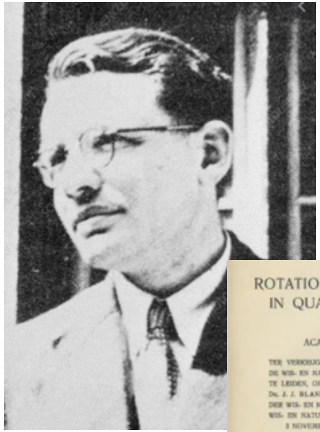
$$Ad_g^{\#}(Q) = Ad_g^{*}(Q) + \theta(g)$$

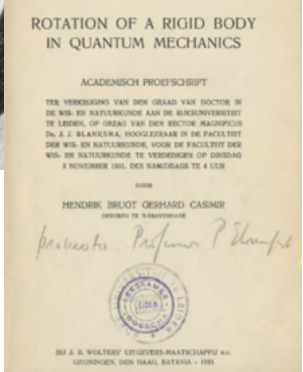
- where $\theta(g)$ is called the Souriau cocyle.

$$Q\left(Ad_g(\beta)\right) = Ad_g^{*}(Q) + \theta(g)$$

$$S\left(Q\left(Ad_g(\beta)\right)\right) = S(Q)$$

New Entropy Definition:
Casimir Function in Coadjoint Representation Invariant under the action of the Group




ROTATION OF A RIGID BODY IN QUANTUM MECHANICS

ACADEMISCH PROEFSCHRIFT

HENDRIK BRUGT GERHARD CASIMIR

Hendrik Casimir (Thesis supervised by Niels Bohr & Paul Ehrenfest)

H.B.G. Casimir, On the Rotation of a Rigid Body in Quantum Mechanics, Doctoral Thesis, Leiden, 1931.

Entropy Invariance under the action of the Group (1/2)

$$\beta \in \mathfrak{g} \to Ad_g(\beta) \Rightarrow \Psi(Ad_g(\beta)) = \int_M e^{-\langle U, Ad_g(\beta)\rangle} d\lambda_\omega$$

$$\Psi(Ad_g(\beta)) = \int_M e^{-\left\langle Ad^*_{g^{-1}} U, \beta \right\rangle} d\lambda_\omega = \int_M e^{-\left\langle U\left(Ad_{g^{-1}}\beta\right) - \theta\left(g^{-1}\right), \beta\right\rangle} d\lambda_\omega$$

$$\Psi(Ad_g(\beta)) = e^{\left\langle \theta\left(g^{-1}\right), \beta\right\rangle} \Psi(\beta)$$

$$\theta\left(g^{-1}\right) = -Ad^*_{g^{-1}}\theta(g) \Rightarrow \Psi(Ad_g(\beta)) = e^{-\left\langle Ad^*_{g^{-1}}\theta(g), \beta\right\rangle} \Psi(\beta)$$

$$\Phi(\beta) = -\log \Psi(\beta)$$

$$\Rightarrow \Phi\left(Ad_g(\beta)\right) = \Phi(\beta) - \left\langle \theta\left(g^{-1}\right), \beta\right\rangle = \Phi(\beta) + \left\langle Ad^*_{g^{-1}}\theta(g), \beta\right\rangle$$

OPEN

THALES

Entropy Invariance under the action of the Group (2/2)

$$S(Q) = \langle Q, \beta \rangle - \Phi(\beta) \Rightarrow S\left(Q\left(Ad_g \beta\right)\right) = \left\langle Q\left(Ad_g \beta\right), Ad_g \beta \right\rangle - \Phi\left(Ad_g \beta\right)$$

$$\begin{cases} Q\left(Ad_g(\beta)\right) = Ad_g^*(Q) + \theta\left(g\right) \\ \Phi\left(Ad_g(\beta)\right) = -\log \Psi(Ad_g(\beta)) = -\left\langle \theta\left(g^{-1}\right), \beta \right\rangle + \Phi\left(\beta\right) \end{cases}$$

$$\Rightarrow S\left(Q\left(Ad_g \beta\right)\right) = \left\langle Ad_g^*(Q) + \theta\left(g\right), Ad_g \beta \right\rangle + \left\langle \theta\left(g^{-1}\right), \beta \right\rangle - \Phi\left(\beta\right)$$

$$\Rightarrow S\left(Q\left(Ad_g \beta\right)\right) = \left\langle Ad_g^*(Q) + \theta\left(g\right), Ad_g \beta \right\rangle - \left\langle Ad_{g^{-1}}^* \theta(g), \beta \right\rangle - \Phi\left(\beta\right)$$

$$\Rightarrow S\left(Q\left(Ad_g \beta\right)\right) = \left\langle Ad_{g^{-1}}^* Ad_g^*(Q) + Ad_{g^{-1}}^* \theta\left(g\right), \beta \right\rangle - \left\langle Ad_{g^{-1}}^* \theta(g), \beta \right\rangle - \Phi\left(\beta\right)$$

$$Ad_{g^{-1}}^* Ad_g^*(Q) = Q \Rightarrow S\left(Q\left(Ad_g \beta\right)\right) = \langle Q, \beta \rangle - \Phi\left(\beta\right) = S\left(\beta\right)$$

Casimir Function and Entropy

- Classically, the Entropy is defined axiomatically as Shannon or von Neumann Entropies without any geometric structures constraints.
- Entropy could be built by Casimir Function Equation:

$$\left(ad^{*}_{\frac{\partial S}{\partial Q}}Q\right)_j+\Theta\left(\frac{\partial S}{\partial Q}\right)_j=C^k_{ij}ad^{*}_{\left(\frac{\partial S}{\partial Q}\right)^i}Q_k+\Theta_j=0$$

$$\tilde{\Theta}(X,Y)=\langle\Theta(X),Y\rangle=J_{[X,Y]}-\{J_X,J_Y\}=-\langle d\theta(X),Y\rangle \quad , \ X,Y\in\mathfrak{g}$$

$$\Theta(X)=T_e\theta\left(X(e)\right)$$

$$\theta(g)=Q\left(Ad_g(\beta)\right)-Ad^{*}_g(Q)$$

Souriau Entropy and Casimir Invariant Function

Demo

- if we consider the heat expression $Q=\frac{\partial\Phi}{\partial\beta}$, that we can write $\delta\Phi-\langle Q,\delta\beta\rangle=0$.
- For each $\delta\beta$ tangent to the orbit, and so generated by an element Z of the Lie algebra, if we consider the relation $\Phi\left(Ad_g(\beta)\right)=\Phi(\beta)-\left\langle\theta\left(g^{-1}\right),\beta\right\rangle$, we differentiate it at $g=e$ using the property that:

$$\tilde{\Theta}(X,Y)=-\langle d\theta(X),Y\rangle \ , \ X,Y\in\mathfrak{g}$$

- we obtain : $\langle Q,[\beta,Z]\rangle+\tilde{\Theta}(\beta,Z)=0$
- From last Souriau equation, if we use the identities $\beta=\frac{\partial S}{\partial Q}$, $ad_\beta Z=[\beta,Z]$ and $\tilde{\Theta}(\beta,Z)=\langle\Theta(\beta),Z\rangle$
- Then we can deduce that: $\left\langle ad^*_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right),Z\right\rangle=0,\quad\forall Z$
- So, Entropy $S(Q)$ should verify:

$$ad^*_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0 \qquad \{S,H\}_{\tilde{\Theta}}(Q)=0\quad\forall H:\mathfrak{g}^*\to R,\ \ Q\in\mathfrak{g}^*$$

$$\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle Q,\left[\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right]\right\rangle+\tilde{\Theta}\left(\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right)=0$$

Entropy as Invariant Casimir Function in Coadjoint Representation

NEW GEOMETRIC DEFINITION OF ENTROPY

$$\{S,H\}_{\tilde{\Theta}}(Q)=0$$

$$ad^*_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0$$

$$\{S,H\}(Q)=\left\langle Q,\left[\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right]\right\rangle=-C_{ij}^{k}Q_k\frac{\partial S}{\partial Q_i}\cdot\frac{\partial H}{\partial Q_j}$$

$$\left[e_i,e_j\right]=C_{ij}^{k}e_k \quad , \quad C_{ij}^{k} \quad \text{structure coefficients}$$

$$\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle Q,\left[\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right]\right\rangle+\tilde{\Theta}\left(\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right)=0 \quad , \quad \forall H:\mathfrak{g}^*\to R,\quad Q\in\mathfrak{g}^*$$

$$\tilde{\Theta}(X,Y)=J_{[X,Y]}-\{J_X,J_Y\} \quad \text{where } J_X(x)=\langle J(x),X\rangle$$

$$\tilde{\Theta}\left(X,Y\right)=\langle\Theta(X),Y\rangle \text{ with } \Theta(X)=T_e\theta\left(X(e)\right)$$

$$\theta\left(g\right)=Q\left(Ad_g(\beta)\right)-Ad_g^*(Q)$$

Souriau Entropy and Casimir Invariant Function

Demo

$$\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle Q,\left[\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right]\right\rangle+\left\langle \Theta\left(\frac{\partial S}{\partial Q}\right),\frac{\partial H}{\partial Q}\right\rangle=0$$

$$\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle Q,ad_{\frac{\partial S}{\partial Q}}\frac{\partial H}{\partial Q}\right\rangle+\left\langle \Theta\left(\frac{\partial S}{\partial Q}\right),\frac{\partial H}{\partial Q}\right\rangle=0$$

$$\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle ad^{*}_{\frac{\partial S}{\partial Q}}Q,\frac{\partial H}{\partial Q}\right\rangle+\left\langle \Theta\left(\frac{\partial S}{\partial Q}\right),\frac{\partial H}{\partial Q}\right\rangle=0$$

$$\forall H,\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle ad^{*}_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right),\frac{\partial H}{\partial Q}\right\rangle=0\Rightarrow ad^{*}_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0$$

Souriau Entropy and Casimir Invariant Function

Link with Souriau development

- Souriau property: $\beta \in Ker\tilde{\Theta}_\beta \Rightarrow \langle Q,[\beta,Z]\rangle + \tilde{\Theta}(\beta,Z) = 0$

$$\Rightarrow \langle Q, ad_\beta Z\rangle + \tilde{\Theta}(\beta,Z) = 0 \Rightarrow \langle ad^*_\beta Q, Z\rangle + \tilde{\Theta}(\beta,Z) = 0$$

$$\beta = \frac{\partial S}{\partial Q} \Rightarrow \left\langle ad^*_{\frac{\partial S}{\partial Q}} Q, Z \right\rangle + \tilde{\Theta}\left(\frac{\partial S}{\partial Q}, Z\right) = \left\langle ad^*_{\frac{\partial S}{\partial Q}} Q, Z \right\rangle + \left\langle \Theta\left(\frac{\partial S}{\partial Q}\right), Z \right\rangle = 0$$

$$\Rightarrow \left\langle ad^*_{\frac{\partial S}{\partial Q}} Q + \Theta\left(\frac{\partial S}{\partial Q}\right), Z \right\rangle = 0, \forall Z$$

$$\Rightarrow ad^*_{\frac{\partial S}{\partial Q}} Q + \Theta\left(\frac{\partial S}{\partial Q}\right) = 0$$

Souriau relation on foliation

- In his 1974 paper, Jean-Marie Souriau has written (without proof):
  $$\langle Q,[\beta,Z]\rangle + \tilde{\Theta}(\beta,Z) = 0$$
- To prove this equation, we have to consider the parametrized curve
  $$t \mapsto Ad_{\exp(tZ)}\beta \text{ with } Z \in \mathfrak{g} \text{ and } t \in R$$
- The parameterized curve $Ad_{\exp(tZ)}\beta$ passes, for $t=0$, through the point β, since $Ad_{\exp(0)}$ is the identical map of the Lie Algebra $\mathfrak{g}$. This curve is in the adjoint orbit of β. So by taking its derivative with respect to t, then for $t=0$, we obtain a tangent vector in β at the adjoint orbit of this point. When Z takes all possible values in $\mathfrak{g}$, the vectors thus obtained generate all the vector space tangent in β to the orbit of this point:

$$\left.\frac{d\Phi\left(Ad_{\exp(tZ)}\beta\right)}{dt}\right|_{t=0} = \left\langle \frac{d\Phi}{d\beta}, \left(\left.\frac{d\left(Ad_{\exp(tZ)}\beta\right)}{dt}\right|_{t=0}\right)\right\rangle = \langle Q, ad_Z\beta\rangle = \langle Q,[Z,\beta]\rangle$$

Souriau relation on foliation

- As we have seen before $\Phi\left(Ad_g\beta\right)=\Phi(\beta)-\left\langle\theta\left(g^{-1}\right),\beta\right\rangle$. If we set $g=\exp(tZ)$, we obtain:

$$\Phi\left(Ad_{\exp(tZ)}\beta\right)=\Phi(\beta)-\left\langle\theta\left(\exp(-tZ)\right),\beta\right\rangle$$

- By derivation with respect to t at $t=0$, we finally recover the equation given by Souriau :

$$\left.\frac{d\Phi\left(Ad_{\exp(tZ)}\beta\right)}{dt}\right|_{t=0}=\left\langle Q,[Z,\beta]\right\rangle=-\left\langle d\theta(-Z),\beta\right\rangle \text{ with } \tilde{\Theta}(X,Y)=-\left\langle d\theta(X),Y\right\rangle$$

Fundamental Equation of Geometric Thermodynamic: Entropy Function is an Invariant Casimir Function in Coadjoint Representation

Entropy S

$$S:\mathfrak{g}^* \to R$$
$$Q \mapsto S(Q)$$

Heat Q, (Planck) Temperature β and Φ Massieu Characteristic Function

$$S(Q)=\langle \beta,Q\rangle-\Phi(\beta)\ ,\ Q=\frac{\partial\Phi(\beta)}{\partial\beta}\in\mathfrak{g}^*\ ,\ \beta=\frac{\partial S(Q)}{\partial Q}\in\mathfrak{g}$$

Invariance of Entropy S Under the action of the Group

$$Q\left(Ad_g(\beta)\right)=Ad_g^*(Q)+\theta(g)$$

$$S\left(Q\left(Ad_g(\beta)\right)\right)=S(Q)$$

$$\Theta(X)=T_e\theta\left(X(e)\right)$$

$$\tilde{\Theta}(X,Y)=\langle\Theta(X),Y\rangle=J_{[X,Y]}-\{J_X,J_Y\}=-\langle d\theta(X),Y\rangle$$

Moment Map J

New Definition of Entropy S as Invariant Casimir Function in Coadjoint Representation

$$ad^*_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0$$

$$\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle Q,\left[\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right]\right\rangle+\tilde{\Theta}\left(\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right)=0$$

Lie Groups Thermodynamic Equations and its extension (1/3)

Q : Heat, element of dual Lie Algebra

β : (Planck) température element of Lie algebra

$$Q\left(Ad_g(\beta)\right) = Ad_g^*(Q) + \theta(g)$$

$\theta(g)$: Souriau Cocyle

Φ : Massieu Characteristic Function

J : Souriau Moment Map

$$\Phi(\beta) = -\log \int_M e^{-\langle \beta, J(\xi) \rangle} d\lambda$$

$$J : M \rightarrow \mathfrak{g}^*$$

S : Entropy

Legendre Transform

$$S(Q) = \langle \beta, Q \rangle - \Phi(\beta) \text{ with } Q = \frac{\partial \Phi(\beta)}{\partial \beta} \in \mathfrak{g}^* \text{ and } \beta = \frac{\partial S(Q)}{\partial Q} \in \mathfrak{g}$$

I : Fisher Information Metric

$$I(\beta) = -\frac{\partial^2 \Phi}{\partial \beta^2}$$

Lie Groups Thermodynamic Equations and its extension (2/3)

Entropy Invariance under the action of the Group !

$$S\left(Ad_g^{\#}(Q)\right)=S(Q)$$

$$Ad_g^{\#}(Q)=Ad_g^{*}(Q)+\theta(g)$$

Souriau characteristic of the foliation

$$\langle Q,[\beta,Z]\rangle+\tilde{\Theta}(\beta,Z)=0$$

Entropy Solution of Casimir Equation

$$ad^{*}_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0$$

Souriau cocycle

$$\Theta(X)=T_e\theta(X(e))$$

$$\theta(g)=Q\left(Ad_g(\beta)\right)-Ad_g^{*}(Q)$$

Entropy & Poisson Bracket

$$\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle Q,\left[\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right]\right\rangle+\tilde{\Theta}\left(\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right)=0$$

$$\tilde{\Theta}(X,Y)=\langle\Theta(X),Y\rangle=J_{[X,Y]}-\{J_X,J_Y\}=-\langle d\theta(X),Y\rangle$$

Lie Groups Thermodynamic Equations and its extension (3/3)

Entropy Production

$$dS = \tilde{\Theta}_\beta\left(\frac{\partial H}{\partial Q}, \beta\right) dt$$

2nd principle is related to positivity of Fisher tensor

$$\frac{dS}{dt} = \tilde{\Theta}_\beta\left(\frac{\partial H}{\partial Q}, \frac{\partial S}{\partial Q}\right) \geq 0$$

Metric Tensor related to Fisher Metric

$$\tilde{\Theta}_\beta\left(\frac{\partial H}{\partial Q}, \beta\right) = \tilde{\Theta}\left(\frac{\partial H}{\partial Q}, \beta\right) + \left\langle Q, \left[\frac{\partial H}{\partial Q}, \beta\right]\right\rangle$$

Time Evolution of Heat wrt to Hamiltonian H

$$\frac{dQ}{dt} = ad^*_{\frac{\partial H}{\partial Q}} Q + \Theta\left(\frac{\partial H}{\partial Q}\right)$$

Stochastic Equation

$$dQ + \left[ad^*_{\frac{\partial H}{\partial Q}} Q + \Theta\left(\frac{\partial H}{\partial Q}\right)\right] dt + \sum_{i=1}^{N}\left[ad^*_{\frac{\partial H_i}{\partial Q}} Q + \Theta\left(\frac{\partial H_i}{\partial Q}\right)\right] \circ dW_i(t) = 0$$

Euler-Poincaré Equation in case of Non-Null Cohomology

$$\frac{dQ}{dt} = ad^*_{\frac{\partial H}{\partial Q}} Q + \Theta\left(\frac{\partial H}{\partial Q}\right)$$

$$Q = \frac{\partial \Phi}{\partial \beta}$$

$$\frac{d}{dt}\frac{\partial \Phi}{\partial \beta} = ad^*_{\frac{\partial H}{\partial Q}} \frac{\partial \Phi}{\partial \beta} + \Theta\left(\frac{\partial H}{\partial Q}\right)$$

$$\left(ad^*_{\frac{\partial H}{\partial Q}} \frac{\partial \Phi}{\partial \beta}\right)_j + \Theta\left(\frac{\partial H}{\partial Q}\right)_j = C^k_{ij}\, ad^*_{\left(\frac{\partial H}{\partial Q}\right)^i} \left(\frac{\partial \Phi}{\partial \beta}\right)_k + \Theta_j$$

de Saxcé, G. Euler-Poincaré equation for Lie groups with non null symplectic cohomology. Application to the mechanics. In GSI 2019. LNCS; Nielsen, F., Barbaresco, F., Eds.; Springer: Berlin, Germany, 2019; Volume 11712

« Ayant eu l'occasion de m'occuper du mouvement de rotation d'un corps solide creux, dont la cavité est remplie de liquide, j'ai été conduit à mettre les équations générales de la mécanique sous une forme que je crois nouvelle et qu'il peut être intéressant de faire connaître » - Henri Poincaré, CRAS, 18 Février 1901

SÉANCE DU LUNDI 18 FÉVRIER 1901,

PRÉSIDENCE DE M. FOUQUÉ.

MEMOIRES ET COMMUNICATIONS

DES MEMBRES ET DES CORRESPONDANTS DE L'ACADÉMIE.

MÉCANIQUE RATIONNELLE. — *Sur une forme nouvelle des équations de la Mécanique*. Note de M. H. Poincaré.

« Ayant eu l'occasion de m'occuper du mouvement de rotation d'un corps solide creux, dont la cavité est remplie de liquide, j'ai été conduit à mettre les équations générales de la Mécanique sous une forme que je crois nouvelle et qu'il peut être intéressant de faire connaître.

$$\frac{d}{dt}\frac{dT}{d\eta_s} = \sum c_{ski}\frac{dT}{d\eta_i}\eta_k + \Omega_s.$$

« Elles sont surtout intéressantes dans le cas où U étant nul, T ne dépend que des η » - Henri Poincaré

Lie-Poisson variational principle

$$\frac{dQ}{dt} = ad^*_{\frac{\partial H}{\partial Q}} Q + \Theta\left(\frac{\partial H}{\partial Q}\right)$$

- This Lie-Poisson equation is equivalent to this **Lie-Poisson variational principle**:

$$\delta \int_0^\tau \left(\left\langle Q(t), \frac{\partial H}{\partial Q}(t) \right\rangle - H\left(Q(t)\right) \right) dt = 0 \quad \text{where} \quad \begin{cases} \dfrac{\partial H}{\partial Q} = g^{-1}\dot{g} \in \mathfrak{g}, g \in G \\ \dfrac{\partial^2 H}{\partial Q^2} \delta Q = \delta\left(\dfrac{\partial H}{\partial Q}\right), \eta = g^{-1}\delta g \\ \left\langle Q, \delta\left(\dfrac{\partial H}{\partial Q}\right)\right\rangle = \left\langle Q, \dot{\eta} + \left[\dfrac{\partial H}{\partial Q}, \eta\right]\right\rangle + \tilde{\Theta}\left(\dfrac{\partial H}{\partial Q}, \eta\right) \end{cases}$$

Lie-Poisson variational principle

- Proof of **Lie-Poisson variational principle**:

$$\delta\int_0^\tau\left(\left\langle Q(t),\frac{\partial H}{\partial Q}(t)\right\rangle - H\left(Q(t)\right)\right)dt=\int_0^\tau\left(\left\langle \delta Q,\frac{\partial H}{\partial Q}\right\rangle+\left\langle Q,\delta\left(\frac{\partial H}{\partial Q}\right)\right\rangle-\left\langle \delta Q,\frac{\partial H}{\partial Q}\right\rangle\right)dt \;=0$$

$$=\int_0^\tau\left(\left\langle Q,\frac{d\eta}{dt}+\left[\frac{\partial H}{\partial Q},\eta\right]\right\rangle+\tilde{\Theta}\left(\frac{\partial H}{\partial Q},\eta\right)\right)dt=\int_0^\tau\left(\left\langle Q,\frac{d\eta}{dt}\right\rangle+\left\langle Q,ad_{\frac{dH}{dQ}}\eta\right\rangle+\left\langle\Theta\left(\frac{\partial H}{\partial Q}\right),\eta\right\rangle\right)dt$$

$$\underset{\substack{Int.\\ by\\ parts}}{=}\int_0^\tau\left\langle -\frac{dQ}{dt}+ad^*_{\frac{dH}{dQ}}Q+\Theta\left(\frac{\partial H}{\partial Q}\right),\eta\right\rangle dt+\left\langle Q,\eta\right\rangle\Big|_0^\tau=0$$

Koszul Poisson Cohomology and Entropy Characterization

Poisson Cohomology was introduced by A. Lichnerowicz and J.L. Koszul.

Koszul Cohomology and seminal work of Elie Cartan. Koszul made reference to seminal E. Cartan paper

- "***Elie Cartan does not explicitly mention $\Lambda(g')$ [the complex of alternate forms on a Lie algebra], because he treats groups as symmetrical spaces and is therefore interested in differential forms which are invariant to both by the translations to the left and the translations to the right, which corresponds to the elements of $\Lambda(g')$ invariant by the prolongation of the coadjoint representation. Nevertheless, it can be said that by 1929 an essential piece of the cohomological theory of Lie algebras was in place.***" – Jean-Louis Koszul

Koszul Poisson Cohomology and Entropy Characterization

- Y. Vorob'ev and M.V. Karasev have suggested cohomology classification in terms of closed forms and de Rham Cohomology of coadjoint orbits Ω (called Euler orbits by authors), symplectic leaves of a Poisson manifold N .
- Let $Z^k(\Omega)$ and $H^k(\Omega)$ be the space of closed k-forms on Ω and their de Rham cohomology classes.
- Considering the base of the fibration of N by these orbits as N/Ω , they have introduced the smooth mapping

$Z^k[\Omega] = C^\infty\left(N/\Omega \rightarrow Z^k(\Omega)\right)$ and $H^k[\Omega] = C^\infty\left(N/\Omega \rightarrow H^k(\Omega)\right)$

- The elements of $Z^k(\Omega)$ are closed forms on Ω , depending on coordinates on N/Ω
- Then $H^0[\Omega] = Casim(N)$ is the set of Casimir functions on N , of functions which are constant on all Euler orbits.
- **Entropy is then characterized by zero-dimensional de Rham Cohomology**.
- The center of Poisson algebra induced from the symplectic structure is the zero-dimensional de Rham cohomology group, the Casimir functions.

Poincaré Unit Disk and SU(1,1) Lie Group

- The group of complex unimodular pseudo-unitary matrices $SU(1,1)$:

$$G = SU(1,1) = \left\{ \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} / |a|^2 - |b|^2 = 1,\ a,b \in C \right\}$$

- the Lie algebra $\mathfrak{g} = \mathfrak{su}(1,1)$ is given by:

$$\mathfrak{g} = \left\{ \begin{pmatrix} -ir & \eta \\ \eta^* & ir \end{pmatrix} / r \in R, \eta \in C \right\}$$

with the following bases $(u_1, u_2, u_3) \in \mathfrak{g}$:

$$u_1 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ u_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ u_3 = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

with the commutation relation:

$$[u_3, u_2] = u_1, [u_3, u_1] = u_2, [u_2, u_1] = -u_3$$

Poincaré Unit Disk and SU(1,1) Lie Group

- Dual base on dual Lie algebra is named $\left(u_1^*, u_2^*, u_3^*\right) \in \mathfrak{g}^*$
- The dual vector space $\mathfrak{g}^* = \mathfrak{su}^*(1,1)$ can be identified with the subspace of $\mathfrak{sl}(2,C)$ of the form:

$$\mathfrak{g}^* = \left\{ \begin{pmatrix} z & x+iy \\ -x+iy & -z \end{pmatrix} = x\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} / \, x, y, z \in R \right\}$$

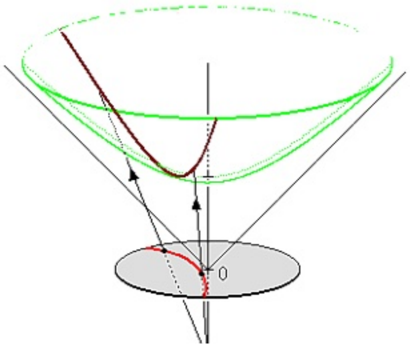
- Coadjoint action of $g \in G$ on dual Lie algebra $\xi \in \mathfrak{g}^*$ is written $g.\xi$

Coadjoint Orbit of SU(1,1) and Souriau Moment Map

$$J(z) = r\left(\frac{z+z^*}{\left(1-|z|^2\right)} u_1^* + \frac{z-z^*}{i\left(1-|z|^2\right)} u_2^* + \frac{1+|z|^2}{\left(1-|z|^2\right)} u_3^* \right) \in \mathrm{O}\left(r u_3^*\right), z \in D$$

- J is linked to the natural action of G on D (by fractional linear transforms) but also the coadjoint action of G on $\mathrm{O}\left(r u_3^*\right) = G / K$
- J^{-1} could be interpreted as the stereographic projection from the two-sphere S^2 onto $C \cup \infty$:

The coadjoint action of $G = SU(1,1)$ is the upper sheet $x_3 > 0$ of the two-sheet hyperboloid



Charles-Michel Marle, Projection stéréographique et moments, hal-02157930, version 1, Juin 2019

$$\left\{ \xi = x_1 u_1^* + x_2 u_2^* + x_3 u_3^* : -x_1^2 - x_2^2 + x_3^2 = r^2 \right\}$$

THALES

Moment Map for SU(1,1)

Invariant Moment Map

- The associated moment map $J : D \to su^*(1,1)$ defined by $J(z).u_i = J_i(z,z^*)$, maps D into a coadjoint orbit in $su^*(1,1)$.
- Then, we can write the moment map as a matrix element of $su^*(1,1)$:

$$J(z) = J_1\left(z,z^*\right)u_1^* + J_2\left(z,z^*\right)u_2^* + J_3\left(z,z^*\right)u_3^*$$

$$J(z) = r\left(\frac{z+z^*}{\left(1-|z|^2\right)}u_1^* + \frac{z-z^*}{i\left(1-|z|^2\right)}u_2^* + \frac{1+|z|^2}{\left(1-|z|^2\right)}u_3^*\right), z \in D$$

$$J(z) = r\begin{pmatrix} \frac{1+|z|^2}{1-|z|^2} & -2\frac{z^*}{1-|z|^2} \\ 2\frac{z}{1-|z|^2} & -\frac{1+|z|^2}{1-|z|^2} \end{pmatrix} \in \mathfrak{g}^*$$

$$\mathfrak{g}^* = \left\{\begin{pmatrix} z & x+iy \\ -x+iy & -z \end{pmatrix} = x\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} / x,y,z \in R\right\}$$

Poisson Bracket for SU(1,1)

- Since the unit disk is Kählerian, it is symplectic and so can be given a phase space structure and interpretation. This Poisson Bracket could be written in terms of the Poincare disk coordinates as:

$$\{f,g\} = \frac{\left(1-|z|^2\right)^2}{2i}\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial z^*} - \frac{\partial f}{\partial z^*}\frac{\partial g}{\partial z}\right)$$

- It is possible to define new coordinates (q,p) that are canonical in the sense that:

$$\{f,g\} = \left(\frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}\right)$$ with coordinates given by: $$\frac{q+ip}{2} = \frac{z}{\sqrt{1-|z|^2}}$$

- The Metriplectic equation is then given by:

$$\frac{\partial f}{\partial t} = \{f,H\} + (f,H) = \frac{\left(1-|z|^2\right)^2}{2i}\left(\frac{\partial f}{\partial z}\frac{\partial H}{\partial z^*} - \frac{\partial f}{\partial z^*}\frac{\partial H}{\partial z}\right) + (f,H)$$

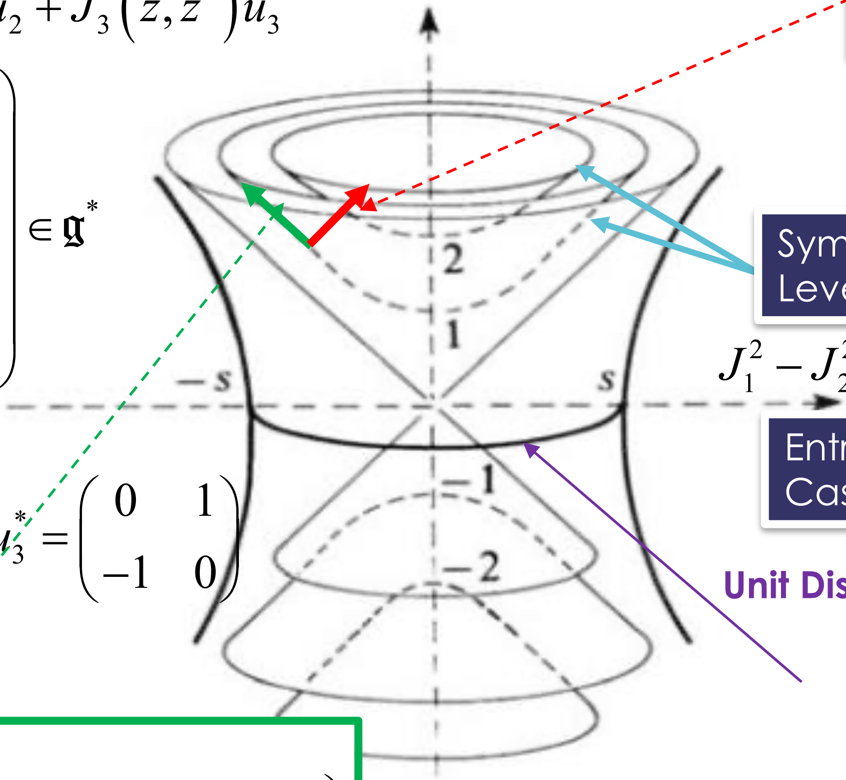
Souriau Lie Groups Thermodynamics Model for SU(1,1) and Unit Disk

$$J(z)=J_1\left(z,z^*\right)u_1^*+J_2\left(z,z^*\right)u_2^*+J_3\left(z,z^*\right)u_3^*$$

$$J(z)=\rho\begin{pmatrix}\dfrac{1+|z|^2}{1-|z|^2} & -2\dfrac{z^*}{1-|z|^2}\\ 2\dfrac{z}{1-|z|^2} & -\dfrac{1+|z|^2}{1-|z|^2}\end{pmatrix}\in\mathfrak{g}^*$$

$$J_1u_1^*+J_2u_2^*+J_3u_3^*\in su^*(1,1)$$

$$u_1^*=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad u_2^*=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix},\quad u_3^*=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$



Dissipative flow (transverse to symplectic leaves)

$$\frac{\partial f}{\partial t}=(f,H)$$

Symplectic Leaves = Level sets of Entropy

$$J_1^2-J_2^2-J_3^2=\rho^2\quad,\quad J_1\geq\rho$$

Entropy = Invariant Casimir Function

Unit Disk as Symplectic Manifold

$$D=\left\{z\in\mathbb{C}\,/\,|z|<1\right\}$$

$$\omega_\rho=2ir\frac{dz\wedge dz^*}{\left(1-|z|^2\right)^2}$$

Non Dissipative flow (along the symplectic leaves)

$$\frac{\partial f}{\partial t}=\{f,H\}=\frac{\left(1-|z|^2\right)^2}{2i}\left(\frac{\partial f}{\partial z}\frac{\partial H}{\partial z^*}-\frac{\partial f}{\partial z^*}\frac{\partial H}{\partial z}\right)$$

THALES

Souriau Gibbs density for SU(1,1)

Covariant Gibbs density

- We can the write the covariant Gibbs density in the unit disk given by moment map of the Lie group $SU(1,1)$ and geometric temperature in its Lie algebra $\beta \in \Lambda_\beta$:

$$p_{Gibbs}(z) = \frac{e^{-\langle J(z),\beta\rangle}}{\int_D e^{-\langle J(z),\beta\rangle} d\lambda(z)} \text{ with } d\lambda(z) = 2ir\frac{dz \wedge dz^*}{\left(1-|z|^2\right)^2}$$

$$p_{Gibbs}(z) = \frac{e^{-\left\langle r\left(2\Im bb^+ - Tr\left(\Im bb^+\right)I\right),\beta\right\rangle}}{\int_D e^{-\langle J(z),\beta\rangle} d\lambda(z)} = \frac{e^{-\left\langle r\begin{pmatrix} \frac{1+|z|^2}{\left(1-|z|^2\right)} & \frac{-2z^*}{\left(1-|z|^2\right)} \\ \frac{2z}{\left(1-|z|^2\right)} & -\frac{1+|z|^2}{\left(1-|z|^2\right)} \end{pmatrix}, \begin{pmatrix} ir & \eta \\ \eta^* & -ir \end{pmatrix}\right\rangle}}{\int_D e^{-\langle J(z),\beta\rangle} d\lambda(z)}$$

$$J(z) = r\left(2Mbb^+ - Tr\left(Mbb^+\right)I\right) \text{ with } M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } b = \frac{1}{1-|z|^2}\begin{bmatrix} 1 \\ -z \end{bmatrix}$$

Souriau Gibbs density

Covariant Gibbs Density

$$p_{Gibbs}(z)=\frac{e^{-\left\langle \rho\begin{pmatrix}\frac{1+|z|^2}{(1-|z|^2)} & \frac{-2z^*}{(1-|z|^2)}\\ \frac{2z}{(1-|z|^2)} & -\frac{1+|z|^2}{(1-|z|^2)}\end{pmatrix},\begin{pmatrix} ir & \eta\\ \eta^* & -ir\end{pmatrix}\right\rangle}}{\int_D e^{-\langle J(z),\beta\rangle}d\lambda(z)}$$

- To write the Gibbs density with respect to its statistical moments, we have to express the density with respect to $Q=E\left[J(z)\right]$
- Then, we have to invert the relation between Q and β, to replace $\beta=\begin{pmatrix} ir & \eta\\ \eta^* & -ir\end{pmatrix}\in\Lambda_\beta$ by $\beta=\Theta^{-1}(Q)\in\mathfrak{g}$ where $Q=\frac{\partial\Phi(\beta)}{\partial\beta}=\Theta(\beta)\in\mathfrak{g}^*$ with $\Phi(\beta)=-\log\int_D e^{-\langle J(z),\beta\rangle}d\lambda(z)$ deduce from Legendre tranform. The mean moment map is given by:

$$Q=E\left[J(z)\right]=E\left[\rho\begin{pmatrix}\frac{1+|w|^2}{(1-|w|^2)} & \frac{-2w^*}{(1-|w|^2)}\\ \frac{2w}{(1-|w|^2)} & -\frac{1+|w|^2}{(1-|w|^2)}\end{pmatrix}\right] \quad \text{where } w\in D$$

Gauss Density on Siegel Unit Disk

Moment Map of SU(n,n)/S(U(n)xU(n))

- The moment map for $SU(n,n)/S(U(n)\times U(n))$ is then given by:

$$J(Z)=\rho n\begin{pmatrix}(I_n-ZZ^+)^{-1}(I_n+ZZ^+) & -2Z^+(I_n-ZZ^+)^{-1}\\ 2(I_n-ZZ^+)^{-1}Z & (I_n+ZZ^+)(I_n-ZZ^+)^{-1}\end{pmatrix}\in\mathfrak{g}^*$$

- The Souriau Gibbs density is then given with $\beta,M\in\mathfrak{g}$ and $Z\in SD_n$ by:

$$p_{Gibbs}(Z)=\frac{e^{-\left\langle \rho n\begin{pmatrix}(I_n-ZZ^+)^{-1}(I_n+ZZ^+) & -2Z^+(I_n-ZZ^+)^{-1}\\ 2(I_n-ZZ^+)^{-1}Z & (I_n+ZZ^+)(I_n-ZZ^+)^{-1}\end{pmatrix},\beta\right\rangle}}{\int\limits_{SD_n} e^{-\langle J(Z),\beta\rangle}d\lambda(Z)}$$

$$\beta=\Theta^{-1}(Q)\in\mathfrak{g}$$

$$Q=E\left[J(Z)\right]$$

$$Q=\frac{\partial\Phi(\beta)}{\partial\beta}=\Theta(\beta)\in\mathfrak{g}^*$$

- Gauss density of SPD matrix is given by Cayley Transform with:

$$Z=(Y-I)(Y+I)^{-1},Y\in Sym(n)^+$$

THALES

THALES

Lars Onsager & Baptiste Coquinot:
Symplectic Foliation Model of METRIPLECTIC FLOW

METRIPLECTIC FRAMEWORK FOR DISSIPATIVE HEAT EQUATION

METRIPLECTIC & GENERIC Models

- Systems that preserve energy throughout the phase are characterized by an Hamiltonian formulation of dynamics.
- Classical Hamiltonian systems cannot take into account dissipative effects, as irreversible changes from a thermodynamic standpoint (dissipative dynamics that do not preserve energy).
- A. N. Kaufman and P.J. Morrison have introduced in 1983, the **metriplectic** bracket by introducing a bracket formalism that ensures both conservation of energy and non-decrease of entropy, and that reduces to the standard Poisson bracket formalism in the limit of no dissipation.
- This model has been axiomatized in parallel by Grmela and his collaborators (**GENERIC method: General Equation for Non-Equilibrium Reversible Irreversible Coupling**).

METRIPLECTIC FRAMEWORK FOR DISSIPATIVE HEAT EQUATION

Metriplectic Model

- Dissipation could take two forms:
  - viscosity removes energy from the system (e.g. Navier-Stokes equation)
  - thermal diffusion with conservation of energy and entropy production by heat transfer (e.g. Boltzmann operator, Transport equations with collision operators,…).
- Metriplectic dynamics includes these kind of systems compliant both with the first and second thermodynamics principles.
- In the metriplectic formalism, evolution equation is given by a new bracket: $\{\{.,.\}\}$

$$\frac{df}{dt} = \{\{f,F\}\} = \{f,F\} + (f,F)$$

Non-Dissipative Bracket (Poisson Bracket) → $\{f,F\}$

Dissipative Bracket (Metric Flow Bracket) → (f,F)

METRIPLECTIC FRAMEWORK FOR DISSIPATIVE HEAT EQUATION

Metriplectic Model

$$\frac{df}{dt} = \{\{f, F\}\} = \{f, F\} + (f, F)$$

- Hamiltonian components is introduced by requiring:

$$F = H + S$$

- The 2nd bracket has 2 constraints :

$$(f, F) = (F, f) \text{ and } (f, f) \geq 0$$

- with **the entropy S selected from the set of Casimir invariants of the non-canonical Poisson bracket**, playing the role of a Lyapunov functional.
- A metriplectic vector field induced by F is given by the dynamics:

$$\frac{dz_i}{dt} = J_{ij}\frac{\partial F}{\partial z_j} + M_{ij}\frac{\partial F}{\partial z_j}$$

Metriplectic Model Compliance to Thermodynamics Principles

Metriplectic Model compliance with two first principles of thermodynamics

- First principle: **Energy conservation**

$$F = H + S$$

$$\frac{dH}{dt} = \{H,F\} + (H,F) = \{H,H\} + \{H,S\} + (H,H) + (H,S) = 0$$

$$\text{because } \begin{cases} \{H,H\} = 0 \text{ by symmetry} \\ \{f,S\} = 0, \quad \forall f \\ (H,f) = 0, \quad \forall f \end{cases}$$

Metriplectic Model Compliance to Thermodynamics Principles

Metriplectic Model compliance with two first principles of thermodynamics

- Second principle: **Entropy production**

$$\frac{dS}{dt} = \{S,F\} + (S,F) = 0 + (S,H) + (S,S) = (S,S) \geq 0$$

$$\text{because } \begin{cases} \{S,f\} = 0, \quad \forall f \quad \text{(Casimir property)} \\ (f,H) = 0, \quad \forall f \\ \text{positive semi-definite} \Rightarrow \quad (S,S) \geq 0 \end{cases}$$

- The choice of thermal equilibrium is induced by selecting **Entropy as Casimir invariant function**.

Metriplectic Model and Casimir invariants

Metriplectic Model

- Finally in metriplectic systems, the geometry is determined by two compatible brackets, a Poisson bracket and a symmetric bracket:

$$\frac{df}{dt}=\{\{f,F\}\}=\{f,H\}+(f,S)$$

- The energy H is a Casimir invariant of the dissipative bracket, and the entropy S is a Casimir invariant of the Poisson bracket:

$$\{S,H\}=0 \quad \forall H$$

$$(H,S)=0 \quad \forall S$$

OPEN

THALES

Foliation Structures of Thermodynamics-Informed Neural Networks

Thermodynamics-Informed Neural Networks and Symplectic Foliation

- TINN are based on metriplectic flow (also called GENERIC flow) modeling:
  - **non-dissipative dynamics (1st thermodynamic principle of energy preservation)**
  - **dissipative dynamics (2nd thermodynamic principle of entropy production).**

$$\frac{dz}{dt}=\{z,E\}+(z,S)$$

Metriplectic Flow →

$$\frac{dz}{dt}=L(z)\frac{\partial E}{\partial z}+M(z)\frac{\partial S}{\partial z}$$

Energy Preservation

$$\frac{dE}{dt}(z)=0$$

Entropy Production

$$\frac{dS}{dt}(z)=\frac{\partial S}{\partial z}M(z)\frac{\partial S}{\partial z}\geq 0$$

- Souriau's Lie Groups Thermodynamics allows to characterize geometrically metriplectic flow by:
  - **Symplectic foliation (non-dissipative part)**
  - **Riemanian Foliations (dissipative part), transverse to symplectic leaves**
- From symmetries of the problem, we can generate **Lie group coadjoint orbits that are symplectic leaves defined as level sets of Entropy**, where Entropy is an invariant Casimir function in coadjoint representation (invariant function on these symplectic leaves).
  - **Dynamics along these symplectic leaves**, given by Poisson bracket, characterize non-dissipative dynamics with Entropy preservation.
  - **Dissipative dynamics are then given by transverse Poisson structure and metric flow bracket**, with evolution from leaf to leaf constrained by entropy production and Energy preservation. Foliation transverse structure are linked with Riemannian foliations given by Souriau-Fisher metric.

Metriplectic Flow on Symplectic Foliation & Transverse Metric Foliation

Foliation Leaves = Level Sets of Energy

Foliation Leaves = Level Sets of Entropy

METRIC FOLIATION $\mathfrak{I}^{\perp} \perp \mathfrak{I}$ **SYMPLECTIC FOLIATION**

1st Principle of Thermodynamics Preservation of Energy

2nd Principle of Thermodynamics Entropy Production

METRIPLECTIC FLOW

$$\frac{dF}{dt} = \{F,H\} + (F,S)$$

Non-dissipative Entropy = Constant

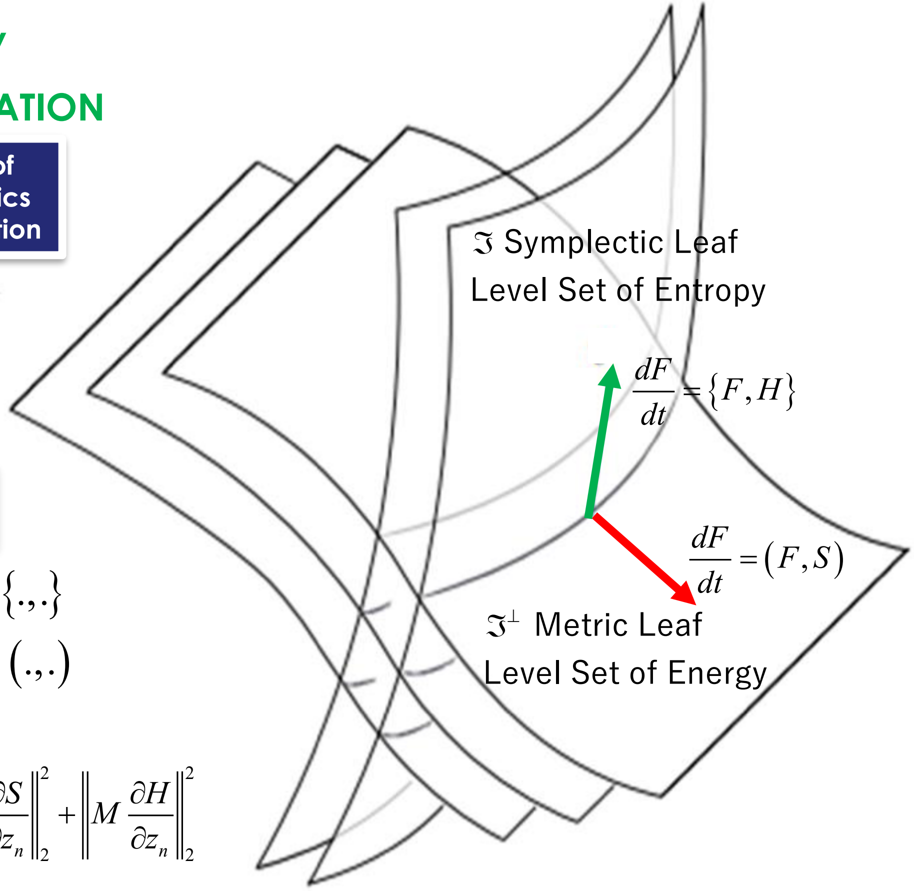
Dissipative Energy = Constant

$\{F,S\} = 0 \quad \forall F : S$ Entropy Casimir Function for $\{.,.\}$

$(F,H) = 0 \quad \forall F : H$ Energy Casimir Function for $(.,.)$

Thermodynamics-Informed Neural Networks

$$\Lambda = \frac{1}{N_{batch}} \sum_{n=0}^{N_{batch}} \left(\lambda \Lambda_n^{data} + \Lambda_n^{\deg} \right) \qquad \Lambda_n^{data} = \left\| \frac{dz^{GT}}{dt} - \frac{dz^{net}}{dt} \right\|_2^2 \qquad \Lambda_n^{\deg} = \left\| L \frac{\partial S}{\partial z_n} \right\|_2^2 + \left\| M \frac{\partial H}{\partial z_n} \right\|_2^2$$



SINN & MINN: Symplectic-Informed NN & Metriplectic-Informed NN

Symplectic Integrator

- Capture of symmetries (Geometry-Informed)
- Capture Noether Invariants via Souriau's Moment Map as Energy, Angular Momentum (Physics-Informed)

$$\frac{dF}{dt} = \{F, H\}$$

non-dissipative

(*) The time evolution of Hamilton's equations

$$\dot{p} = -\frac{\partial H}{\partial q} \quad , \quad \dot{q} = \frac{\partial H}{\partial p}$$

is a symplectomorphism, meaning that it conserves the symplectic 2-form $dp \wedge dq$. A numerical scheme is a symplectic integrator if it also conserves this 2-form and equation are reduce to:

$$z = (p,q) \Rightarrow \dot{z} = \{z, H(z)\}$$

Metriplectic Integrator

- Capture 1st principle (Energy preservation, Moment map)
- Capture 2nd principle (Entropy Production)

$$\frac{dF}{dt} = \{F, H\} + (F, S)$$

dissipative

$$F = H + S$$

(*) 1st principle: Preservation of Energy

$$\frac{dH}{dt} = 0$$

2nd Principle Entropy production

$$\frac{dS}{dt} \geq 0$$

THALES
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Metriplectic Flow and Symplectic Complete Integrable System

- They form what is well known in **symplectic geometry** under a variety of names: **dual pair, symplectically complete foliation, bifoliation, bifibration.**
- **These foliations are fibrations**, then the space of the leaves of the polar foliation is an affine manifold which has the same dimension of the invariant tori and plays the role of 'action manifold' of the system. Furthermore, the base of the fibration by the invariant tori is a Poisson manifold and the space of **its symplectic leaves can be identified with the action manifold**.
- A submanifold of M is said to be isotropic (resp. coisotropic) if its tangent spaces are contained in (resp. contain) their own symplectic complements. **Lagrangian submanifolds are both isotropic and coisotropic**. Let F be a foliation of M. The polar of $\mathfrak{I}$, if it exists, is the unique foliation $\mathfrak{I}^{\perp}$ of M with the property that the tangent spaces of its leaves are the **symplectic orthogonals** of the tangent spaces of the leaves of $\mathfrak{I}$. Furthermore, $\left(\mathfrak{I}^{\perp}\right)^{\perp}=\mathfrak{I}$. If $\mathfrak{I}$ is isotropic then $\mathfrak{I}^{\perp}$ is coisotropic.
- A foliation $\mathfrak{I}$ which has a polar is called **symplectically complete**. If $\mathfrak{I}$ is symplectically complete, then the pair $\left(\mathfrak{I},\mathfrak{I}^{\perp}\right)$ is called a **bifoliation**. If, moreover, both $\mathfrak{I}$ and $\mathfrak{I}^{\perp}$ are fibrations, then $\left(\mathfrak{I},\mathfrak{I}^{\perp}\right)$ is also called a **bifibration** A bifibration is a particular case of **dual pair.**

Symplectic potentials and Guillemin Metric

- We consider **Action-Angle coordinates:** $\omega = dx^i \wedge d\theta_i$
- We also consider Moment Map $\mu : M \to \mathfrak{g}^*$ where $\left(x^1, \cdots, x^n\right)$ are coordinates on $\mathfrak{g}^*$ given by $x^i = \langle X_i, . \rangle$ where $\left(X_1, X_2, ..., X_n\right)$ is a base of vectors field of group action: $dx^i = -i_{X_i}\omega$. We can select angular coordinates such that $X_i = \dfrac{\partial}{\partial \theta_i}$
- Symplectic coordinates and complex structure: Consider a complexe structure J

$$J dx^i = G^{ij} d\theta_j \quad \text{and} \quad J d\theta_i = -G_{ij} dx^j \quad \text{where} \quad G_{ij} = \left(G^{ij}\right)^{-1}$$

- We can then deduce the metric:

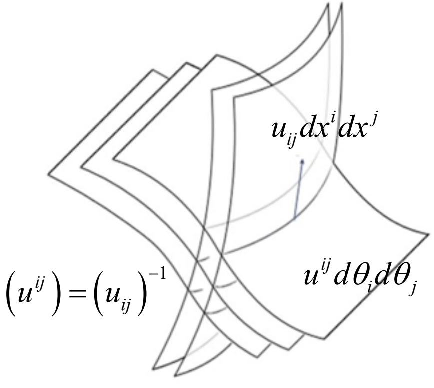
$$g = G_{ij} dx^i dx^j + G^{ij} d\theta_i d\theta_j \quad \text{where} \quad \left(G_{ij}\right) \text{ is symetric positive definite}$$

- We can make appear a symplectic potential:

$$dJd\theta_i = -\frac{\partial G_{ij}}{\partial x^k} dx^k \wedge dx^j \;\; \text{of type } (1,1)$$

$$dJd\theta_i = 0 \Rightarrow \frac{\partial G_{ij}}{\partial x^k} = \frac{\partial G_{ik}}{\partial x^j}, \exists u \text{ convex, } G_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j} = u_{ij}$$

$$g = u_{ij} dx^i dx^j + u^{ij} d\theta_i d\theta_j \quad \text{where} \quad \left(u^{ij}\right) = \left(u_{ij}\right)^{-1}$$



Transversally oriented foliation & the Godbillon–Vey characteristic class

GODBILLON–VEY CLASSES OF SYMPLECTIC FOLIATIONS by Kentaro Mikami

- Each transversally oriented foliation has the Godbillon–Vey characteristic class, and regular Poisson structures define symplectic foliations.
- K. Mikami has given a new interpretation and the explicit formula for a representative of the Godbillon–Vey characteristic classes of symplectic foliations in the context of Poisson geometry.
- For each transversely oriented foliation, we have the famous Godbillon-Vey characteristic class. When the symplectic foliations of regular Poisson structures are transversely oriented, they have the Godbillon-Vey characteristic classes. K. Mikami has given a formula defining their Godbillon-Vey classes in terms of Poisson structure.
- Transversely oriented foliations have secondary characteristic classes which are called the Godbillon-Vey class.
  - Reference: **J.-L. Koszul, Crochet de Schouten-Nijenhuis et cohomologie, in ‘Elie Cartan et les Mathématiques d’aujourd’hui’, 257-271; Société Math. de France, Astérisque, hors série, 1985.**

THALES
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Canonical /Noncanonical Hamiltonian Structures and Poisson Bracket

Non-dissipative part of Metriplectic Model

- The 1st part of a metriplectic vector field relative to non-dissipative part is given by Poisson bracket. Considering the Hamiltonian function $H(q,p)$ depending on the canonical coordinates q and momenta p , with $z=(p,q)$, Hamiltonian equations:

$$\frac{dz_i}{dt}=J_{ij}\frac{\partial H}{\partial z_j}=\{z_i,H\},\quad i,j=1,...,2N$$

$$J=\begin{pmatrix}0_N & I_N\\ -I_N & 0_N\end{pmatrix} \text{ and } \{f,g\}=\frac{\partial f}{\partial z_j}J_{ij}\frac{\partial g}{\partial z_j}$$

- We note the symplectic 2-form $\omega=\sum dp^i\wedge dq_i$ such that $\omega^{ik}J_{kj}=\delta_{ij}$

Metric Flow Structures and Symmetric Bracket

Dissipative part of Metriplectic Model

- The 2nd part of a metriplectic vector field relative to dissipative part, is a flow in a metric space. A metric flow on a finite dimensional phase space manifold, has the following form in coordinates:

$$\frac{dz_i}{dt} = M_{ij}\frac{\partial S}{\partial z_j} = (z_i, S), \quad i = 1,\dots,M$$

$$(f,g) = \frac{\partial f}{\partial z_j} M_{ij}(z) \frac{\partial g}{\partial z_j}, \quad i,j = 1,\dots,M \ \text{ with } \ (f,g) = (g,f)$$

- where S is an Entropy function. We should have the properties for the metric:

$$M \text{ positive semi-definite} \Rightarrow \frac{dS}{dt} = (S,S) \geq 0$$

$$\text{Energy Conservation } H \text{ (Hamiltonian)} \Rightarrow (H,f) = 0, \quad \forall f$$

Metriplectic Model

- **The symmetry condition is a generalization of the Onsager symmetry of linear irreversible thermodynamics to non-linear problems**, but classically in Metriplectic model, they don't consider the possibility of Casimir symmetry.
- In the Lie-Poisson framework, different dissipation Bracket $(.,.)$ that can be defined by the Lie algebra have been proposed in the literature for Metriplectic system: the double bracket, Cartan-Killing bracket, and Casimir dissipation bracket.
- For Lie-Poisson systems, Lie-Poisson bracket for two functions f, h is given by:

$$\{f,h\}(z)=\left\langle z,\left[\frac{\partial f}{\partial z},\frac{\partial h}{\partial z}\right]\right\rangle=\left\langle z,-ad_{\frac{\partial h}{\partial z}}\frac{\partial f}{\partial z}\right\rangle=\left\langle ad^*_{\frac{\partial h}{\partial z}}z,\frac{\partial f}{\partial z}\right\rangle$$

- Hamiltonian dynamics is given by :

$$\frac{df}{dt}=\{f,h\}(z)=\left\langle ad^*_{\frac{\partial h}{\partial z}}z,\frac{\partial f}{\partial z}\right\rangle\Rightarrow\frac{dz}{dt}=ad^*_{\frac{\partial h}{\partial z}}z$$

Metriplectic Model

- In coordinate realization, with a coordinate chart (z_i), the Poisson bivector is represented by a set of coefficient function determining the Poisson bracket:

$$\{f,h\} = J_{ij} \frac{\partial f}{\partial z_i} \frac{\partial h}{\partial z_j} \Rightarrow \frac{dz_i}{dt} = J_{ij} \frac{\partial h}{\partial z_j}$$

- **For Lie-Poisson structure defined on the dual of a finite dimensional Lie algebra, we can introduce structure constants** with an N dimensional Lie algebra admitting a basis $\{e_1,...,e_N\}$: $[e_i, e_j] = c_{ij}^k e_k$ (with summation convention over the repeated indices).
- The Lie-Poisson dynamics is given by:

$$J_{ij} = c_{ij}^k z_k \Rightarrow \frac{dz_j}{dt} = c_{ij}^k z_k \frac{\partial h}{\partial z_i}$$

METRIPLECTIC MODEL

Dissipation bracket as double bracket

- The double bracket is given by:

$$(f,h)=\sum_j J_{ij}J_{lj}\frac{\partial f}{\partial z_i}\frac{\partial h}{\partial z_l}=\sum_j c_{ij}^k c_{lj}^r z_k z_r \frac{\partial f}{\partial z_i}\frac{\partial f}{\partial z_l}$$

- with the metriplectic dynamics :

$$\frac{dz_j}{dt}=c_{ij}^k z_k\frac{\partial h}{\partial z_i}+\sum_i c_{ji}^k c_{li}^r z_r z_n\frac{\partial h}{\partial z_l}$$

METRIPLECTIC MODEL

The symmetric dissipative metriplectic bracket as Euclidean metric tensor on the symplectic leaves foliated by the Casimir invariants

- ***Sato observed that the canonical form of the symmetric dissipative part of the metriplectic bracket is identified in terms of a 'canonical metric tensor' corresponding to an Euclidean metric tensor on the symplectic leaves foliated by the Casimir invariants.***
- It makes the link with Symplectic model of Lie groups Thermodynamics. A single generating function $\Phi = \langle \beta, Q \rangle - S$ **is sufficient to generate the dynamics by the action of the metriplectic bracket:**

$$\frac{dF}{dt} = \{\{F, \Phi\}\} = -\beta^{-1}\{F, \Phi\} + (F, \Phi)$$

Variational Principles of Thermodynamics

Variational Principles

- **D'Alembert's variational principles** and Gauss' principle of the least constraint, which are differential principles
- **Maupertuis' principle of the least action** and the Hamilton principle, which are integral principles

Thermodynamics can be embraced with variational principles

- **ONSAGER's principle of least dissipation of energy** restricted to the particular case of heat conduction in anisotropic continua
- **ONSAGER & MACHLUP's principle and TISZA & MANNING's principle of least dissipation of energy** extended for the case of adiabatically isolated, non-continuous systems.
- **PRIGOGINE's principle of minimum production of entropy** with ONO clarification of the relation with least dissipation of energy.
- **GLANSDORFF and PRIGOGINE's integrale principle** when the conductivity coefficients are not constant

COQUINOT NON-EQUILIBRIUM THERMODYNAMIC THEORY OF DISSIPATIVE BRACKETS

- Coquinot deduced that the **dynamics of out-of-equilibrium thermodynamics** on the phase space can be expressed **with a symmetric bracket** for any two functionals f and g :

$$(f,g)=\frac{1}{\Upsilon}\int_{\Omega}\nabla\left(\frac{\partial f}{\partial\zeta_a(y)}\right)L_{\alpha\beta}\nabla\left(\frac{\partial g}{\partial\zeta_\beta(y)}\right)d^3y$$

- Coquinot has observed that this equation is a **pure geometric object, independent of the basis** $\{\zeta_\alpha\}$, where the functional derivatives can be seen as functional gradients, and both functional gradients are contracted thanks to the pseudometric $[L_{\alpha\beta}]$ and where **the bracket is symmetric thanks to the Onsager-Casimir relations.**

- As previously, has been demonstrated that:

$$\frac{\partial\sigma(x,t)}{\partial t}=\int_{\Omega}\left[\nabla\left(\frac{\partial\sigma(x,t)}{\partial\zeta_\alpha(y,t)}\right)L_{\alpha\beta}(y,t)\nabla\left(\frac{\partial S(t)}{\partial\zeta_\beta(y,t)}\right)\right]d^3y=\Upsilon(\sigma,S)$$

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COQUINOT NON-EQUILIBRIUM THERMODYNAMIC THEORY OF DISSIPATIVE BRACKETS

- Baptiste Coquinot proves a **formal equivalence between the classical out-of-equilibrium thermodynamics and a subclass of metriplectic dynamical systems**, showing that the pseudometric nature of the dissipative bracket, usually an ad hoc hypothesis, is the **exact transcription of the well-known second law of thermodynamics and Onsager's relations** through this equivalence.
- Coquinot's construction shows that the dissipative brackets are completely natural for non-equilibrium thermodynamics, just as Poisson brackets are natural for Hamiltonian dynamics, **deriving a general dissipative bracket, for the first time, from basic thermodynamic first principles**.
- Baptiste Coquinot has considered **the role of entropy, a Casimir invariant, as counterpart to the role of the Hamiltonian in analytical mechanics**, the non-negativity of the pseudometric ensures the entropy growth, as given by the second law of thermodynamics.

Baptiste Coquinot, Philip J. Morrison, A General Metriplectic Framework With Application To Dissipative Extended Magnetohydrodynamics, Journal of Plasma Physics (2020)

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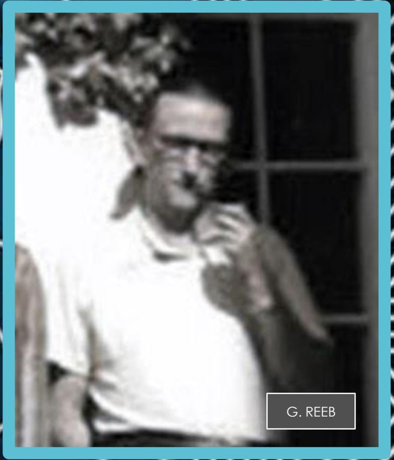
Transverse Symplectic Foliation Structure



Foliation

Folation Theory Inventors: Ehresmann & Reeb

- André Haefliger passed away 7th March 2023 : **Haefliger, A.: Naissance des feuilletages d'Ehresmann-Reeb à Novikov. Journal 2(5), 99–110 (2016)**
- G. REEB, **Sur certaines propriétés topologiques des variétés feuilletées (Thèse)**, Hermann 1952. supervised by Charles Ehresmann
- " ***Sur une durée de quarante années l'immeuble s'est édifié; des centaines d'ouvriers ont œuvré. L'édifice n'est pas achevé, mais on peut visiter. Oui, visiter est le mot*** " - Georges REEB

Charles Ehresmann

Georges Henri Reeb

Sergei Petrovich Novikov

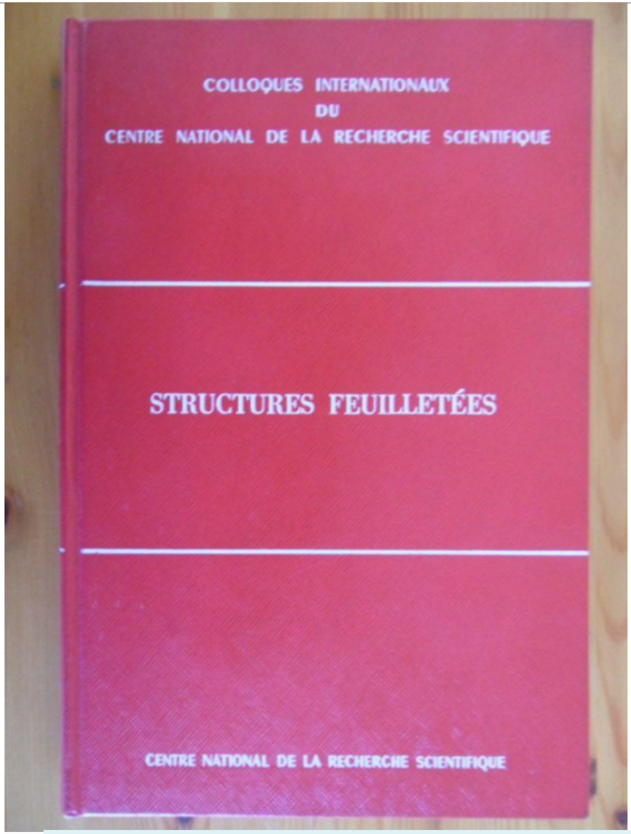
André Haefliger

OPEN

THALES

1963, naissance des « Structures feuilletées »

C. EHRESMANN, Structures feuilletées, Proceedings of the Fifth Canadian Mathematical Congress, 109-172., 1963



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STRUCTURES FEUILLETEES

CHARLES EHRESMANN, *Institut Henri Poincaré*

Introduction

Cet article a pour but la définition précise et l'étude des structures feuilletées dans le cadre de la théorie des structures locales telle qu'elle est exposée dans (**3**; **5**; **6**). Les résultats connus dans le cas des variétés feuilletées sont précisés et généralisés au cas d'un feuilletage topologique localement simple. Les notions d'holonomie, de déroulements et de tubes analysées ici permettent d'étudier les questions de stabilité. Seuls des problèmes généraux sont abordés, les applications étant réservées pour une publication ultérieure.

La plupart des idées contenues dans ce travail ont été exposées dans mes cours (en particulier, Paris 1955–56, 1958, 1961) et dans des conférences (par exemple Princeton 1953, Buenos-Aires 1959–60, Montréal 1961). Rappelons que la notion de variété feuilletée a été introduite dans une Note en collaboration avec Reeb (**1**), puis étudiée d'une façon approfondie par Reeb (**13**; **14**) dans différentes publications. Les structures feuilletées d'espèce $\mathfrak{B} \# \mathfrak{F}$ élargie et de seconde espèce ont été définies dans (**3**). Les Γ-structures étudiées par Haefliger (**16**), qui sont étroitement liées aux feuilletages de seconde espèce, ne seront pas considérées ici. Les feuilletages localement simples ont été introduits dans une Note en collaboration avec Shih Weishu (**2**).

I. Définitions de diverses espèces de structures feuilletées

1. *Feuilletages topologiques*

Soit E un ensemble muni de deux topologies T et T'. On dira que (T, T') définit sur E un *feuilletage topologique* ou une structure d'*espace feuilleté topologique* si la condition suivante est vérifiée : Pour tout $x \in E$, il existe un voisinage ouvert U' de x relativement à T' sur lequel T et T' induisent la même topologie.

Si (T, T') est un feuilletage topologique sur E, alors T' est une topologie plus fine que T. Nous supposerons désormais que T' est

109

Structures feuilletées - Grenoble, 25-30 juillet 1963 - Georges Reeb
Auteur(s) : Georges Reeb - Charles Ehresmann - René Thom - Paulette Libermann
Editeur : Centre National De La Recherche ScientifiqueCollection : Colloques Int. Du Cnrs, 1964

Georges Reeb & Analytical Mechanics

Symplectic Model of Analytical Mechanics

- Symplectic Geometry in conjunction with Analytical Mechanics has grown considerably over the past decades; inspired by the work of S. Lie and E. Cartan, A. Lichnerowicz, G. Reeb, J.M. Souriau, as well as F. Gallissot, who were the initiators of this revival of Analytical Mechanics.

Foliation in Lie Groups Actions and Thermodynamics Integrable Pfaff forms

- In paper “Structures feuilletées”, G. Reeb asked the following questions about foliation structures “Why have they been studied. How were they studied? Is it "profitable" to continue these investigations?” and proposed motivations for studying foliations. Among motivations, G. Reeb identified two key use-cases, Lie Groups action and integrable Pfaff forms of Thermodynamics:
  - **“Lie groups action theory (theory much older than that of foliations) often leads to consider generated foliations. Similarly, the "moving frame" theory (Cartan) ("dual" in a rather vague sense of the previous one) suggests classes of foliations with a remarkable transverse structure.”**
  - **“Thermodynamics has long accustomed mathematical physics [cf. Duhem Pierre] to the consideration of completely integrable Pfaff forms: the elementary heat dQ [notation of thermodynamicists] representing the elementary heat yielded in an infinitesimal reversible modification is such a completely integrable form. This point does not seem to have been explored since then.”.**
- Foliation and Cartan’s moving frame was developed by Reeb PhD student, Edmond Fédida.

THALES

Structures feuilletées – Georges Henri REEB
Motivations for Folations Studies

G. Reeb, Structures feuilletées, Differential Topology, Foliations and Gelfand-Fuks cohomology, Rio de Janeiro, 1976, Springer Lecture Notes in Math. 652 (1978), 104-113.

STRUCTURES FEUILLETEES

Pourquoi les a-t-on étudiées. Comment les a-t-on étudiées. Est-il "rentable" de continuer ces investigations ?

Orbites coadjointes (action d'un groupe sur le dual de l'algèbre de Lie)

M_3 : La théorie des actions de groupes de Lie (théorie bien plus ancienne que celle des feuilletages) conduit souvent à considérer des feuilletages engendrés. De même la théorie du "repère mobile" (CARTAN) ("duale" en un sens assez vague de la précédente) suggère des classes de feuilletages à structure transversale remarquable.

Thermodynamiques et formes de Pfaff

M_4 : La thermodynamique a habitué de longue date la physique mathématique [cf. DUHEM P.] à la considération de formes de Pfaff complètement intégrables : la chaleur élémentaire dQ [notation des thermodynamiciens] représentant la chaleur élémentaire cédée dans une modification infinitésimale réversible est une telle forme complètement intégrable. Ce point ne semble guère avoir été creusé depuis lors.

"Sur une durée de quarante années l'immeuble s'est édifié; des centaines d'ouvriers ont œuvré. L'édifice n'est pas achevé, mais on peut visiter. Oui, visiter est le mot" - Georges REEB

Georges REEB contribution to Analytical Mechanics (in « *Œuvres complètes de Charles Ehresmann* »)

La Géométrie Symplectique en liaison avec la Mécanique Analytique, a pris un extension considérable ces trente dernières années ; inspirés par les travaux de S. Lie et E. Cartan, A. Lichnerowicz [27], G. Reeb [44, 45], J.M. Souriau [48], ainsi que F. Gallissot [8], ont été les initiateurs de ce renouveau de la Mécanique Analytique.

References:

- G. REEB, Propriétés topologiques des trajectoires des systèmes dynamiques, Mém. Acad. Se. Bruxelles 27 (1952).
- G. REEB, Varietés symplectiques, varietés presque-complexes et systèmes dynamiques, Comptes Rendus de l'Académie de Sciences, 235, (1952) 776–778
- G, REEB, Espaces de Finsler et espaces de Cartan, Coll. Int. CNRS Géom. Diff. Strasbourg (1953), 35-40.
- G. REEB, Problèmes relatifs aux variétés presque symplectiques et systèmes dynamiques, Convegno Int. Geom. Diff. Venise-Bologne-Pise (1953), 104-113

Symplectic Leaves
& transverse Metric Leaves

PHYSICAL REVIEW E **91**, 042138 (2015)

Essential equivalence of the general equation for the nonequilibrium reversible-irreversible coupling (GENERIC) and steepest-entropy-ascent models of dissipation for nonequilibrium thermodynamics

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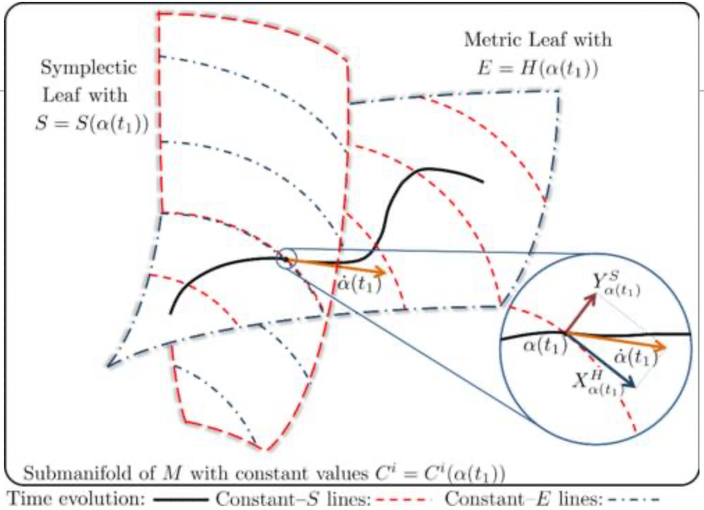
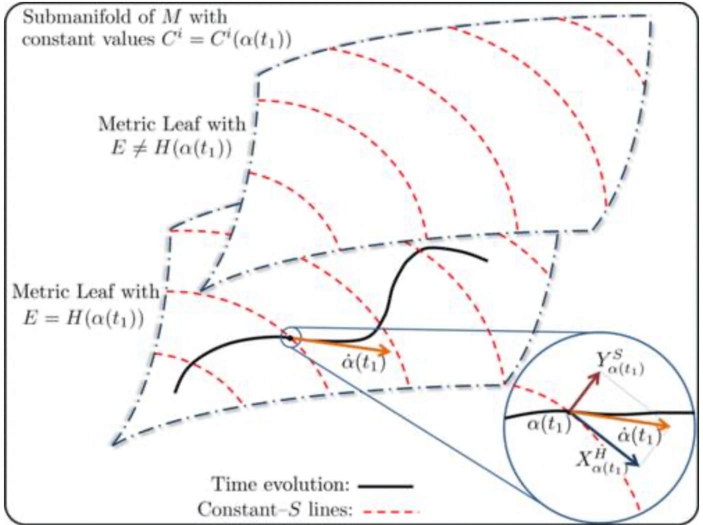
(Received 19 November 2014; published 28 April 2015)

$$\{S,A\} = P(dS,dA) = dA[P^\sharp(dS)] = 0 \quad \forall A \in C^\infty(\mathcal{M}),$$
$$\text{or, equivalently,} \quad P^\sharp_{\alpha(t)}(dS_{\alpha(t)}) = 0. \tag{60}$$

$$[H,A] = D(dH,dA) = dA[D^\sharp(dH)] = 0 \quad \forall A \in C^\infty(\mathcal{M}),$$
$$\text{or, equivalently,} \quad D^\sharp_{\alpha(t)}(dH_{\alpha(t)}) = 0. \tag{61}$$

- In the context of GENERIC dynamics, the degeneracy condition (60) implies that **symplectic leaves are at constant entropy** (and the other distinguished functionals of the Poisson bracket) while the degeneracy condition (61) implies that **metric leaves are at constant energy** (and the other distinguished functionals of the dissipative bracket).
- Each trajectory is effectively constrained on a single metric leaf.
- The relationship between metric leaves, where GENERIC dynamics (of an overall closed and isolated thermodynamic system) takes place, and symplectic leaves, where purely Hamiltonian dynamics takes place.
- **Metric leaves are surfaces with constant energy, while the symplectic leaves are surfaces with constant entropy** (because Hamiltonian dynamics is reversible).
- **The intersection of symplectic leaves on a metric leaf produces isentropic contours** and the GENERIC non dissipative vector (for an overall closed and isolated thermodynamic system) is always contained in such an intersection.





Transverse Structure of Foliations

LISTE DES CONFÉRENCES ET COMMUNICATIONS

N.A'CAMPO - Une mesure SL(n, $\mathbb{R}$)-invariante sur l'espace des couples de drapeaux sur $\mathbb{R}^n$.

R.BARRE - Théorie des Q-variétés et structures de Hodge mixtes.

R.BLUMENTHAL - Transverse curvature of foliated manifolds.

Y.CARRIERE - Flots riemanniens.

P.CARTIER - Variétés quotients : regards rétrospectifs sur leur développement.

L.CONLON - Holonomy pseudogroup and GV (work in progress).

A.CONNES - K-théorie, théorie de l'indice et feuilletages.

D.B.A.EPSTEIN - Foliations of 3-manifolds with transverse hyperbolic structure.

A.HAEFLIGER - Feuilletages avec feuilles minimales et courants invariants.

G.HECTOR - titre non parvenu

F.W.KAMBER - Duality theorems for harmonic foliations.

C.LAMOUREUX - Etude géométrique directe des feuilletages transverses sur les fibrés en cercles et en droites.

D.LEHMANN - Feuilletages avec "suffisamment" de formes basiques.

J.LEHMANN-LEJEUNE - Dérivations d'une algèbre de Lie sur le fibré transverse à un feuilletage.

K.MILLETT - Can R^3 be foliated by circles ?

P.MOLINO - Espace des feuilles des feuilletages riemanniens.

J.PRADINES - Equivalence transverse et groupoïdes différentiables.

B.REINHART - Comprendre la structure transverse, c'est comprendre les groupes de polynômes tronqués.

C.ROGER - Cohomologie (p,q) des feuilletages et applications.

R.SACKSTEDER - Foliations and separation of variables.

G.W.SCHWARZ - Base-like cohomology of foliations.

T.TSUBOI - Cobordismes de feuilletages.

W.T.VAN EST - Rapport sur les schémas de variété.

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http://www.numdam.org/item/AST_1984__116_/

Transverse Structure of Foliations

PRÉFACE

Issue de la rencontre de W.T. Van Est et de P. Molino, la première "Journée Transverse", consacrée à la structure de l'espace des feuilles d'un feuilletage, fut organisée en 1979 à Paris VII par P. Libermann.

L'intérêt d'une rencontre élargie, suggérée par P. Molino, apparut rapidement, renforcé par la convergence des points de vue de W.T. Van Est et de A. Haefliger.

L'impulsion décisive devait venir de P. Cartier, à qui sa culture universelle fit apercevoir la confluence des méthodes des géomètres différentiels avec celles introduites par A. Connes, en liaison avec les travaux de Ruelle-Sullivan, et reconnaître la convergence de courants mathématiques lointains issus de domaines variés (Géométrie Algébrique, topos de Grothendieck, sous-groupes virtuels de Mackey, etc...).

Son appui assura le succès de ces Journées, dépassant les prévisions, attesté par la quantité et la qualité des conférenciers et des participants. Un horaire très chargé, rançon de ce succès, ne fit pas faiblir le nombre et l'attention des auditeurs.

L'Université Paul Sabatier de Toulouse a accueilli le Colloque sur son campus et assuré la base du financement, notamment grâce à des crédits sur programme accordés à l'Equipe de Topologie Algébrique et Différentielle.

La Société Mathématique de France, avec le concours du C.N.R.S., a accordé son soutien moral et financier, complété par une subvention de la D.C.R.I.

On trouvera ci-après les textes rédigés des 2/3 des conférences prononcées à ce Colloque, la plupart avec des développements et améliorations considérables, fruits des discussions tenues au cours de ces Journées.

Il était prévu d'ouvrir et de fermer ce recueil par le texte de l'Introduction et de la Conclusion prononcées par P. Cartier, qui devait être rédigé en collaboration avec A. Connes. Il a fallu malheureusement y renoncer, sous peine de retarder exagérément la publication.

Cette lacune est partiellement comblée par les exposés de W.T. Van Est et de A. Haefliger, qui fondent la théorie, tout en l'illustrant d'applications concrètes. La structure de l'espace des feuilles y apparaît comme une classe d'équivalence, en un sens naturel, de pseudogroupes, ou plus généralement de groupoïdes (dits d'holonomie), munis d'une structure (topologique, différentielle, etc...) invariante.

Il résulte d'une construction de G. Hector que ce point de vue équivaut exactement à celui de P. Molino, très brièvement rappelé dans son article, qui consiste à considérer une certaine classe d'équivalence de feuilletages, qu'il appelle F-variété. Notons que les QF-variétés sont définies par une relation d'équivalence plus large, mais conduisent aux mêmes invariants continus que les F-variétés, du fait que le groupoïde d'holonomie est dense dans le groupoïde d'holonomie transverse utilisé par C. Godbillon.

Les autres conférences illustrent la variété et la richesse des méthodes d'étude de cette structure transverse et des applications géométriques que l'on en tire, tant pour les propriétés globales du feuilletage que pour celles des feuilles. On y trouvera une moisson de résultats inédits.

Cet échantillonnage ne saurait cependant prétendre être exhaustif, ni statistiquement représentatif des applications des propriétés transverses. C'est par un concours de circonstances, et non par suite d'un choix prémédité, que les thèmes le plus souvent abordés sont les feuilletages riemanniens et la cohomologie basique, alors que les propriétés de croissance des feuilles et d'ergodicité le sont beaucoup plus rarement.

Il semble intéressant de souligner le fait que certaines notions de variétés singulières, apparues dans des contextes éloignés

des feuilletages, se sont trouvées être des cas particuliers importants d'espaces de feuilles : les Q-variétés de Barre, et les V-variétés de Satake, redécouvertes comme orbifolds de Thurston, que l'on voit fréquemment apparaître dans ce qui suit.

Je ne saurais terminer sans rendre hommage à la mémoire du grand géomètre Ch. Ehresmann, disparu peu d'années avant la tenue de ce Colloque, fondateur avec G. Reeb, de la théorie des feuilletages. Beaucoup des notions fondamentales, qui sont à la base des travaux que l'on va lire, lui sont dues, notamment notamment les diverses variantes du groupoïde d'holonomie et la notion de groupoïde différentiable (et structuré), qui revient en surface après un long cheminement souterrain. Ce colloque, auquel participaient nombre de ses disciples, illustre l'actualité de sa pensée.

J. Pradines

Transverse Structure of Foliations

LISTE DES PARTICIPANTS

(Journées SMF, Toulouse 17-18-19 février 1982)

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BLUMENTHAL Robert (St-Louis University, Missouri)
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BOYOM N'Guiffo (Montpellier)
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CASCON Ana (Dijon)
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CORNU Philippe (Montpellier)
COSTE Alain (Lyon)
CRUMEYROLLE Albert (Toulouse)
CUMENGE Christian (Toulouse)
DADJO Jean (Toulouse)
DUMINY Gérard (Lille)
EL KACIMI Alaoui Aziz (Lille)
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MASA Xosé (Santiago de Compostela)
MATTEI Jean-François (Toulouse)
MILLETT Kenneth (University of California, Santa Barbara)
MOLINO Pierre (Montpellier)
MONNA Gilbert (Montpellier)
MOREL Francis (Valenciennes)
MORVAN Jean-Marie (Avignon)
MULLER Marie-Paule (Strasbourg)
NORE Thérèse (Limoges)
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PICOUX Alain (Lille)
PINTO DE CARVALHO Sonia (Dijon)
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PRADINES Jean (Toulouse)
RABINOWICZ Michel (CNES, Toulouse)
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REINHART Bruce (Univ. of Maryland, College Park)
RENAULT Jean
ROCHE Claude (Grenoble)
ROGER Claude (Metz)
ROSENBERG Harold (Paris 7)
SACKSTEDER Richard (New-York)
SALHI Ezzedine (Strasbourg)
SCHWARZ Gérald (Brandeis University)
SERGERAERT Francis (Grenoble)
SERGIESCU Vladimir (Lille)
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TAPIA Joseph (Toulouse)
TAQUET Bernadette (Mulhouse)
TSUBOI Takashi (Tokyo)
TURIEL Francisco (Santander, Espagne)
VALETTE Alain (IHES, Paris)
VAN EST W.T. (Amsterdam)
VER EECKE Paul (Amiens)
VIVIENTE José-Luis (Saragosse, Espagne)
WOUAFO KAMGA Jean (Yaoundé, Cameroun)

Transverse Foliation Structures

- Dazord, P., Molino, P. : Γ-Structures poissonniennes et feuilletages de Libermann, Publications du Département de Mathématiques de Lyon, fascicule 1B, « Séminaire Sud-Rhodanien 1ère partie », chapitre II , p. 69-89 (1988)

- Sabourin, H. : Quelques aspects de la méthode des orbites en théorie de Lie, HDR spécialité Mathématique, Université de Poitiers, 15 Décembre 2005

Libermann Foliation and Haefliger Γ-Structure

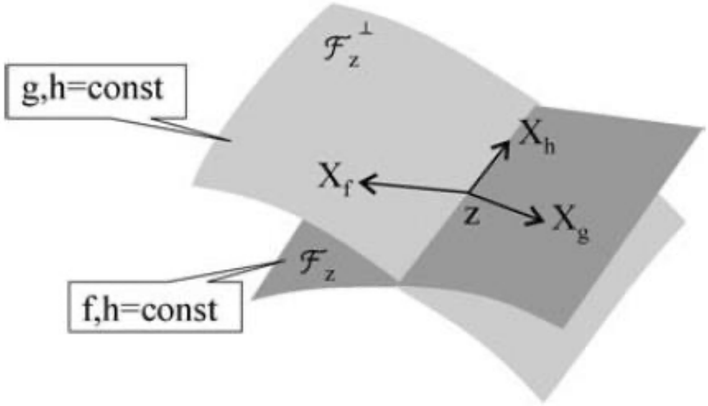
Hervé Sabourin Transverse Poisson Structure

Foliation based on Elie Cartan Moving Frame

Polar Foliation, Bifoliation and Bifibration

- Fedida, E. : Sur la théorie des feuilletages associée au repère mobile : cas des feuilletages de Lie. Lecture Notes in Mathematics, vol 652. Springer (1978)

- Arutyunov, G., Elements of Classical and Quantum Integrable Systems, Springer, (2019)



OPEN

THALES

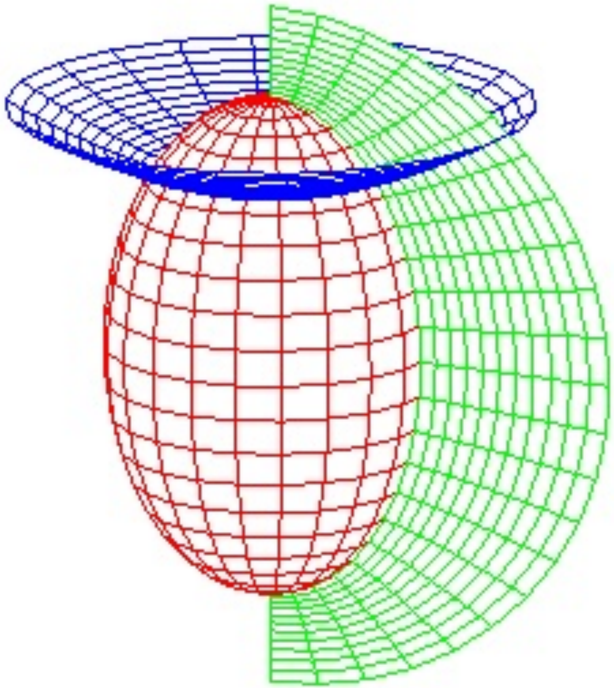
André Lichnerowicz, Collège de France

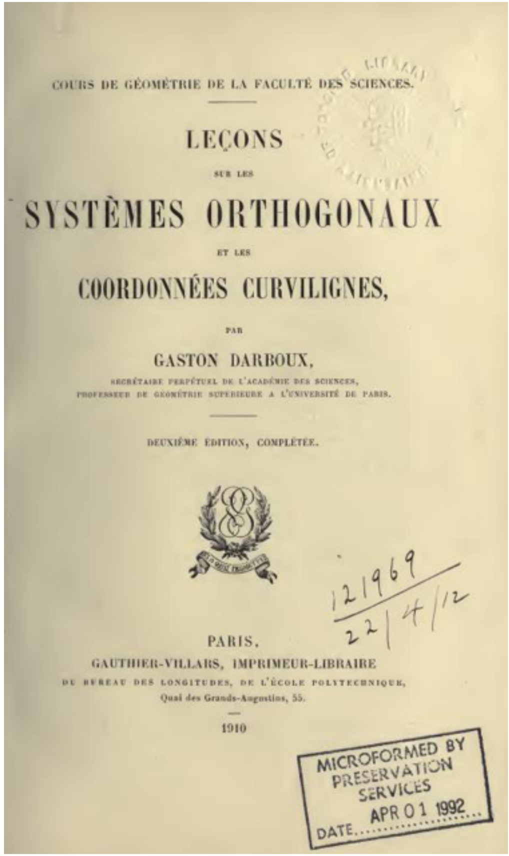
Cours 1982-1983 – Étude systématique des rapports entre feuilletages et variétés de Poisson | Étude de cohomologies d'algèbres de Lie attachées à une variété de contact

- https://www.college-de-france.fr/sites/default/files/media/document/2023-05/1982-1983_lichnerowicz.pdf

Soit (M, F) une variété symplectique munie d'un *feuilletage lagrangien* $\mathscr{F}$. On a montré qu'il existe toujours sur M une connexion adaptée au feuilletage qui induit sur chaque feuille une *connexion plate* sans torsion. Si la variété admet une métrique riemannienne de type fibré pour $\mathscr{F}$, elle admet une métrique riemannienne qui induit sur chaque feuille une métrique plate. On a ainsi précisé et généralisé des résultats récents d'A. Weinstein et P. Dazord. Les mêmes résultats sont valables si, au lieu d'un feuilletage lagrangien, on considère un feuilletage isotrope de (M, F) tel que le champ des plans orthogonaux symplectiques soit un feuilletage coisotrope.

Triple Orthogonal Systems : Darboux and Lamé





COURS DE GÉOMÉTRIE DE LA FACULTÉ DES SCIENCES.

LEÇONS

SUR LES

SYSTÈMES ORTHOGONAUX

ET LES

COORDONNÉES CURVILIGNES,

PAR

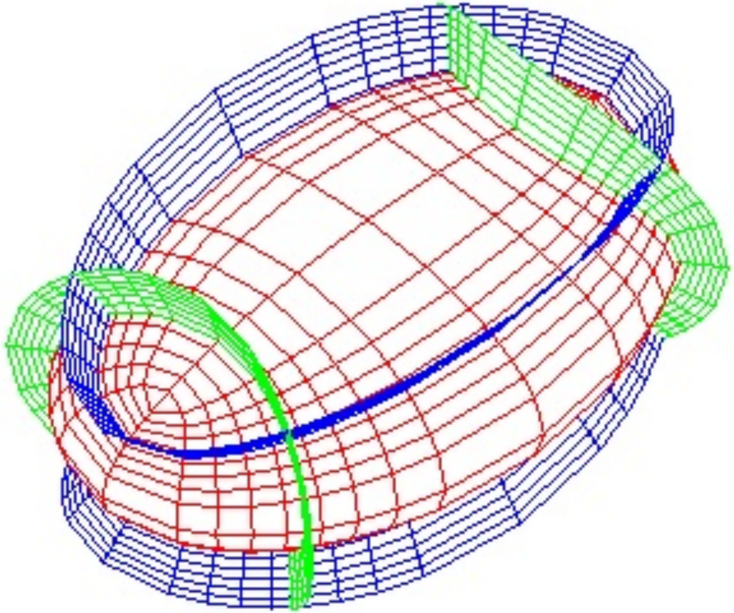
GASTON DARBOUX,

SECRÉTAIRE PERPÉTUEL DE L'ACADÉMIE DES SCIENCES,
PROFESSEUR DE GÉOMÉTRIE SUPÉRIEURE A L'UNIVERSITÉ DE PARIS.

DEUXIÈME ÉDITION, COMPLÉTÉE.

PARIS,
GAUTHIER-VILLARS, IMPRIMEUR-LIBRAIRE
DU BUREAU DES LONGITUDES, DE L'ÉCOLE POLYTECHNIQUE,
Quai des Grands-Augustins, 55.

1910



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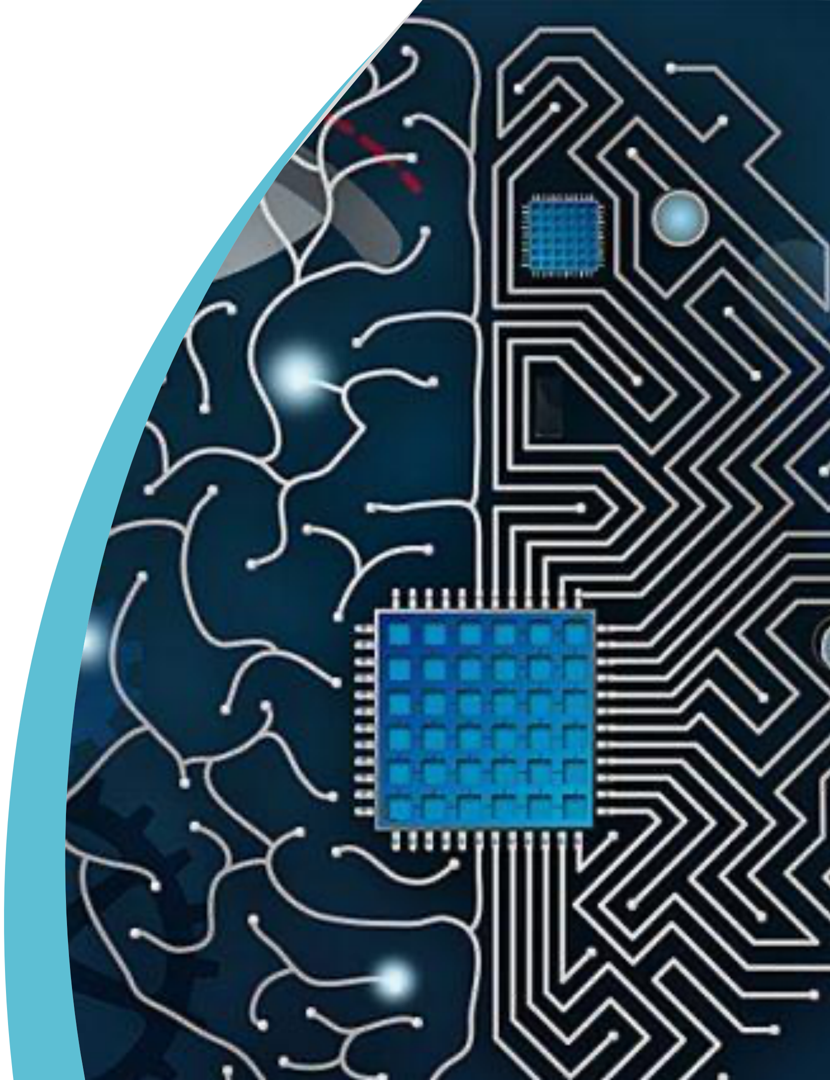
HERVE SABOURIN TRANSVERSE POISSON STRUCTURE



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Transverse Foliation Structures

- Dazord, P., Molino, P. : Γ-Structures poissonniennes et feuilletages de Libermann, Publications du Département de Mathématiques de Lyon, fascicule 1B, « Séminaire Sud-Rhodanien 1ère partie », chapitre II , p. 69-89 (1988)
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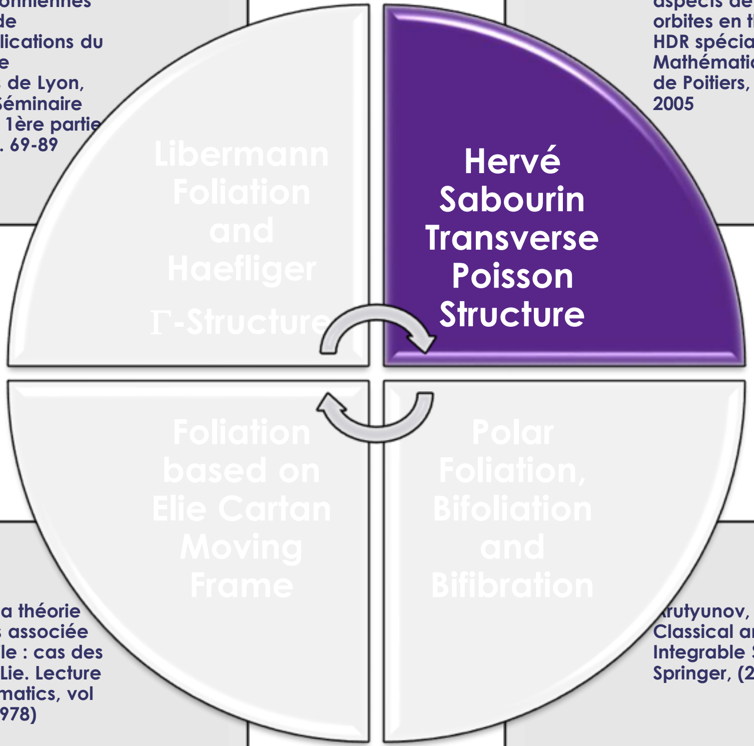

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- Arutyunov, G., Elements of Classical and Quantum Integrable Systems, Springer, (2019)

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THALES

Hervé SABOURIN HDR 2005, Université de Poitiers

DIPLOME D'HABILITATION

A

DIRIGER DES RECHERCHES

présenté à l'Université de Poitiers

par

Hervé Sabourin,

Le 15 Décembre 2005.

Spécialité : Mathématiques.

Quelques aspects de la méthode des orbites en théorie de Lie.

JURY

Michel DUFLO, Professeur Émerite, Université de Paris 7		Examinateur.
Thierry LEVASSEUR, Professeur, Université de Brest,		Rapporteur.
Bent ORSTED, Professeur, Université de Aarhus, Danemark,		Rapporteur.
Hubert RUBENTHALER, Professeur, Université de Strasbourg,		Rapporteur.
Pierre TORASSO, Professeur, Université de Poitiers,		Examinateur.
Pol VANHAECKE, Professeur, Université de Poitiers,		Examinateur.

0.Introduction

L'essentiel de mon travail de recherche a pour cadre général l'espace des orbites nilpotentes co-adjointes d'une algèbre de Lie semi-simple réelle ou complexe.

Considérons, plus précisément, un groupe de Lie réel ou complexe G, d'algèbre de Lie $\mathfrak{g}$, qui opère sur $\mathfrak{g}$ par la représentation adjointe, puis sur le dual $\mathfrak{g}^*$ de $\mathfrak{g}$ suivant l'action "duale". On obtient ainsi un ensemble d'orbites "coadjointes", noté $\mathfrak{g}^*/G$. Lorsque G est semi-simple, on peut identifier $\mathfrak{g}$ à son dual, via la forme de Killing $\mathcal{K}$, et donc l'action co-adjointe à l'action adjointe. On considère alors l'ensemble $\mathcal{N}$ des éléments nilpotents de $\mathfrak{g}$, puis l'ensemble $\mathcal{N}/G$ des orbites nilpotentes adjointes. Cet ensemble joue en fait un rôle essentiel en théorie de Lie.

• L'une des questions importantes à envisager concerne l'étude et la classification éventuelle de l'ensemble des représentations d'un groupe de Lie réel et, plus précisément, de l'ensemble des représentations unitaires irréductibles, appelé dual unitaire . Historiquement cette étude a été initiée dans le cadre des groupes nilpotents; une réponse complète a été apportée dans ce cas par A.A.Kirillov ([30]), qui a établi une bijection naturelle entre le dual unitaire et l'ensemble des orbites coadjointes introduisant de ce fait , pour la première fois, la notion de *méthode des orbites.*

Cette méthose a été ensuite généralisée aux groupes exponentiels par les travaux de P.Bernat ([12]), aux groupes résolubles de type 1 par L.Auslander et B.Kostant ([11]) et enfin aux groupes résolubles quelconques par L.Pukanszky ([40]). L'ensemble des orbites coadjointes sert encore à donner dans ce cas une paramétrisation du dual unitaire mais, cette fois, il ne suffit pas à tout décrire.

Il est naturel donc de s'intéresser au cas d'un groupe réductif et de tenter, entre autres, d'y développer une méthode des orbites. Un tel travail est actuellement en cours et de nombreux auteurs ont apporté une contribution considérable à cette question difficile. En voici une liste non exhaustive : D.Vogan ([53], [54], [55]), R.Brylinski et B.Kostant ([16]) T.Kobayashi et B.Orsted ([31], [32], [33]), M.Duflo ([23]) et P.Torasso ([51], [52]). Comme on le décrira de manière plus précise ultérieurement, la question consiste souvent à essayer de relier de manière la plus naturelle possible le dual unitaire et l'ensemble des orbites nilpotentes coadjointes.

• L'espace $\mathcal{N}/G$ peut aussi servir à l'étude de certains invariants algébriques d'une algèbre de Lie complexe $\mathfrak{g}$. Soit $\mathcal{C}(\mathfrak{g}) = \{(X,Y) \in \mathfrak{g} \times \mathfrak{g} \ / \ [X,Y] = 0\}$ la *variété commutante* de $\mathfrak{g}$. La question se pose de déterminer les composantes irréductibles de cette variété algébrique. Dans [42], W.Richardson montre que, lorsque $\mathfrak{g}$ est semi-simple, $\mathcal{C}(\mathfrak{g})$ est irréductible; le résultat s'obtient par récurrence sur la dimension de $\mathfrak{g}$ en utilisant en particulier les propriétés de certaines orbites nilpotentes dites *orbites distinguées.*

On peut alors s'intéresser à la situation suivante : l'algèbre $\mathfrak{g}$ se décompose, sous l'action d'une involution θ, en sous-espaces propres suivant les valeurs ± 1, soit $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

L'espace propre $\mathfrak{k}$ associé à la valeur propre 1 est une algèbre de Lie et l'on désigne par K le sous-groupe de G correspondant, qui agit sur l'espace $\mathfrak{p}$. On peut considérer la variété commutante de $\mathfrak{p}$ et se demander si celle-ci est aussi irréductible; une fois encore, la structure des K-orbites nilpotentes de $\mathfrak{p}$ joue un rôle prépondérant et permet de conclure dans certains cas.

• On sait enfin que le dual $\mathfrak{g}^*$ d'une algèbre de Lie complexe peut-être muni d'une structure de Poisson canonique, appelée *structure de Lie-Poisson.* Selon des résultats de A.Weinstein ([56]), on peut associer, canoniquement, à chaque feuille symplectique d'une variété de Poisson quelconque, une sous-variété transverse munie d'une structure de Poisson, appelée *structure de Poisson transverse à la feuille symplectique.* Or, les feuilles symplectiques de $\mathfrak{g}^*$ sont les orbites coadjointes, dont il semble donc naturel d'étudier la structure de Poisson transverse. Un cas s'avérera particulièrement intéressant, celui des orbites nilpotentes d'une algèbre de Lie semi-simple.

Voici donc les trois thèmes sur lesquels portent mes travaux et mes résultats que je vais maintenant détailler. Je ne peux terminer cette introduction sans exprimer toute ma reconnaissance aux nombreux collègues avec qui j'ai pu partager mes questionnements mathématiques et de qui j'ai recu tant de réponses utiles. Je commencerai par Michel Duflo et Pierre Torasso dont les travaux ont été si essentiels pour moi et dont la disponibilité a toujours été sans faille tout au long de ces dernières années, David Vogan ensuite pour ses suggestions si souvent déterminantes. Je pense aussi à Toshi Kobayashi, Thierry Levasseur et Mustapha Rais pour les nombreuses et fructueuses discussions que nous avons pu avoir sur ces différents sujets. Je n'oublierai pas mes collègues de Poitiers, Abderrazak Bouaziz, Patrice Tauvel, Pol Vanhaecke et Rupert Yu, leur présence, nos échanges réguliers et nos collaborations.

Hervé SABOURIN HDR 2005, Université de Poitiers

Transverse Poisson Structure

• On sait enfin que le dual $\mathfrak{g}^*$ d'une algèbre de Lie complexe peut-être muni d'une structure de Poisson canonique, appelée *structure de Lie-Poisson*. Selon des résultats de A. Weinstein ([56]), on peut associer, canoniquement, à chaque feuille symplectique d'une variété de Poisson quelconque, une sous-variété transverse munie d'une structure de Poisson, appelée *structure de Poisson transverse à la feuille symplectique*. Or, les feuilles symplectiques de $\mathfrak{g}^*$ sont les orbites coadjointes, dont il semble donc naturel d'étudier la structure de Poisson transverse. Un cas s'avérera particulièrement intéressant, celui des orbites nilpotentes d'une algèbre de Lie semi-simple.

OPEN

THALES

SABOURIN TRANSVERSE POISSON STRUCTURE FOR DISSIPATIVE HEAT EQUATION

- Hervé Sabourin has studied more deeply the transverse Poisson structure to coadjoint orbits **in a complex semisimple Lie algebra**, **by reducing to the case of nilpotent orbits.**
- For subregular nilpotent orbits, Hervé Sabourin showed that **the transverse Poisson structure could be described by a determinantal formula** based on the Chevalley's restriction of the invariants on the slice.
- Based on Slodowy slice model, the transverse Poisson structure is reduced to a three dimensional Poisson bracket.

Notes: Nilpotent means $\left[ad_g(X)\right]^n = 0$ for large enough *n*

Equivalently, X is nilpotent if its characteristic polynomial $P_{ad_X}(t)$ is equal to $t^{\dim g}$.

SABOURIN TRANSVERSE POISSON STRUCTURE FOR DISSIPATIVE HEAT EQUATION

$$\mathfrak{g} = \mathfrak{g}(x) \oplus n_e \oplus n_s \qquad \mathfrak{g}(x) = \left\{ y \in \mathfrak{g} / [x, y] = 0 \right\} \qquad n = n_e \oplus n_s \qquad N_x = x + n^{\perp}$$

- Considering the basis vectors $X_1, \cdots, X_{2r}$ of n such that $X_1, \cdots, X_{2p} \in n_e$ and $X_{2p+1}, \cdots, X_{2r} \in n_s$ and as $\langle n, [\mathfrak{g}(x), n_s] \rangle = \{0\}$ and $\langle n, [n_e, n_s] \rangle = \{0\}$, then the Poisson Matrix takes at $n \in N_x$ the form :

$$\Lambda(n) = \begin{pmatrix} A(n) & B_e(n) & 0 \\ -B_e(n)^T & C_e(n) & 0 \\ 0 & 0 & C_s(n) \end{pmatrix} \text{ where } \begin{cases} A_{i,j}(n) = \langle n, [Z_i, Z_j] \rangle \ , \ \text{for } 1 \le i, j \le k \\ B_{e;i,m}(n) = \langle n, [Z_i, X_m] \rangle \ , \ \text{for } 1 \le i \le k, \ 1 \le m \le 2p \\ C_{e;l,m}(y) = \langle n, [X_i, X_m] \rangle \ , \ \text{for } 1 \le l, m \le 2p \\ C_{s;l,m}(y) = \langle n, [X_i, X_m] \rangle \ , \ \text{for } 2p+1 \le l \ , \ m \le 2r \end{cases}$$

- Sabourin has deduced from it that the Poisson matrix of the transverse Poisson structure on N_x is given by: $\Lambda_{N_x}(n) = A(n) + B_e(c) C_e(n)^{-1} B_e(n)^T$
- Sabourin has also proved that:

$$\Lambda_N = A(n) + B_e(c) C_e(n)^{-1} B_e(n)^T \quad \text{where } n \in N$$

THALES

SABOURIN TRANSVERSE POISSON STRUCTURE FOR DISSIPATIVE HEAT EQUATION

- **M. Saint-Germain** has proved in his PhD that, **in the Lie–Poisson case, the transverse Poisson structure is always rational**. This result has been completed by P. Damianou for coadjoint orbits in a semisimple Lie algebra, and by Hervé Sabourin in 2005, with a more general class of complements having a polynomial transverse structure.
- Hervé Sabourin has studied the **transverse Poisson structure** to any adjoint orbit $G.x$ and has proved that it can be reduced, via the Jordan–Chevalley decomposition of $x \in \mathfrak{g}$ to the case of an adjoint nilpotent orbit.
- As the transverse structure to the regular nilpotent orbit O_{reg} of $\mathfrak{g}$ is always trivial, Hervé Sabourin has considered the case of the **subregular nilpotent orbit** O_{sr} of $\mathfrak{g}$ with dimension of O_{sr} two less than the dimension of the regular orbit, so that the transverse Poisson structure has rank 2. **Sabourin replaced, for the transverse Poisson structure, the complicated Dirac constraints, by a simple determinantal formula**:

$$\{f,g\}_{\det} = \det\left(\nabla f, \nabla g, \nabla \chi_1, \ldots, \nabla \chi_l\right) = \frac{df \wedge dg \wedge d\chi_1 \wedge \ldots \wedge d\chi_l}{dq_1 \wedge dq_2 \wedge \ldots \wedge dq_{l+2}}$$

where $\chi_1, \ldots, \chi_l$ are **independent polynomial Casimir functions with** χ_i **the restriction of the i-th Chevalley invariant** G_i **to the slice** N, and $q_1, \ldots, q_{l+2}$ linear coordinates on N.

SABOURIN TRANSVERSE POISSON STRUCTURE FOR DISSIPATIVE HEAT EQUATION

- **Sabourin Theorem**: Let O_{sr} be the subregular nilpotent adjoint orbit of a complex semisimple Lie algebra $\mathfrak{g}$, and let (h,e,f) be the canonical triple associated to O_{sr}. Let $N = e + n^{\perp}$ be a slice transverse to O_{sr}, where n is an ad_h-invariant complementary subspace to $\mathfrak{g}(e)$. Let $\{.,.\}_N$ and $\{.,.\}_{\det}$ denote respectively **the adjoint transverse Poisson structure and the determinantal structure on** N. Then:

$$\{.,.\}_N = c\{.,.\}_{\det}$$

for some $c \in \mathbb{C}^*$.

SABOURIN TRANSVERSE POISSON STRUCTURE FOR DISSIPATIVE HEAT EQUATION

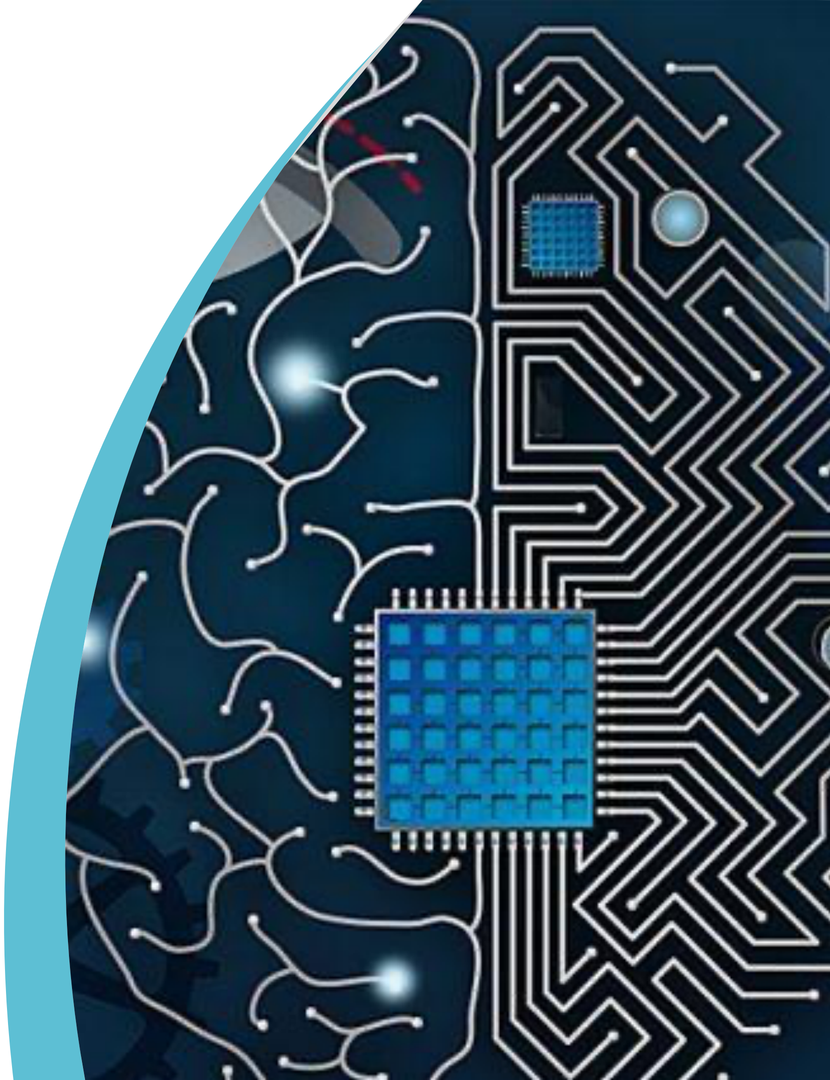
- From this determinantal formula, Sabourin has deduced that the Poisson matrix of the transverse Poisson on N takes, in suitable coordinates, the block form:

$$\tilde{\Lambda} = \begin{pmatrix} 0 & 0 \\ 0 & \Omega \end{pmatrix} \quad \text{where} \quad \Omega = c' \begin{pmatrix} 0 & \frac{\partial F}{\partial q_{l+2}} & -\frac{\partial F}{\partial q_{l+1}} \\ -\frac{\partial F}{\partial q_{l+2}} & 0 & \frac{\partial F}{\partial q_l} \\ \frac{\partial F}{\partial q_{l+1}} & -\frac{\partial F}{\partial q_l} & 0 \end{pmatrix}$$

- where F **is the polynomial** $F\left(u_1,\ldots,u_{l-1},q_l,q_{l+1},q_{l+2}\right)$ with $u_1,\ldots,u_{l-1}$ the deformation parameters, given by **Casimir for the Poisson structure on** N, of simple singularity of the singular surface $N \cap \aleph$, where $\aleph$ is the nilpotent cone of $\mathfrak{g}$.

THALES

LIBERMANN FOLIATION & HAEFLIGER Γ-STRUCTURE



Transverse Foliation Structures

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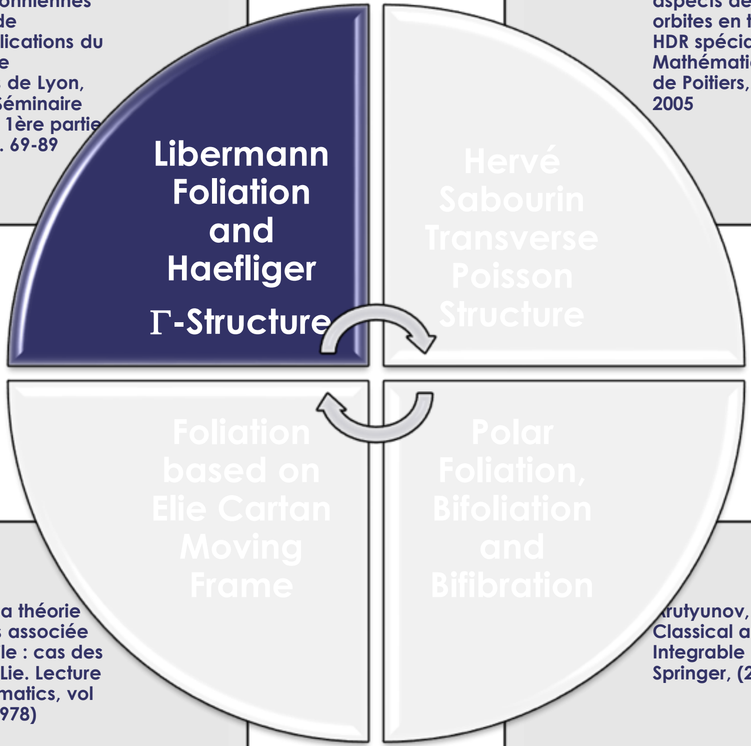

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THALES

LIBERMANN FOLIATION & HAEFLIGER Γ-STRUCTURE

- **In the framework of Stefan foliations, Libermann's foliations are the generalization of foliations studied in regular case by P. Libermann under the name of symplectically complete foliations.**
- The study of these foliations has been clarified by the introduction of the notion of **Poisson Γ-structure**, which is **the maximum extension of the notion of moment map in the sense of J.M. Souriau. This notion of foliation then appears as dual to that of Poisson Γ-structure.**
- **Generalized Moment** has been introduced by P. Molino, M. Condevaux et P. Dazord in papers of "Séminaire Sud-Rhodanien de Geometrie », with the translation in terms of **symplectic duality between Haefliger Γ-structures and Libermann singular foliations of the notion of Souriau's moment of a Hamiltonian action.**

LIBERMANN FOLIATION & HAEFLIGER Γ-STRUCTURE

- We define a map J from M to $\mathfrak{g}^*$, moment of Hamiltonian action ϕ such that:

$$\left\langle J(x), \sum_{i=1}^{p} \lambda^i \xi_i \right\rangle = \sum_{i=1}^{p} \lambda^i f_{\xi_i}(x) \quad , \quad \forall x \in M \quad \text{and} \quad \lambda^1, \ldots, \lambda^p \in \mathbb{R}$$

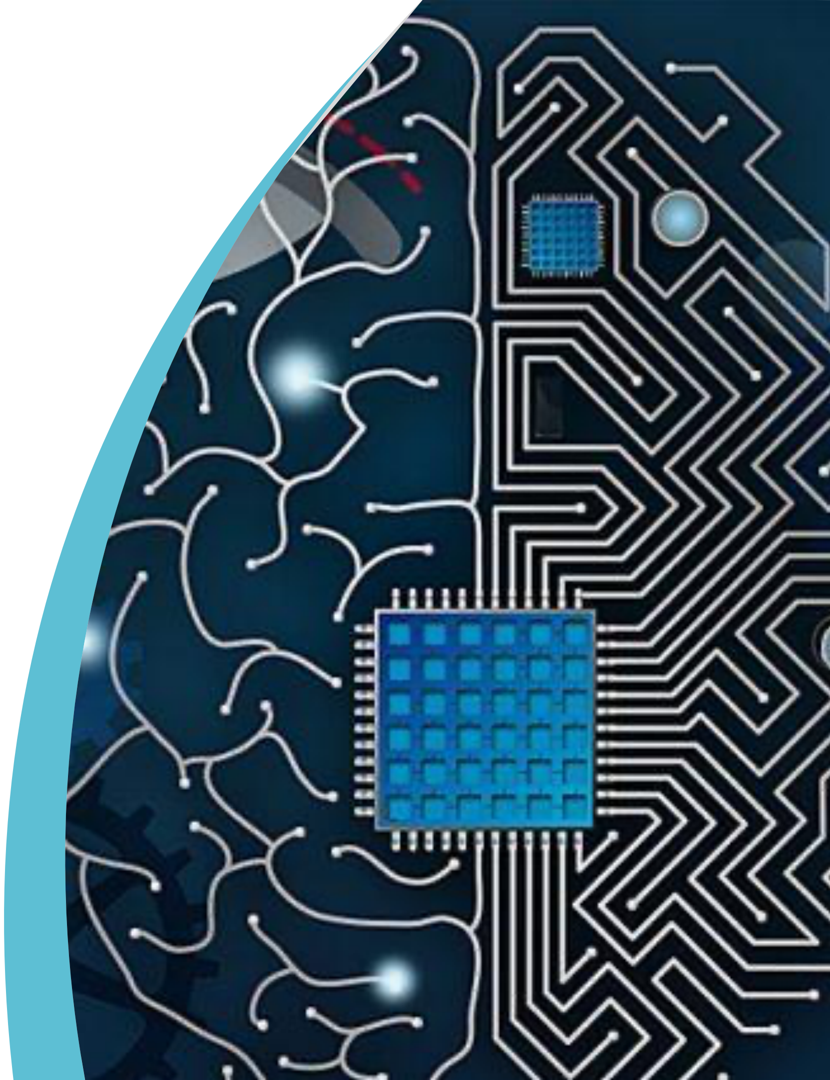
- In the particular case where the orbits of ϕ define a foliation $\mathfrak{I}$ of dimension r on M, the moment map J has constant rank r.
- The connected components of the level submanifolds of J then form on M a new foliation $\mathfrak{I}^{\perp}$, of codimension r.
- The foliations $\mathfrak{I}$ and $\mathfrak{I}^{\perp}$ are orthogonal in the symplectic sense, and the local first integrals of each of them define Hamiltonian fields tangent to the other.
- Based on work of Paulette Libermann, a Libermann Foliation is a foliation $\mathfrak{I}$ on the symplectic manifold (M, ω), with the following properties that are equivalent:
  - **(P1) The field of contact elements orthogonal to $\mathfrak{I}$ is completely integrable.**
  - **(P2) $\mathfrak{I}$ is locally generated by Hamiltonian fields.**
  - **(P3) The Poisson bracket of 2 local first integrals of $\mathfrak{I}$ is again a local first integral.**

LIBERMANN FOLIATION & HAEFLIGER Γ-STRUCTURE

- If $\mathfrak{I}$ is a Libermann foliation on (M,ω), the fact that the Poisson bracket of two local first integrals is still a local first integral makes it possible to define a natural structure of Poisson manifold on the transversals to the foliation.
- **The transverse structure of a Libermann foliation is a Poisson manifold structure, and the local projection onto a transverse is a Poisson morphism.**
- An important Poisson structure is the **Lie-Poisson structure** $(\mathfrak{g}^*,\Lambda_0)$ on the dual of the Lie algebra $\mathfrak{g}$ of the Lie group G by setting an **affine Poisson structure** $\{\eta,\xi\}_{\Lambda_0}(\upsilon)=\{\eta,\xi\}(\upsilon)+\tilde{\Theta}$ where $\tilde{\Theta}$ is a **2-cocycle on the Lie algebra** $\mathfrak{g}$.
- We can observe that **the Souriau moment of a Hamiltonian action is a Poisson morphism endowed with such an affine structure**. If we consider the Hamiltonian action of a Lie group on (M,ω), its orbits in general no longer form a foliation, but a singular foliation. Similarly, **the Souriau moment no longer determines a foliation, but simply a Poisson morphism on a transverse Poisson manifold**.

THALES

POLAR FOLIATION, BIFOLIATION & BIFIBRATION



Transverse Foliation Structures

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Libermann Foliation and Haefliger Γ-Structure

Hervé Sabourin Transverse Poisson Structure

Foliation based on Elie Cartan Moving Frame

Polar Foliation, Bifoliation and Bifibration

- Fedida, E. : Sur la théorie des feuilletages associée au repère mobile : cas des feuilletages de Lie. Lecture Notes in Mathematics, vol 652. Springer (1978)

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POLAR FOLIATION, BIFOLIATION & BIFIBRATION

- We study bifoliation as described in:
  - **Arutyunov, G., Elements of Classical and Quantum Integrable Systems, Springer, (2019)**
- Consider P as a symplectic manifold equipped with a form ω and consider Ψ be foliation of P such that the quotient space N_{Ψ} of P is a manifold over an equivalence relation set up by Ψ. Let $\mathfrak{I}_{\Psi}$ the space of functions on P which are constant along the leaves of Ψ .
- Consider $T\Psi$ be a bundle of vectors tangent to the leaves of Ψ, and consider $T\Psi^{\perp}$ be its orthogonal complement with respect to ω. Let χ_f be the hamiltonian vector field of $f \in \mathfrak{I}_{\Psi}$. It follows as f is constant along Ψ, for any $v \in T\Psi$, that $\omega\left(v, \chi_f\right) = vf = 0$. Then χ_f lies in $T\Psi^{\perp}$. We observe that $T\Psi^{\perp}$ is spanned at each point by the hamiltonian vector fields of functions in $\mathfrak{I}_{\Psi}$.

Symplectic Transverse Foliation Structure

Lie-Poisson Bracket

$\{e_i\}$ basis of $\mathfrak{g}$ and $\{e^i\}$ basis of $\mathfrak{g}^*$

$[e_i, e_j] = c_{ij}^k e_k$ and $\langle e^i, e_j \rangle = \delta_{ij}$

$l = l_i e^i$

and l_i coordinates of l on $\mathfrak{g}^*$

$\nabla f = e_i \frac{\partial f}{\partial l_i}$ so that $\nabla l_i = e_i$

$\{l_i, l_j\} = \langle l, [e_i, e_j] \rangle = c_{ij}^k l_k$

Foliation $\Psi^\perp$ skew-orthogonal to Ψ at any point

Foliation Ψ by levels of a function set $\Im_\Psi$

Casimir Function

$i_{\xi_f}\omega + df = 0$ with

$\omega(\xi_f, \xi_h) = \{f, h\} = \xi_f h = -\xi_h f$

Casimir function C such that

$\{C, f\} = 0, \forall f \in \Im_\Psi$

$\Lambda_c = \{x \in P : C(x) = c\}$

$\Rightarrow \xi_f C = \{f, C\} = 0, \forall f \in \Im_\Psi$

$v \in T\Psi$

$\xi_f = \{f, .\}$

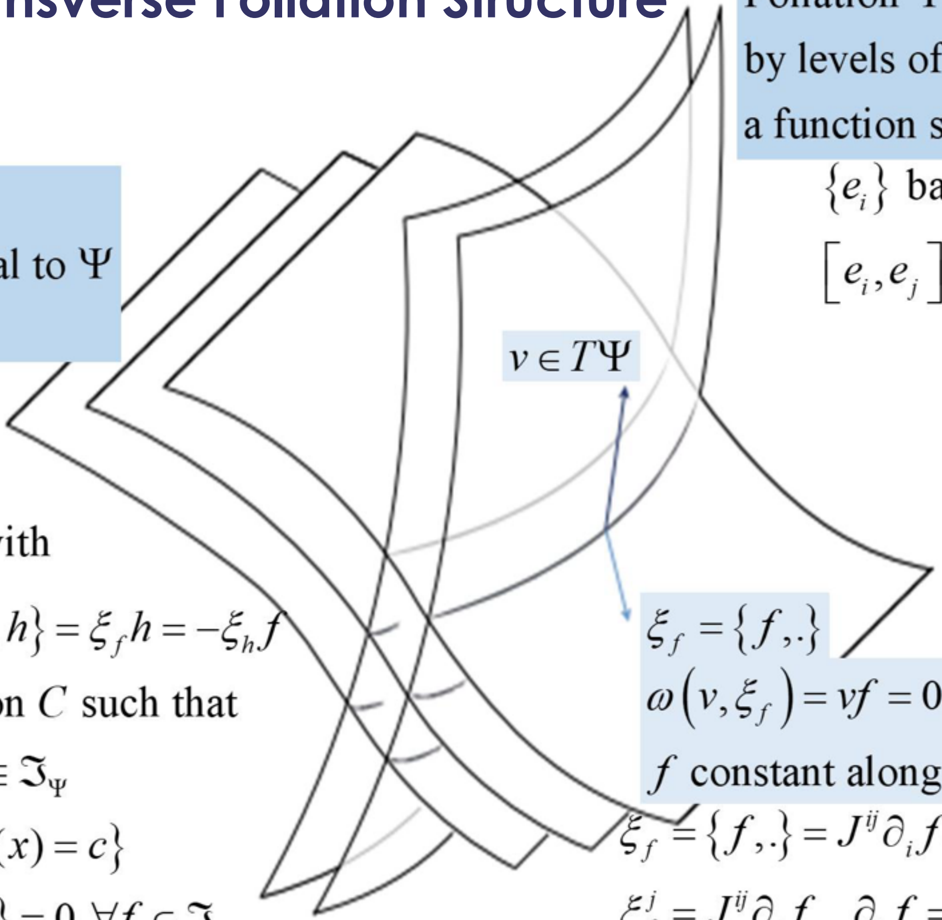
$\omega(v, \xi_f) = vf = 0$

f constant along Ψ

Symplectic Transverse Foliation Structure

Foliation Ψ
by levels of
a function set $\Im_\Psi$

Foliation $\Psi^\perp$
skew-orthogonal to Ψ
at any point



$\{e_i\}$ basis of $\mathfrak{g}$ and $\{e^i\}$ basis of $\mathfrak{g}^*$

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$\nabla f = e_i \dfrac{\partial f}{\partial l_i}$ so that $\nabla l_i = e_i$

$\{l_i, l_j\} = \langle l, [e_i, e_j] \rangle = c_{ij}^k l_k$

$i_{\xi_f}\omega + df = 0$ with

$\omega(\xi_f, \xi_h) = \{f, h\} = \xi_f h = -\xi_h f$

Casimir function C such that

$\{C, f\} = 0, \forall f \in \Im_\Psi$

$\Lambda_c = \{x \in P : C(x) = c\}$

$\Rightarrow \xi_f C = \{f, C\} = 0, \forall f \in \Im_\Psi$

$\xi_f = \{f, .\}$

$\omega(v, \xi_f) = vf = 0$

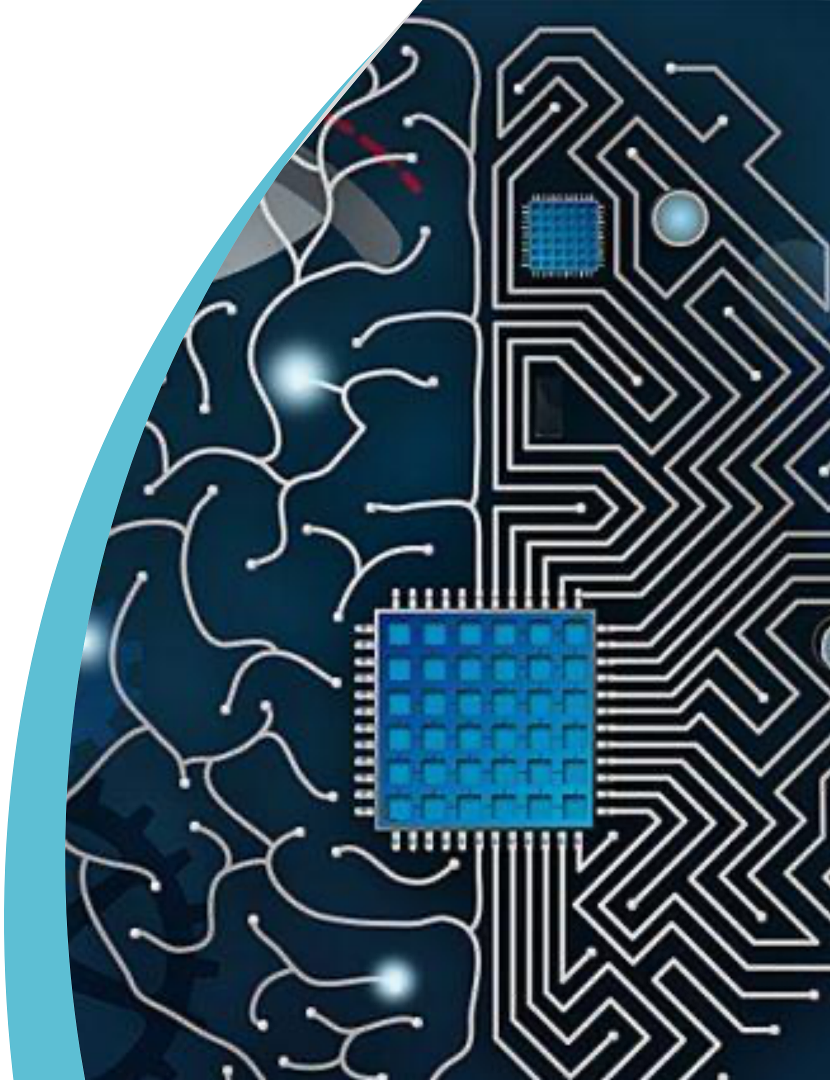
f constant along Ψ

$\xi_f = \{f, .\} = J^{ij} \partial_i f \partial_j = \xi_f^j \partial_j$

$\xi_f^j = J^{ij} \partial_i f,\ \partial_i f = \omega_{ij} \xi_f^i$ and $\xi_{\{f,g\}} = [\xi_f, \xi_g]$

THALES

FOLIATION BASED ON ELIE CARTAN MOVING FRAME



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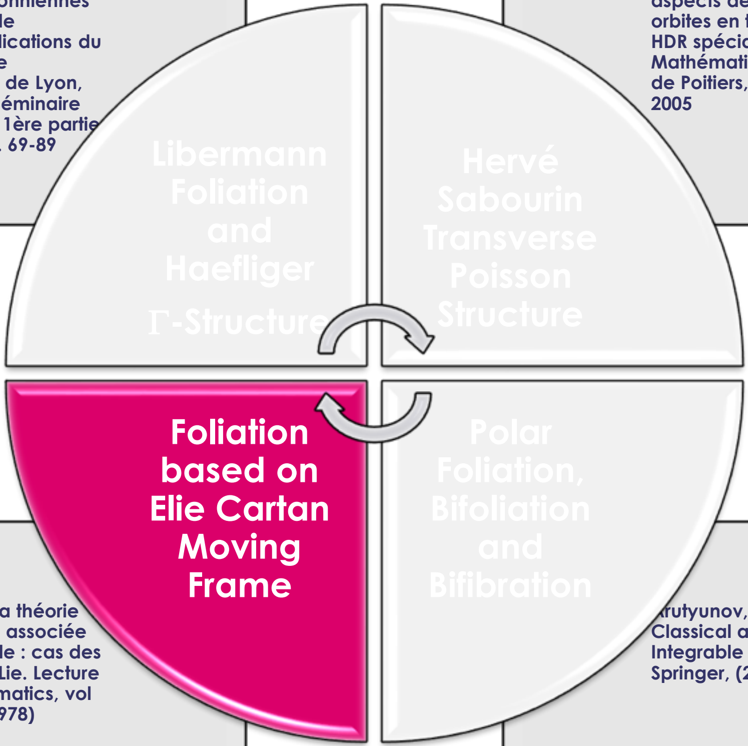

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THALES

FOLIATION BASED ON ELIE CARTAN MOVING FRAME

- Based on the duality between the algebra of exterior differential forms and the Lie algebra of vector fields, **Edmond Fédida** has explored the theory of foliation structures, either in the language of **fully integrable Pfaff systems**, or in that of **involution vector fields**. Within the theory of foliation structures, **Lie groups transformations, which correspond to systems of vector fields in involution, associated with a Lie algebra**, has a counterpart, which is the **theory of foliations associated with the Cartan's moving frame**.
- For group of Lie transformation, a Manifold M of dimension n is equipped with vector fields X_i and structure coefficients given by C_{ij}^k by $\left[X_i, X_j\right] = C_{ij}^k X_k$, and **as counterpart for moving frame, we consider a system of Pfaff form ω_i with structure coefficients where $d\omega_i = C_{ij}^k \omega_j \wedge \omega_k$.**
- The ω_i form a constant rank system at all points. **The classes form a foliation M of whose transverse structure is modeled on that of a subspace of G**. On the counterpart, The X_i form a constant rank system at all points: trajectories, the trajectories then constitute a foliation of M ; leaves are homogeneous spaces. The ω_i are linearly independent at any point: **the associated foliation has a transverse structure modeled on G**. Such a foliation then deserves the name of **Lie foliation**.

FOLIATION BASED ON ELIE CARTAN MOVING FRAME

- We can always come back to this case, a structure is given on M by a 1-form m with values in a Lie algebra of dimention q that verify **Maurer-Cartan equation**:

$$d\omega + \frac{1}{2}[\omega, \omega] = 0$$

- One can "desingularize" the foliation defined by ω on M by considering on the trivial principal bundle $M \times G$, the form of **connection** Ω induced by ω ; implying that Ω **is a form of flat connection**. In particular Ω is a 1-form on $M \times G$ satisfying the **Maurer-Cartan equation**:

$$d\Omega + \frac{1}{2}[\Omega, \Omega] = 0$$

- $\omega : T_x(M) \rightarrow \mathfrak{g}$ is surjective for all $x \in M$. Under these conditions ω **determines a foliation of codimension q of M , and is a Lie G-foliation of M .**
- **Molino has proved that this foliation is transversely parallelizable. We can then complete in a Riemanian structure of M by an "horizontal" metric subject to the only condition that at any point $x \in M$, the space tangent to the leaf is orthogonal, to the transverse space. Hence the foliation has a quasi-fibered metric.**

Cartan-Reeb-Godbillon-Vey: Maurer-Cartan Algebra for sl(2,R)

- Maurer-Cartan form with coordinates $\left[X_i, X_j\right] = \sum_k c_{ij}^k X_k \Rightarrow d\omega_k = -\sum_{i<j} c_{ij}^k \omega_i \wedge \omega_j$
- Maurer-Cartan Algebra bases of $sl(2,\mathbb{R})$

$$d\omega_0 = \omega_0 \wedge \omega_1 \;\; , \;\; d\omega_1 = \omega_0 \wedge \omega_2 \;\; , \;\; d\omega_2 = \omega_1 \wedge \omega_2$$

$$X \in SL(2,\mathbb{R}) \;\; , \;\; X^{-1}dX = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}$$

$$\omega_0 = \omega_{12} \;\; , \;\; \omega_1 = 2\omega_{11} \;\; , \;\; \omega_2 = -2\omega_{21}$$

$$X \in SL(2,\mathbb{R}) \text{ such that } X = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \rho & 0 \\ \sigma & \rho^{-1} \end{pmatrix} = \begin{pmatrix} \rho\cos\varphi - \sigma\sin\varphi & -\rho^{-1}\sin\varphi \\ \rho\sin\varphi + \sigma\cos\varphi & \rho^{-1}\cos\varphi \end{pmatrix} \;\; , \;\; \rho > 0$$

$$\omega_0 = \omega_{12} = -\frac{d\varphi}{\rho^2}$$

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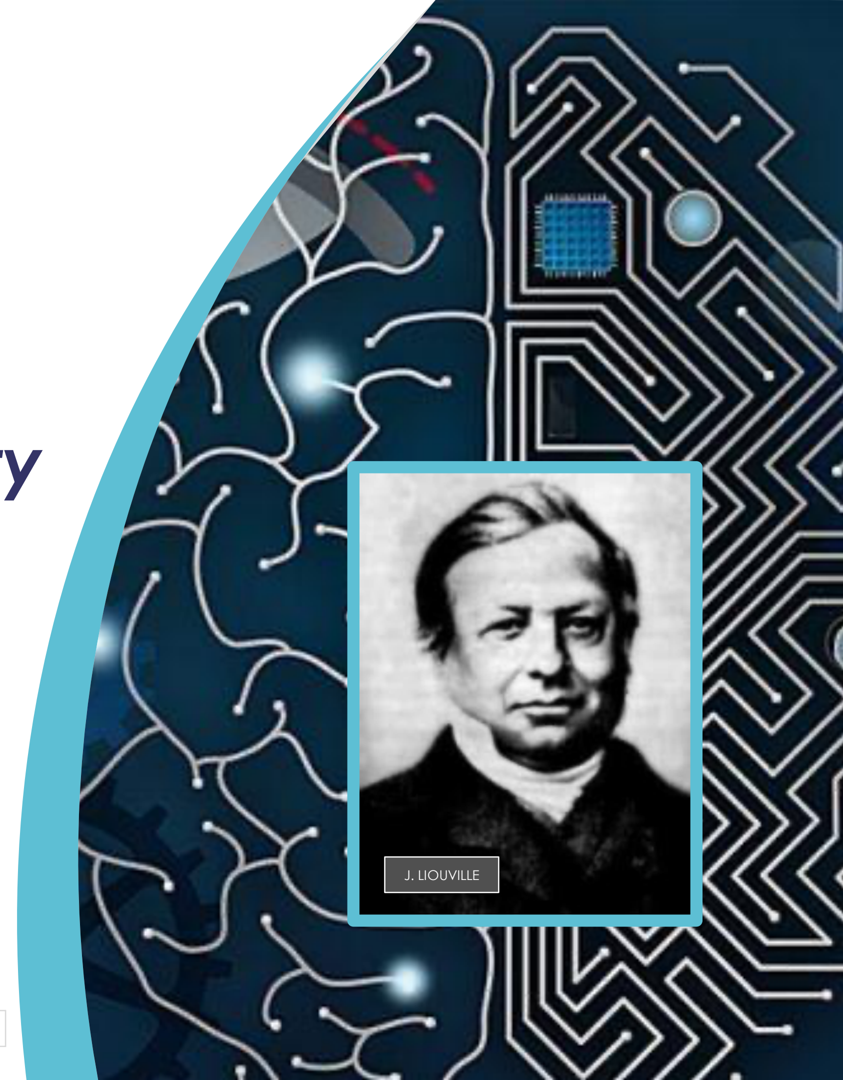
Liouville Complete Integrability and Information Geometry



Liouville Complete Integrability

Integrability

- Integrability refers to the existence of invariant, regular foliations, related to the degree of integrability, depending on the dimension of the leaves of the invariant foliation.
- In the case of Hamiltonian systems, this integrability is called "complete integrability" in the sense of Liouville (Liouville-Mineur Theorem).
- In case of Liouville integrability, a regular foliation of the phase space by invariant manifolds such that the Hamiltonian vector fields associated to the invariants of the foliation span the tangent distribution. In this case, a maximal set of Poisson commuting invariants exist (functions on the phase space whose Poisson brackets with the Hamiltonian of the system, and with each other, vanish).
- If the phase space is symplectic, the leaves of the foliation are totally isotropic with respect to the symplectic form and such a maximal isotropic foliation is called Lagrangian.

Liouville Complete Integrability on statistical manifolds

In 1993, in the framework of Liouville Completely Integrable Systems on Statistical Manifolds, Y. Nakamura developed gradient systems on manifolds of various probability distributions. He proved the following results

- the gradient systems on the even-dimensional manifolds of Gaussian and multinomial distributions are completely integrable Hamiltonian systems.
- the gradient systems can always be linearized by using Information Geometry dual coordinate system of the original coordinates.
- the gradient flows on the statistical manifolds converge to equilibrium points exponentially.
- the gradient systems are completely integrable Hamiltonian systems if the manifolds have even dimensions. There is a 2m-dimensional Hamiltonian System which is integrable so that the Dynamics restricted to the common level sets of first integrals is a gradient.
- the gradient system associated with the Gaussian distribution can be related to an Orstein-Ulhenbeck process

Nakamura Theorem

We consider the statistical Manifolds with Fisher Metric

$$p(x,\theta) \text{ with } \theta = (\theta_1,...,\theta_n)^T \text{ and } l(x,\theta) = \log p(x,\theta)$$

$$G = [g_{i,j}] \text{ with } g_{i,j} = E\left[\frac{\partial l(x,\theta)}{\partial \theta_i}\frac{\partial l(x,\theta)}{\partial \theta_j}\right] = \frac{\partial^2 \Psi(\theta)}{\partial \theta_i \partial \theta_j} \quad (\otimes)$$

Gradient flow on Statistical Manifolds

$$(¤) \quad \frac{d\theta}{dt} = -G^{-1}\frac{\partial \Psi}{\partial \theta} \text{ with } \frac{\partial \Psi}{\partial \theta} = \left(\frac{\partial \Psi}{\partial \theta_1},...,\frac{\partial \Psi}{\partial \theta_n}\right)^T$$

First Y. Nakamura Theorem

- The gradient system $(¤)$ is always linearizable. The induced flow on an open subset of the Riemannian Statistical Manifold such that there exists a potential function satisfying $(\otimes)$ converges to equilibrium points exponentially.

$$\eta = \frac{\partial \Psi(\theta)}{\partial \theta} \text{ with } \eta = (\eta_1,...,\eta_n)^T \text{ and } (¤) \Rightarrow \frac{d\eta}{dt} = -\eta \ , \ \eta(t) = e^{-t}\eta(0)$$

Liouville Completely Integrable Systems on Statistical Manifolds

Second Y. Nakamura Theorem

- If n is even $(n = 2m)$, then the gradient system (¤) is a completely integrable Hamiltonian system:

$$P_j = \eta_j^{-1} \quad , \quad Q_j = \eta_{m+j} \quad \text{and} \quad H_j = P_j Q_j \quad \text{for} \quad j = 1, \ldots, m$$

$$\frac{d\eta}{dt} = -\eta \Rightarrow \frac{dH_j}{dt} = \frac{dP_j}{dt} Q_j + P_j \frac{dQ_j}{dt} = \left(-\eta_j^{-2} \frac{d\eta_j}{dt} \right) \eta_{m+j} + \eta_j^{-1} \frac{d\eta_{m+j}}{dt} = \eta_j^{-1}\eta_{m+j} - \eta_j^{-1}\eta_{m+j} = 0$$

$$\Rightarrow \frac{dH_j}{dt} = 0$$

- Define the Hamiltonian $H = \sum_{j=1}^{m} H_j$, then the corresponding equations of motion,

$$\frac{d\Xi_k}{dt} = \left\{ \Xi_k , H \right\} \text{ with } \begin{cases} \Xi_k = P_k \ , k = 1, \ldots, m \\ \Xi_k = Q_{k-m} \ , k = m+1, \ldots, n \end{cases}$$ coincides with $\frac{d\eta}{dt} = -\eta$ and $\begin{cases} \left\{ H_j , H \right\} = 0 \\ \left\{ H_i , H_j \right\} = 0 \end{cases}$

Liouville Completely Integrable Systems on Gaussian Manifolds

Gaussian distributions use-case

$$p(x,\theta)=\frac{1}{\sqrt{2\pi\sigma}}e^{-\frac{(x-m)^2}{2\sigma^2}}$$

$$\theta=(\theta_1,\theta_2)=\left(\frac{m}{\sigma^2},\frac{1}{2\sigma^2}\right)\ \text{and}\ E[x]=\frac{1}{2}\theta_1\theta_2^{-1}\ ,\ E\left[x^2\right]=\frac{1}{4}\theta_1^2\theta_2^{-2}+\frac{1}{2}\theta_2^{-1}$$

$$G=\frac{1}{2}\begin{pmatrix}\theta_2^{-1} & \theta_1\theta_2^{-2}\\ \theta_1\theta_2^{-2} & \theta_2^{-2}+\theta_1^2\theta_2^{-3}\end{pmatrix}=\left[\frac{\partial^2\Psi}{\partial\theta_i\partial\theta_j}\right]\ ,\ \Psi(\theta)=\frac{\theta_1^2}{4\theta_2}-\log\left(\sqrt{\theta_2}\right)+\log\sqrt{\pi}$$

$$\frac{d\theta}{dt}=-G^{-1}\frac{\partial\Psi}{\partial\theta}\Rightarrow\begin{pmatrix}\frac{d\theta_1}{dt}\\ \frac{d\theta_2}{dt}\end{pmatrix}=-\frac{1}{2}\begin{pmatrix}\theta_1^3\theta_2^{-1}\\ \theta_1^2-2\theta_2\end{pmatrix}\Rightarrow\begin{cases}\theta_1(t)=\dfrac{\eta_1(0)}{\eta_2(0)-\eta_1(0)^2e^{-t}}\\ \theta_2(t)=\dfrac{1}{2\left(\eta_2(0)e^{-t}-\eta_1(0)^2e^{-2t}\right)}\end{cases}$$

$$\frac{dH}{dt}=0\ \text{where}\ H=PQ=\theta_1^{-1}+\frac{1}{2}\theta_1\theta_2^{-1}\ \text{with}\ P=2\theta_2\theta_1^{-1}=E[x]^{-1}\ ,Q=\frac{1}{4}\theta_1^2\theta_2^{-2}+\frac{1}{2}\theta_2^{-1}=E\left[x^2\right]$$

$$\frac{dQ}{dt}=-\frac{\partial H}{\partial P}\ ,\ \frac{dP}{dt}=\frac{\partial H}{\partial Q}$$

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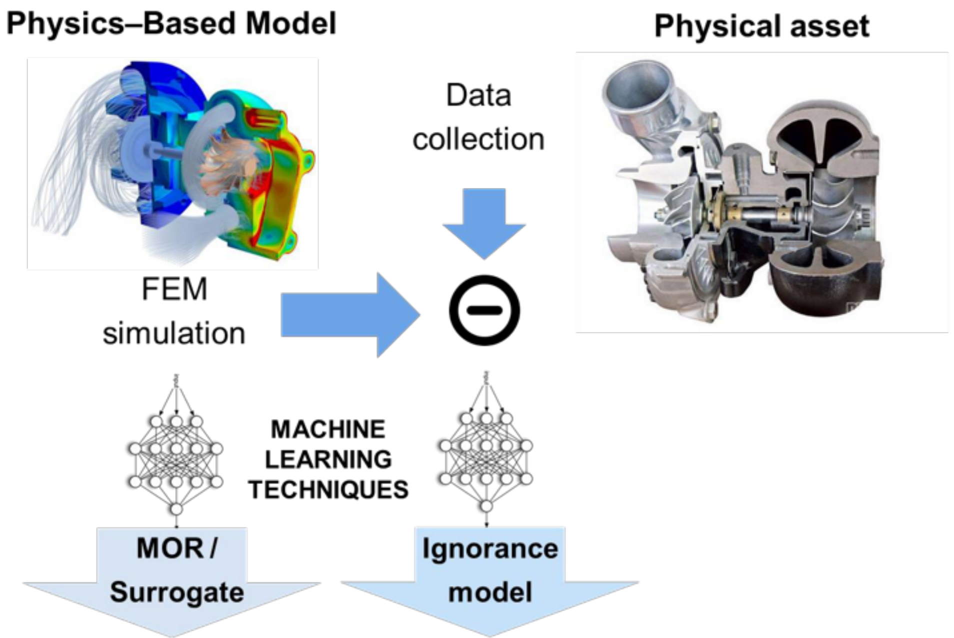
Thermodynamics-Informed Neural Network and Symplectic Foliation Structure

PINN Physics-Informed Neural Networks



$$\frac{dx}{dt} = f^{\texttt{physics-based}}(x, \mathbf{z}) + f^{\texttt{data-driven}}(x, \mathbf{z})$$

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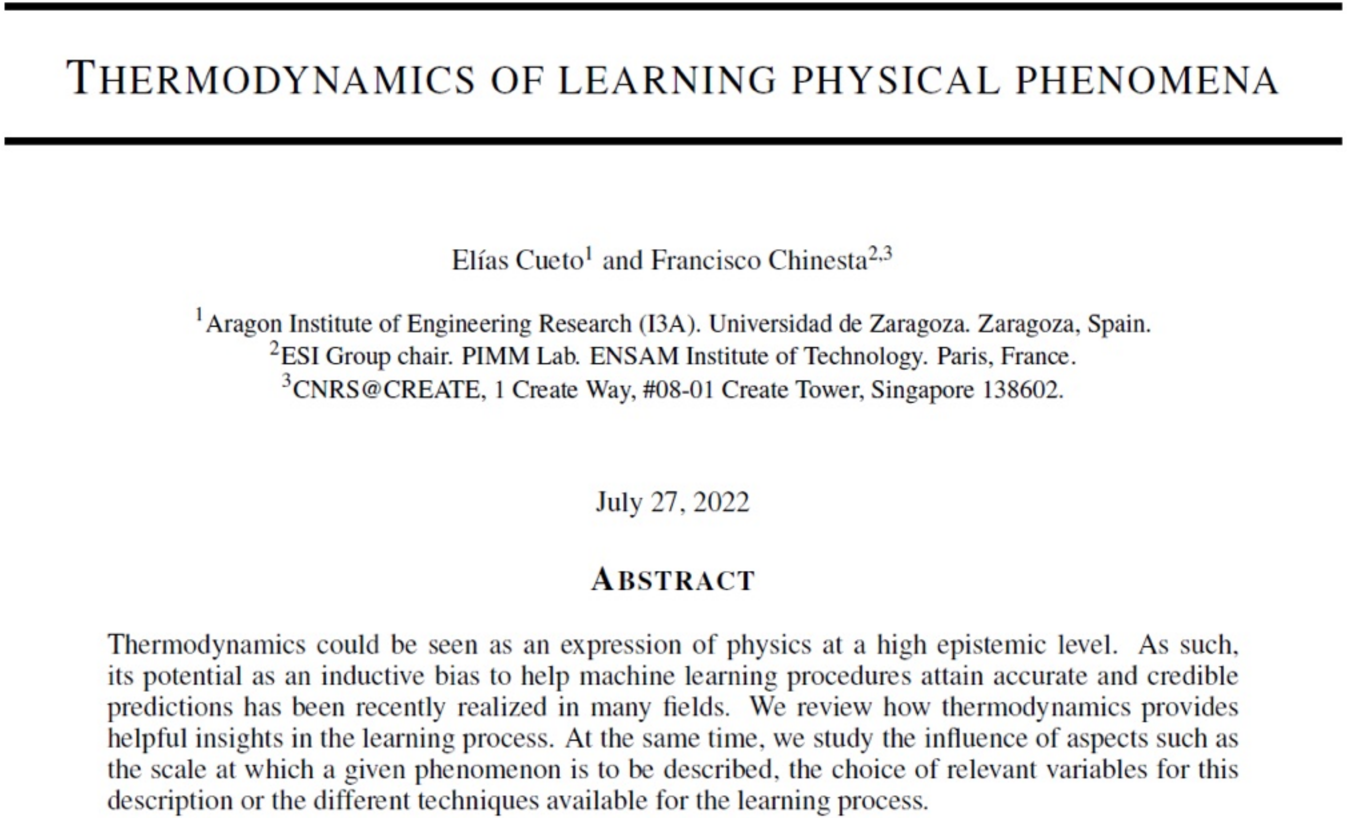
Thermodynamics-Informed Neural Network based on Metriplectic Flow


THERMODYNAMICS OF LEARNING PHYSICAL PHENOMENA

Elías Cueto[1] and Francisco Chinesta[2,3]

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[2]ESI Group chair. PIMM Lab. ENSAM Institute of Technology. Paris, France.
[3]CNRS@CREATE, 1 Create Way, #08-01 Create Tower, Singapore 138602.

July 27, 2022

ABSTRACT

Thermodynamics could be seen as an expression of physics at a high epistemic level. As such, its potential as an inductive bias to help machine learning procedures attain accurate and credible predictions has been recently realized in many fields. We review how thermodynamics provides helpful insights in the learning process. At the same time, we study the influence of aspects such as the scale at which a given phenomenon is to be described, the choice of relevant variables for this description or the different techniques available for the learning process.

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Viscosity Simulation of water and honey

Computational sensing, understanding, and reasoning: an artificial intelligence approach to physics-informed world modeling

Beatriz Moya[1*], Alberto Badias[2], David Gonzalez[3], Francisco Chinesta[1,4], Elias Cueto[3]

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[3]ESI Group Chair, Aragon Institute of Engineering Research, Universidad de Zaragoza, Maria de Luna, s.n., Zaragoza, 50018, Spain.
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Algorithm 1 Hybrid twin

Require: Free surface $\mathbf{x}_n \in X$, full state $\mathbf{s}_n \in S$, source model π_θ
Ensure: Next state $\mathbf{s}_{n+1}$

```
for Iterations do
    for n = 1 to N sequences do
        Encode z_n ← φ_GRU(x_n);
        Compute SPNN a_n ← π_θ(z_n), with a_n = [L_n, M_n, DE_n, DS_n];
        Determine next integration step ẑ_{n+1} ← Δt(L_n DE_n + M_n DS_n) + z_n;
        Decode ŝ_{n+1} ← ψ(ẑ_{n+1});
    end for
    Extract free surface of ŝ_{n+1};
    Compute loss L_correction;
    Update π_θ;
end for
return Optimized hybrid twin π_θ
```

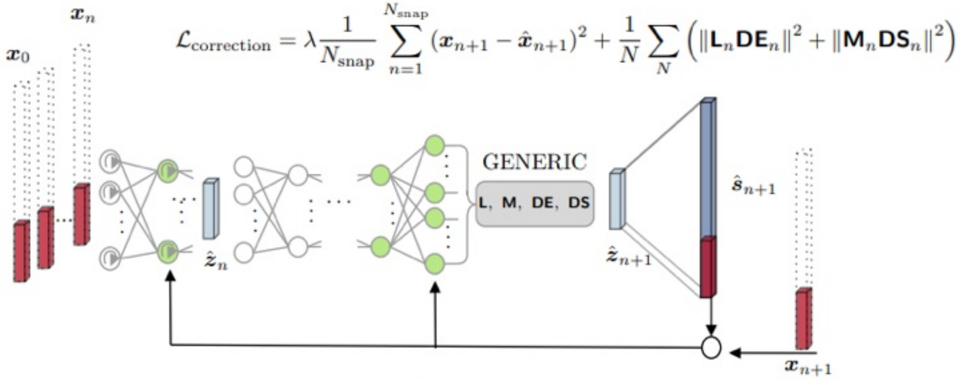
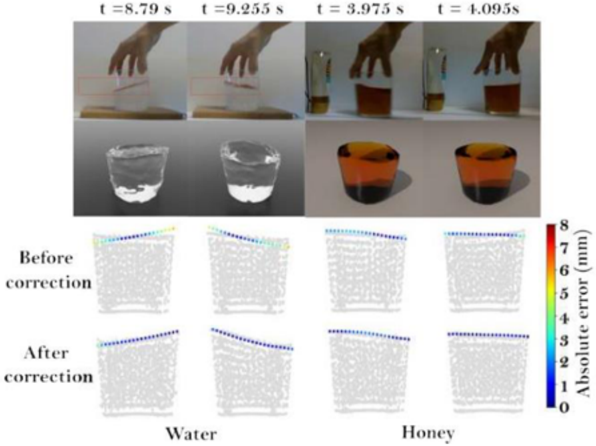


Fig. 5 Hybrid twin algorithm for the adaptation to new physics. The network as a whole adapts to the new data by fine-tuning only part of the layers, highlighted in green. The result of the integration algorithm is compared with the ground truth liquid by evaluating the reconstruction of the free surface in time. As a result the twin adapts when it detects any deviation between the prediction and the ground trut



Port-Metriplectic Neural-Network

Port-metriplectic neural networks: thermodynamics-informed machine learning of complex physical systems

Quercus Hernández[1] · Alberto Badías[2] · Francisco Chinesta[3,4] · Elías Cueto[1]

Received: 3 November 2022 / Accepted: 17 February 2023 / Published online: 21 March 2023

Abstract
We develop inductive biases for the machine learning of complex physical systems based on the port-Hamiltonian formalism. To satisfy by construction the principles of thermodynamics in the learned physics (conservation of energy, non-negative entropy production), we modify accordingly the port-Hamiltonian formalism so as to achieve a port-metriplectic one. We show that the constructed networks are able to learn the physics of complex systems by parts, thus alleviating the burden associated to the experimental characterization and posterior learning process of this kind of systems. Predictions can be done, however, at the scale of the complete system. Examples are shown on the performance of the proposed technique.

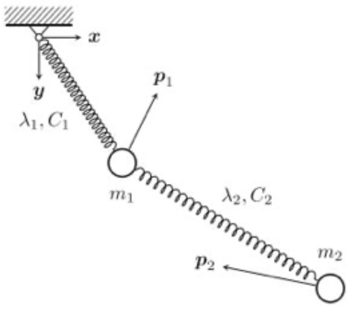
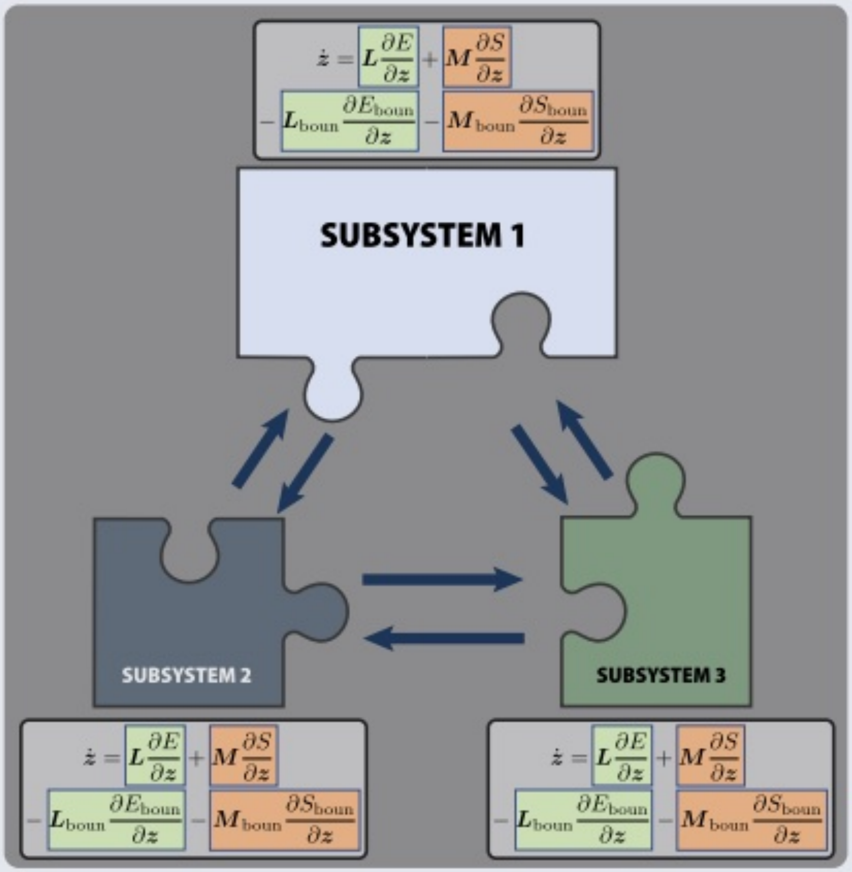


Fig. 2 Double thermoelastic pendulum. A single pendulum with external perturbations is learned, and the coupling between both systems is achieved via the port-metriplectic framework



Thermodynamics-Informed Neural Network

- The Fokker-Planck equation and, more particularly, its equivalent Itô stochastic differential equation takes the form:

$$dz = \left[L(z)\frac{\partial E}{\partial z} + M(z)\frac{\partial S}{\partial z} + k_B \nabla M(z) \right] dt + d\tilde{z}$$

- where k_B is the Boltzmann constant, $M(z)$ is a symmetric, positive semi-definite dissipation matrix, S is a second potential (the so-called Massieu potential, entropy at this level of description) and $d\tilde{z}$ is a Wiener process that satisfies

$$d\tilde{z} = B(z)dW(t)$$

- with B a non-square matrix satisfying

$$B(z)B(z)^T = 2k_B M(z)$$

- The importance of thermal fluctuations is controlled by the relative value of the Boltzmann constant k_B with respect to the average value of entropy. Given that E, S, L and M do not depend on k_B, if these effects are of low importance, we can take the limit $k_B \to 0$, resulting in

$$dz = L(z)\frac{\partial E}{\partial z} + M(z)\frac{\partial S}{\partial z}$$

Thermodynamics-Informed Neural Network

$$\frac{dz}{dt} = L(z)\frac{\partial E}{\partial z} + M(z)\frac{\partial S}{\partial z}$$

$$\frac{dz}{dt} = \{z, E\} + (z, S)$$ **Metriplectic Flow**

- This same assumption, $k_B \to 0$, induces two additional consequences:

$$L(z)\frac{\partial S}{\partial z} = 0 \quad \text{and} \quad M(z)\frac{\partial E}{\partial z} = 0$$

- which constitute the ingredients of the celebrated General Equation for the non-Equilibrium Reversible-Irreversible Coupling, **GENERIC equations**. This type of formulations are also known as metriplectic formulations, since they combine metric and symplectic terms. However, in GENERIC equations, known as degeneracy conditions, play a fundamental role. They are key ingredients in the demonstration of the a priori satisfaction of the two laws of thermodynamics:

  - **Conservation of energy** in closed systems Given the **anti-symmetry of L**:

$$\frac{dE}{dt}(z) = \frac{\partial E}{\partial z}\frac{\partial z}{\partial t} = 0$$

  - **Non-negative entropy production**, given the **positive semi-definiteness of M**:

$$\frac{dS}{dt}(z) = \frac{\partial S}{\partial z}\frac{\partial z}{\partial t} = \frac{\partial S}{\partial z}M(z)\frac{\partial S}{\partial z} \geq 0$$

Thermodynamics-Informed Neural Network

- Therefore, the GENERIC structure consistently guarantees the satisfaction of the laws of thermodynamics by construction. This makes GENERIC a very appealing choice for the construction of inductive biases in the learning of physical phenomena.
- For Thermodynamics of learning physical phenomena, we assume that D_i data sets contain labelled pairs of a single-step state vector z_t and its time evolution z_{t+1}

$$D = \{D_i\}_{i=1}^{N_{sim}}, D_i = \{(z_t, z_{t+1})\}_{t=0}^{T}$$

- so that a neural network can be constructed by means of two loss terms, a data loss term that takes into account the correct prediction of the state vector time evolution using the GENERIC integrator, defined as:

$$\Lambda_n^{data} = \left\| \frac{dz^{GT}}{dt} - \frac{dz^{net}}{dt} \right\|_2^2$$

Thermodynamics-Informed Neural Network

- where $\|\cdot\|_2$ denotes the L2-norm. The choice of the time derivative instead of the state vector itself is to regularize the global loss function to a uniform order of magnitude with respect to the degeneracy terms. A second loss term takes into account the fulfillment of the degeneracy equations,

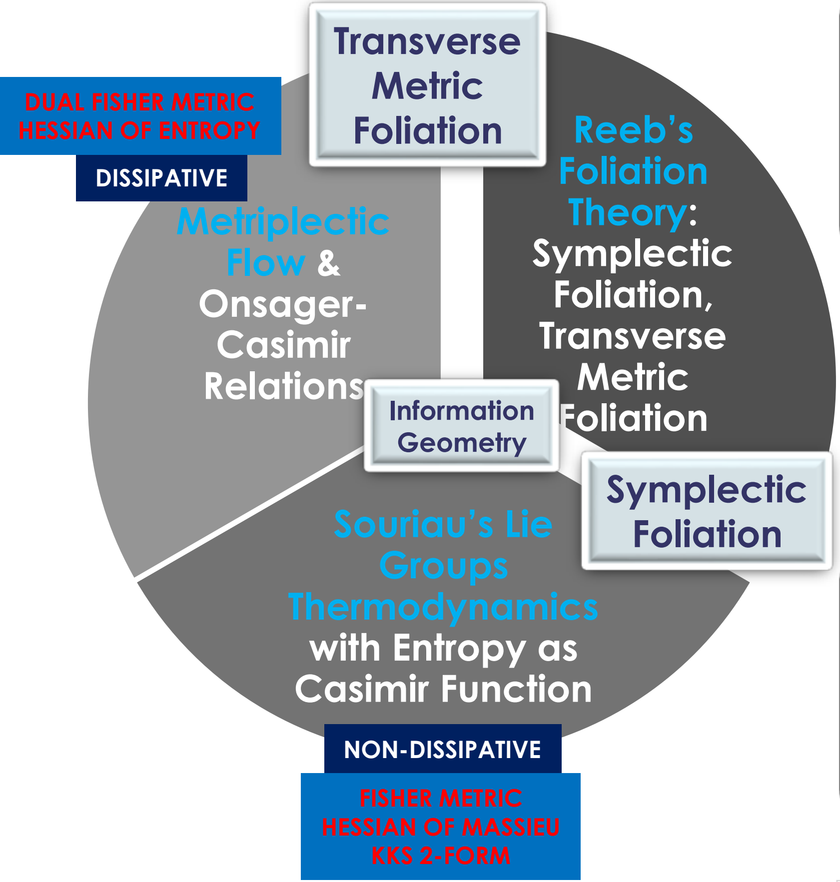
$$\Lambda_n^{\deg} = \left\| L \frac{\partial S}{\partial z_n} \right\|_2^2 + \left\| M \frac{\partial E}{\partial z_n} \right\|_2^2$$

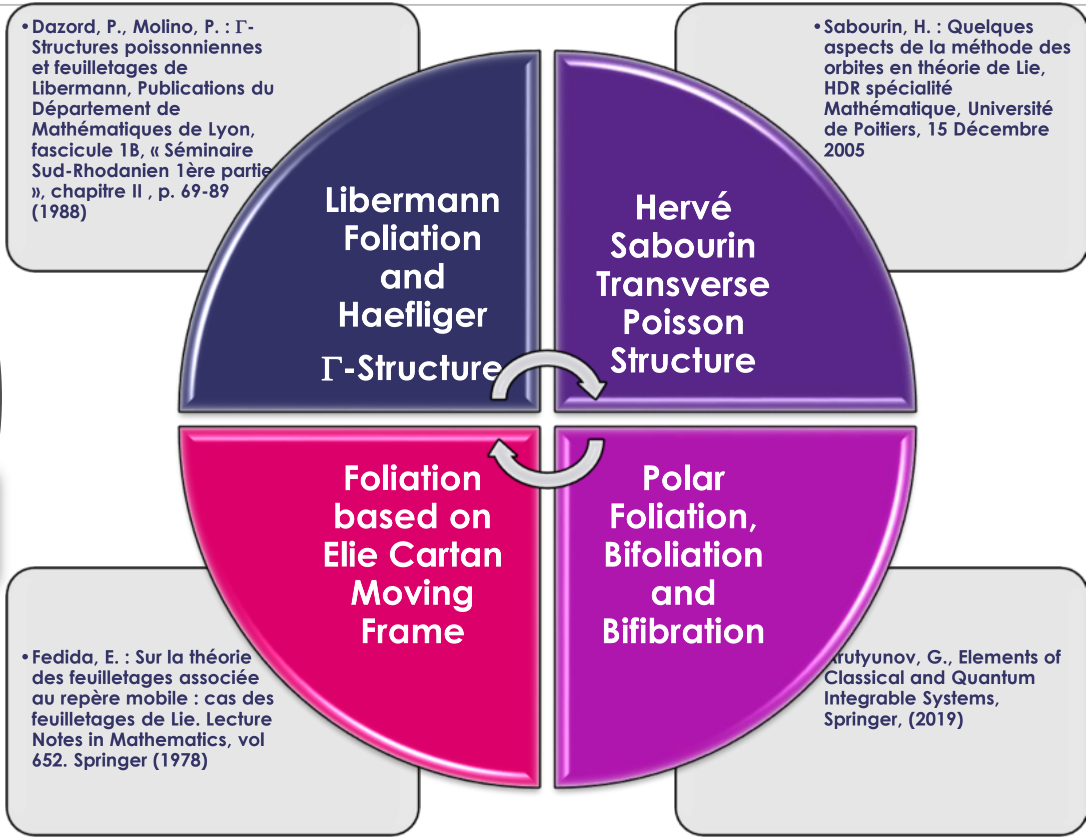
- This formulation gave rise to the so-called structure-preserving neural networks and thermodynamics-informed neural networks. These networks have been employed recently in the development of physics perception with the help of computer vision techniques.
- The global loss term is a weighted mean of the two terms over the shuffled Nbatch batched snapshots.

$$\Lambda = \frac{1}{N_{batch}} \sum_{n=0}^{N_{batch}} \left(\lambda \Lambda_n^{data} + \Lambda_n^{\deg} \right)$$

Conclusion

Symplectic Foliation Transverse Structure of Dissipative Thermodynamics





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Information Geometry, Toric Manifold, Delzant Polytope, Symplectic Potentials and Guillemin Metric

Notation.

- $\mathbb{P}_n(c)$ is the complex projective space of complex dimension n and holomorphic sectional curvature $c > 0$.
- $\mathbb{D}(c)$ is the unit disk in $\mathbb{C}$ endowed with the Hyperbolic metric of constant holomorphic sectional curvature $c < 0$.

Similar examples

Toric manifold	Statistical manifold
$S^2 = \mathbb{P}_1(\frac{1}{n})$	Binomial
$\mathbb{C}$	Poisson
$\mathbb{P}_n(1)$	Categorical
$\mathbb{P}_n(\frac{1}{m})$	Multinomial
$\mathbb{D}(-\frac{1}{r})$	Negative Binomial

Recap

- $\mathcal{B}(n) \cong (0, n)$.
- Fisher metric $h_F(x) = \frac{n}{x(n-x)}$.
- Potential $\phi(x) = x\ln(x) + (n-x)\ln(n-x)$.

Key observation:

$$\phi = 2\phi_G,$$

where ϕ_G is the Guillemin's potential associated to the polytope $\Delta = [0, n]$.

Thus:

By Delzant's correspondence:

- $[0, n] \rightsquigarrow (S^2, \frac{n}{2}\omega_{round})$
- Momentum map $J(x, y, z) = \frac{n}{2}(z + 1)$.

By Guillemin's results and the fact that $\phi = 2\phi_G$:

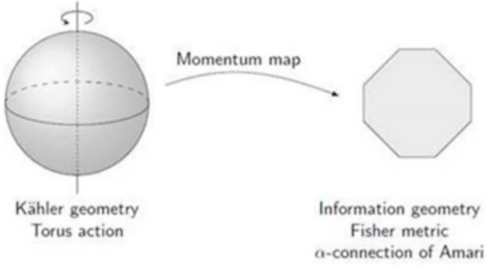
- $J : (S^2, \frac{n}{2}g_{round}) \to ((0, n), \frac{1}{2}h_F)$ is a Riemannian submersion.

Rescale by a factor 2:

- $J : (S^2, ng_{round}) \to ((0, n), h_F)$ is a Riemannian submersion.

Summary

- $(\mathcal{B}(n), h_F)$ is isometric to the interior of a Delzant polytope endowed with (a multiple of) the Guillemin metric.
- (S^2, ng_{round}) is the corresponding toric manifold.



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Variables actions-angles non commutatives et exemples d'images convexes de l'application moment, Thomas Delzant PhD, Paris 6 1986, supervised by CM. Marle

Symplectic versus complex points of view

Delzant, T.: Hamiltoniens périodiques et images convexes de l'application moment. Bull. Soc. Math. France 116(3), 315–339 (1988), https://irma.math.unistra.fr/~delzant/BSMF.pdf

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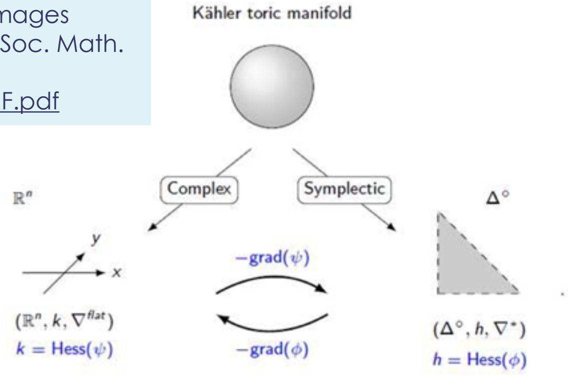

Proposition

1. The map $-\text{grad}(\psi) = (\frac{\partial\psi}{\partial x_1}, \ldots, \frac{\partial\psi}{\partial x_n})$ is an isometry from $(\mathbb{R}^n, k)$ onto (Δ°, h) with inverse $-\text{grad}(\phi)$.
2. ϕ and ψ are Legendre dual to each other:

$$\phi(x) + \psi(y) + \langle x, y\rangle = 0, \qquad x = -\text{grad}(\phi)(y), \qquad y \in \Delta^\circ.$$

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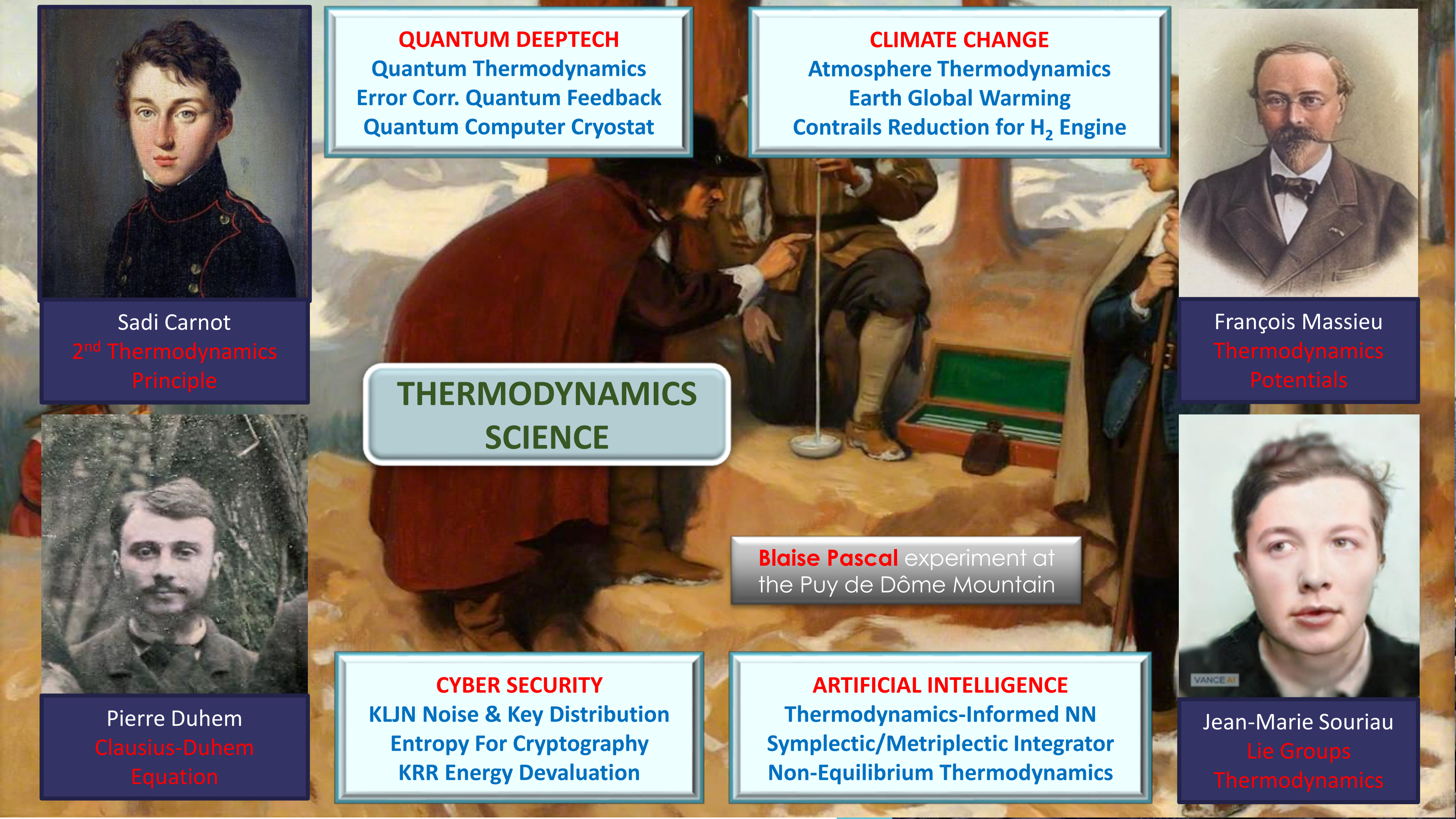
"We haven't yet discovered the foliations, but it will come. I am convinced that the fundamental problem posed by quantum mechanics is the following problem. … it will be necessary to take into consideration more complicated mathematical structures, such as foliations … It's a bit my hope that one day or another, we will manage to develop models where a phenomenon will be defined as a leaf of foliation in a product of spaces of vision by a space of observer positions." – René Thom in "DETERMINISME ET INNOVATIONS"
https://www.youtube.com/watch?v=BXxKQVQFnRo

"At the start, the theory of structural stability had seemed to me of such breadth and generality, that with it I could hope in some way to replace thermodynamics by geometry, to geometrize in a certain sense thermodynamics, eliminate from thermodynamic considerations all aspects of a measurable and stochastic nature to retain only the corresponding geometric characterization of the attractors." – René Thom


QUANTUM DEEPTECH
Quantum Thermodynamics
Error Corr. Quantum Feedback
Quantum Computer Cryostat
CLIMATE CHANGE
Atmosphere Thermodynamics
Earth Global Warming
Contrails Reduction for H_2 Engine
Sadi Carnot
2nd Thermodynamics Principle
François Massieu
Thermodynamics Potentials
THERMODYNAMICS SCIENCE
Blaise Pascal experiment at the Puy de Dôme Mountain
Pierre Duhem
Clausius-Duhem Equation
CYBER SECURITY
KLJN Noise & Key Distribution
Entropy For Cryptography
KRR Energy Devaluation
ARTIFICIAL INTELLIGENCE
Thermodynamics-Informed NN
Symplectic/Metriplectic Integrator
Non-Equilibrium Thermodynamics
Jean-Marie Souriau
Lie Groups Thermodynamics

Lille Carnot 2024

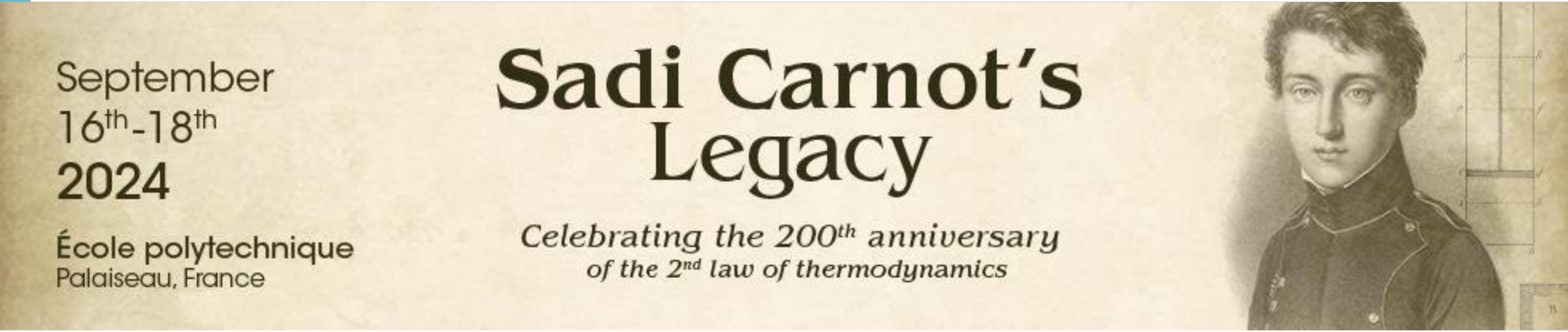
Celebration of 200 years since Sadi Carnot's Réflexions sur la puissance motrice du Feu, 1824–2024 & les Carnots

Hosted by History of Physics and Applied Science & Technologies Team (HOPAST) at IEMN, France - Patronage by French Académie des Sciences

Official Website URL: www.carnotlille2024.com

Talk: Symplectic Foliation Model of Sadi Carnot's Thermodynamics: from Carathéodory's seminal idea to Souriau's Lie Groups Thermodynamics

Ecole Polytechnique: Sadi Carnot's Legacy



SADI CARNOT'S LEGACY - "CELEBRATING THE 200TH ANNIVERSARY OF THE 2ND LAW OF THERMODYNAMICS"

When Sadi Carnot published his "Reflections on the motive power of fire" in 1824, there was no sign that one of the greatest scientific revolutions was about to take place, in a world then dominated by mechanics and optics. Yet, by bringing a conceptual analysis to the practical problem of the steam engine, Sadi Carnot wrote the birth certificate of thermodynamics, and, in particular, its second principle.

Today, thermodynamics has branched out into a multitude of fields and applications, from industrial processes to microscopic systems, and continues to renew our view of science.

Since its origins, thermodynamics has raised as many questions as it has answered.

To celebrate the bicentenary of the "Réflexions", this colloquium aims to bring together members of the thermodynamics community and to invite them to take a critical look at modern thermodynamics and the open questions it raises. The colloquium will be structured around pedagogical presentations introducing the various fields of the discipline. Poster sessions will allow participants to share their work.

https://carnot-legacy.sciencesconf.org/

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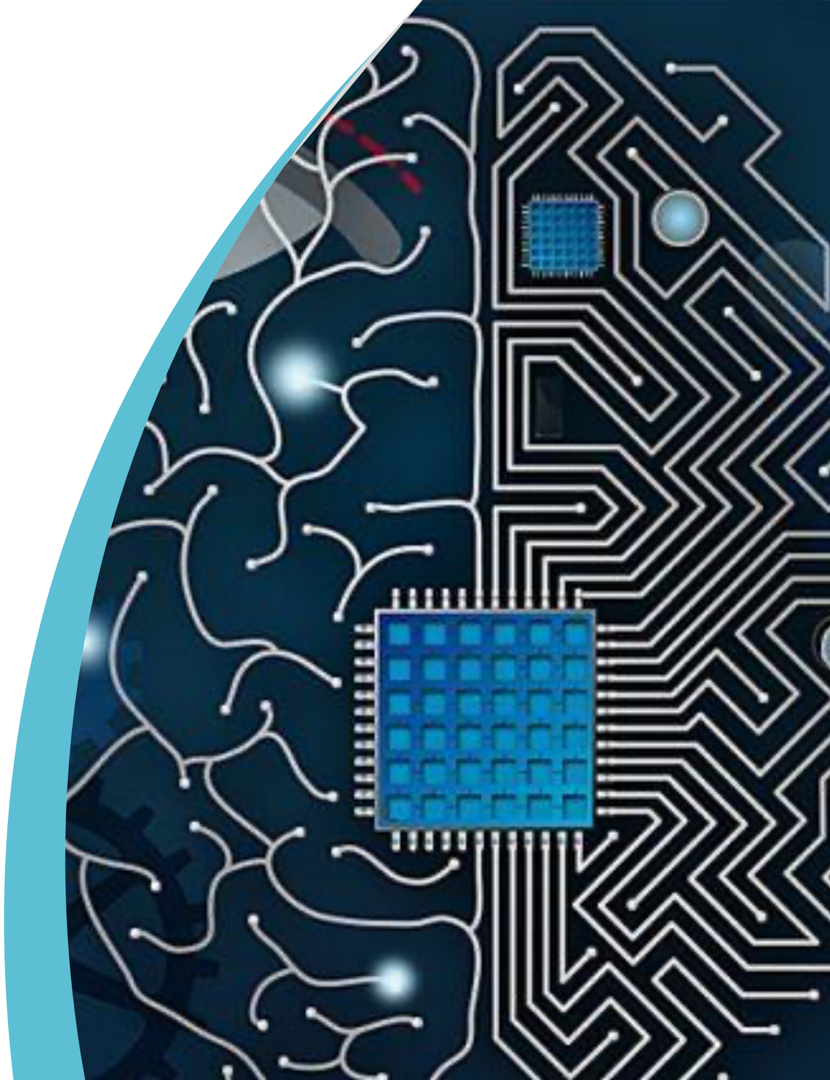
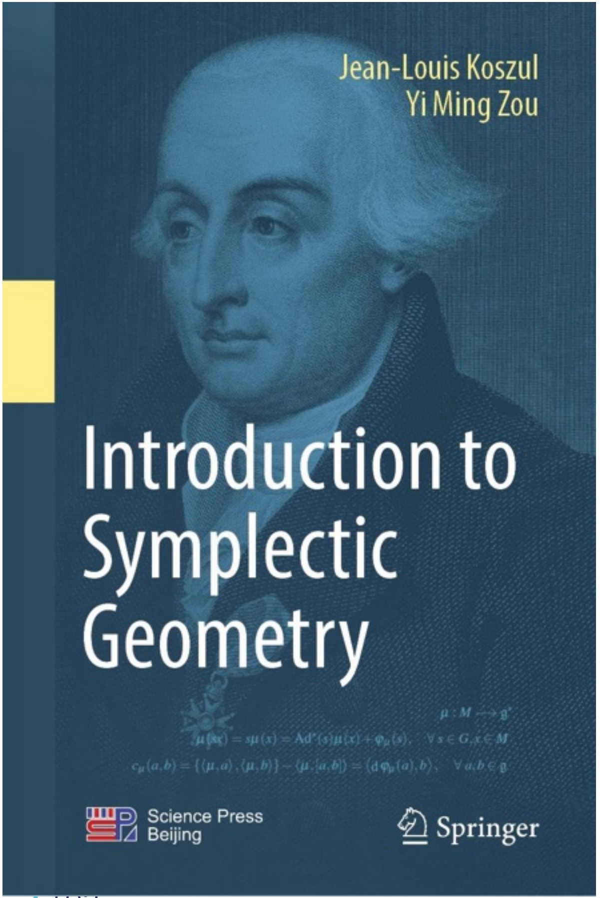
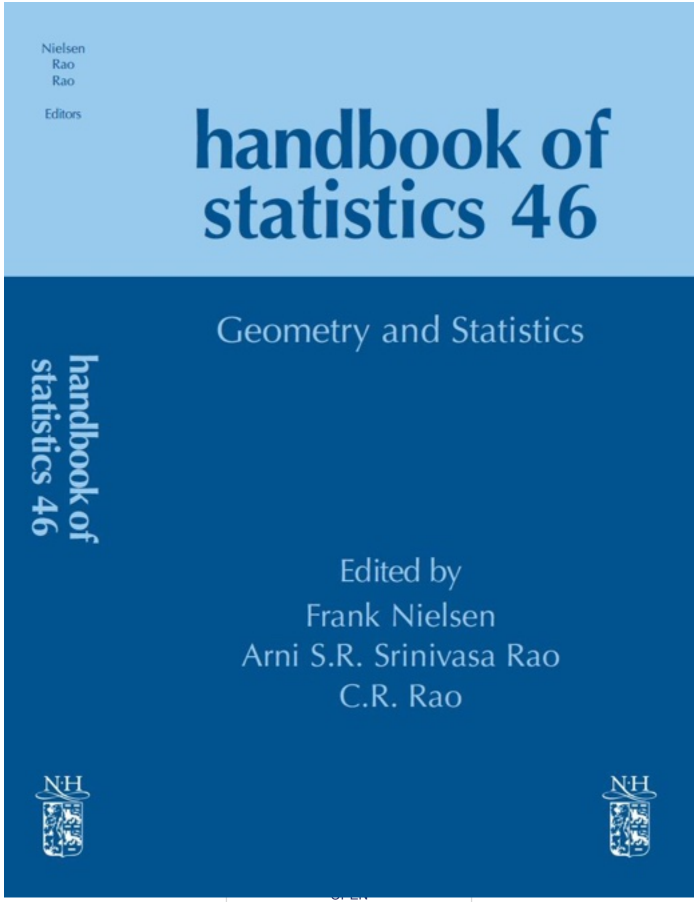
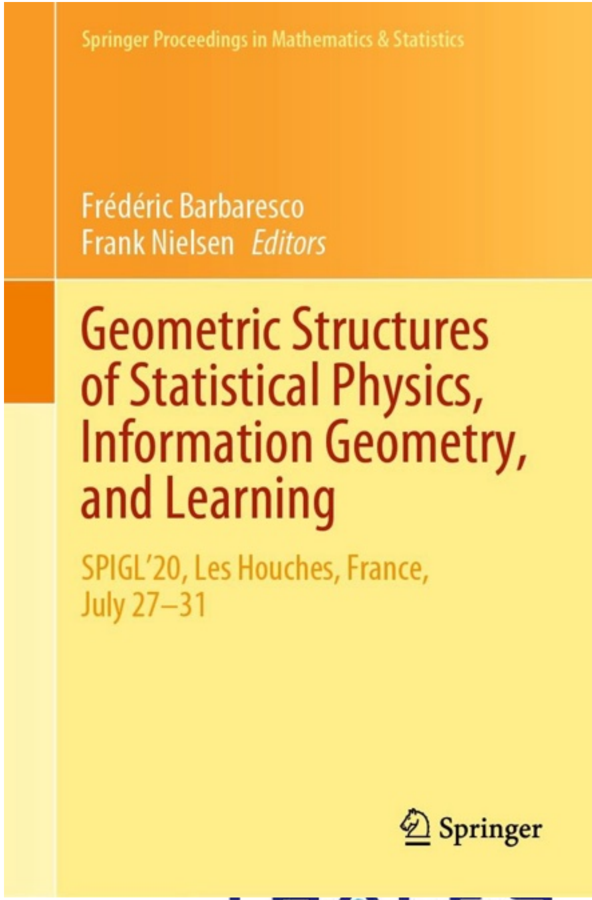
References



To Go Further on Information Geometry and Symplectic Geometry







Transverse Structure of Foliations

http://www.numdam.org/item/AST_1984__116_/

LISTE DES CONFÉRENCES ET COMMUNICATIONS

N.A'CAMPO - Une mesure $SL(n,\mathbb{R})$-invariante sur l'espace des couples de drapeaux sur $\mathbb{R}^n$.

R.BARRE - Théorie des Q-variétés et structures de Hodge mixtes.

R.BLUMENTHAL - Transverse curvature of foliated manifolds.

Y.CARRIERE - Flots riemanniens.

P.CARTIER - Variétés quotients : regards rétrospectifs sur leur développement.

L.CONLON - Holonomy pseudogroup and GV (work in progress).

A.CONNES - K-théorie, théorie de l'indice et feuilletages.

D.B.A.EPSTEIN - Foliations of 3-manifolds with transverse hyperbolic structure.

A.HAEFLIGER - Feuilletages avec feuilles minimales et courants invariants.

G.HECTOR - titre non parvenu

F.W.KAMBER - Duality theorems for harmonic foliations.

C.LAMOUREUX - Etude géométrique directe des feuilletages transverses sur les fibrés en cercles et en droites.

D.LEHMANN - Feuilletages avec "suffisamment" de formes basiques.

J.LEHMANN-LEJEUNE - Dérivations d'une algèbre de Lie sur le fibré transverse à un feuilletage.

K.MILLETT - Can R^3 be foliated by circles ?

P.MOLINO - Espace des feuilles des feuilletages riemanniens.

J.PRADINES - Equivalence transverse et groupoïdes différentiables.

B.REINHART - Comprendre la structure transverse, c'est comprendre les groupes de polynômes tronqués.

C.ROGER - Cohomologie (p,q) des feuilletages et applications.

R.SACKSTEDER - Foliations and separation of variables.

G.W.SCHWARZ - Base-like cohomology of foliations.

T.TSUBOI - Cobordismes de feuilletages.

W.T.VAN EST - Rapport sur les schémas de variété.

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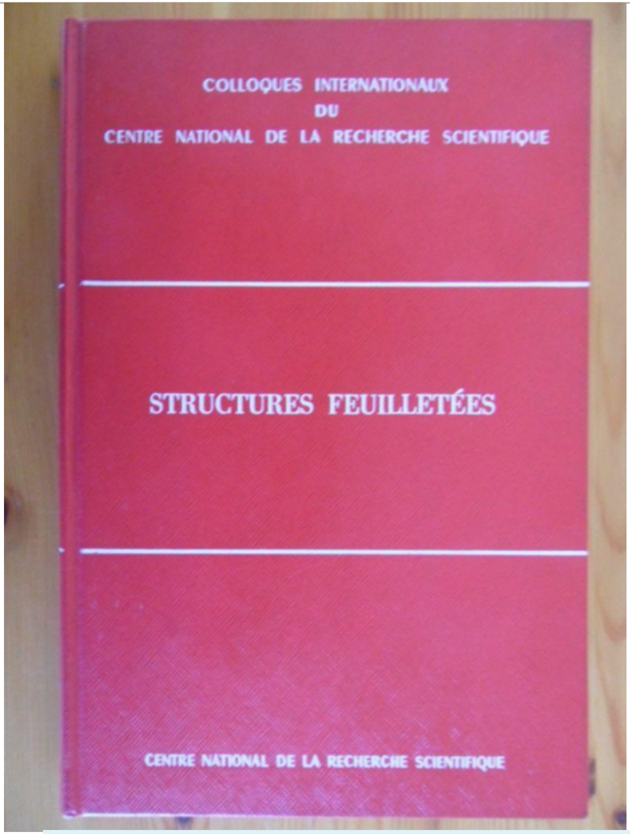
1963, naissance des « Structures feuilletées »

C. EHRESMANN, Structures feuilletées, Proceedings of the Fifth Canadian Mathematical Congress, 109-172., 1963



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STRUCTURES FEUILLETEES

CHARLES EHRESMANN, *Institut Henri Poincaré*

Introduction

Cet article a pour but la définition précise et l'étude des structures feuilletées dans le cadre de la théorie des structures locales telle qu'elle est exposée dans (**3**; **5**; **6**). Les résultats connus dans le cas des variétés feuilletées sont précisés et généralisés au cas d'un feuilletage topologique localement simple. Les notions d'holonomie, de déroulements et de tubes analysées ici permettent d'étudier les questions de stabilité. Seuls des problèmes généraux sont abordés, les applications étant réservées pour une publication ultérieure.

La plupart des idées contenues dans ce travail ont été exposées dans mes cours (en particulier, Paris 1955–56, 1958, 1961) et dans des conférences (par exemple Princeton 1953, Buenos-Aires 1959–60, Montréal 1961). Rappelons que la notion de variété feuilletée a été introduite dans une Note en collaboration avec Reeb (**1**), puis étudiée d'une façon approfondie par Reeb (**13**; **14**) dans différentes publications. Les structures feuilletées d'espèce $\mathfrak{B} \# \mathfrak{F}$ élargie et de seconde espèce ont été définies dans (**3**). Les Γ-structures étudiées par Haefliger (**16**), qui sont étroitement liées aux feuilletages de seconde espèce, ne seront pas considérées ici. Les feuilletages localement simples ont été introduits dans une Note en collaboration avec Shih Weishu (**2**).

I. Définitions de diverses espèces de structures feuilletées

1. *Feuilletages topologiques*

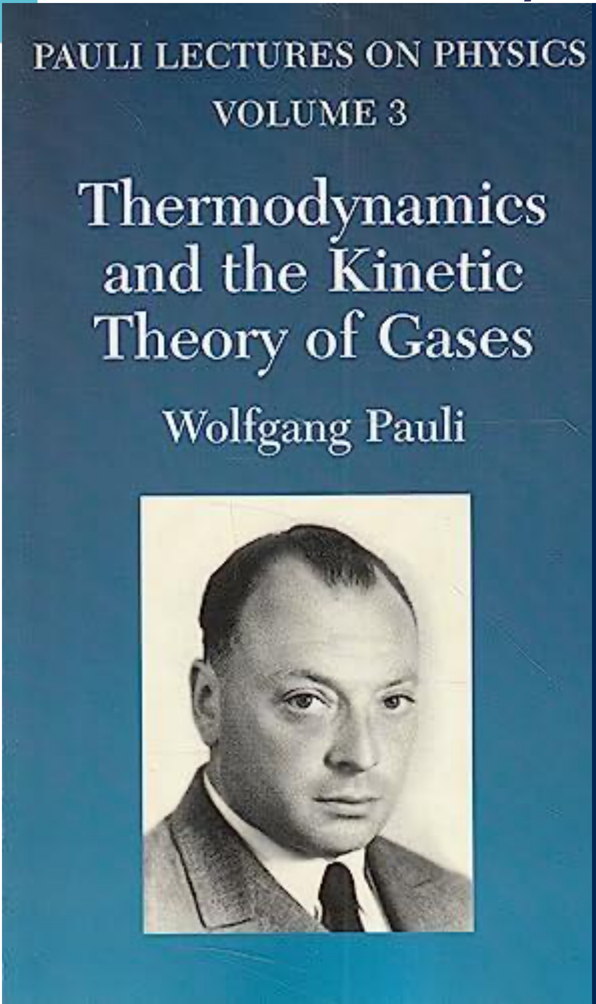
Soit E un ensemble muni de deux topologies T et T'. On dira que (T, T') définit sur E un *feuilletage topologique* ou une structure d'*espace feuilleté topologique* si la condition suivante est vérifiée : Pour tout $x \in E$, il existe un voisinage ouvert U' de x relativement à T' sur lequel T et T' induisent la même topologie.

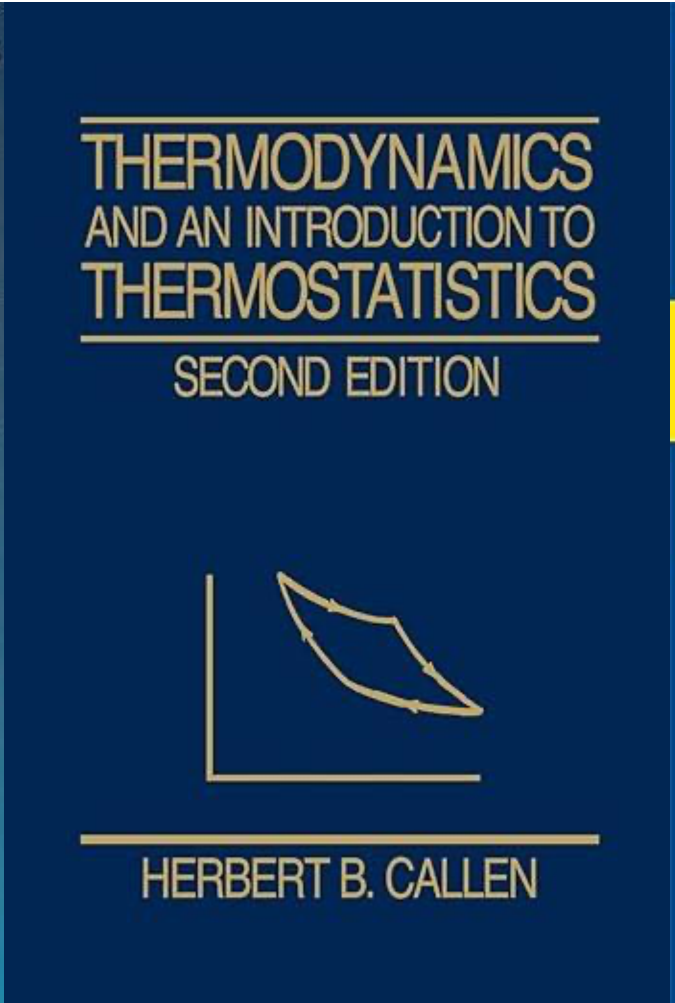
Si (T, T') est un feuilletage topologique sur E, alors T' est une topologie plus fine que T. Nous supposerons désormais que T' est

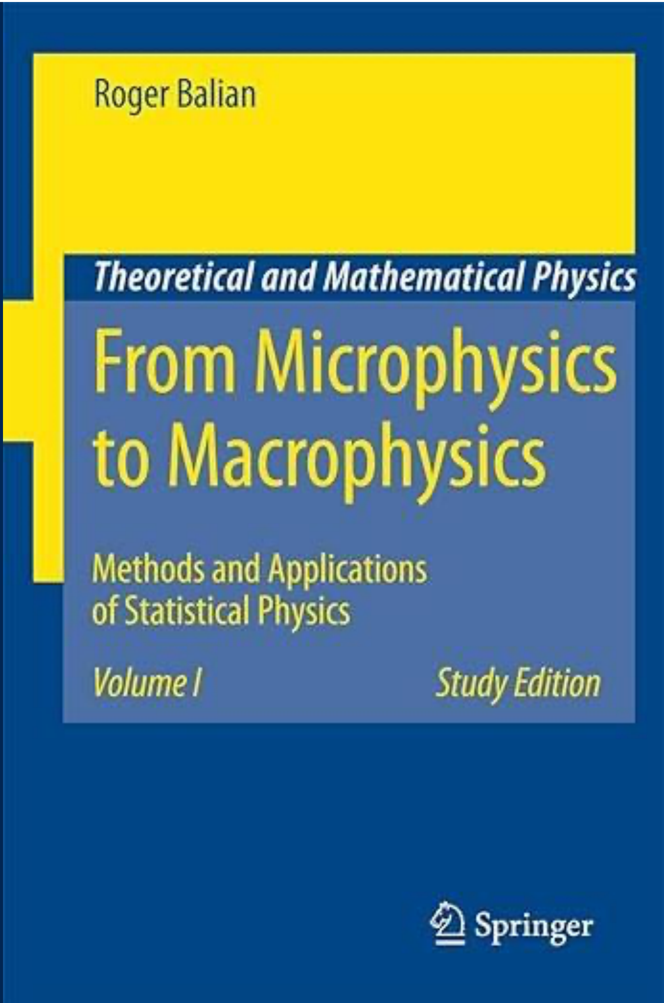
109

Structures feuilletées - Grenoble, 25-30 juillet 1963 - Georges Reeb
Auteur(s) : Georges Reeb - Charles Ehresmann - René Thom - Paulette Libermann
Editeur : Centre National De La Recherche ScientifiqueCollection : Colloques Int. Du Cnrs, 1964

Books on Thermodynamics







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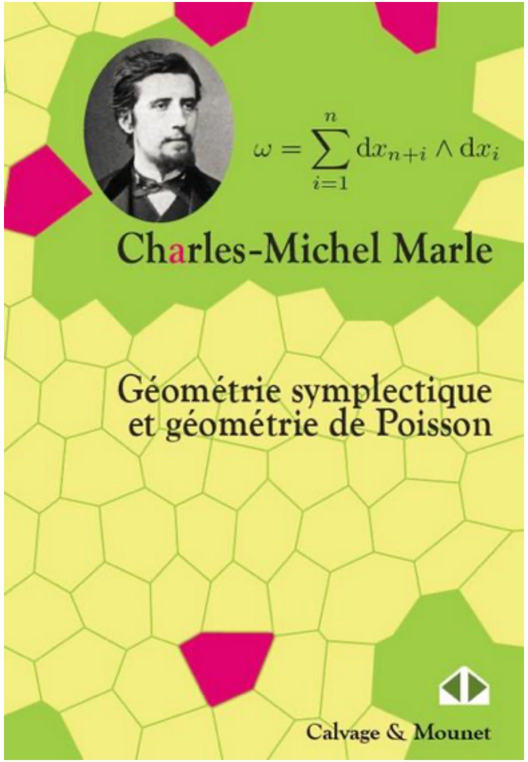

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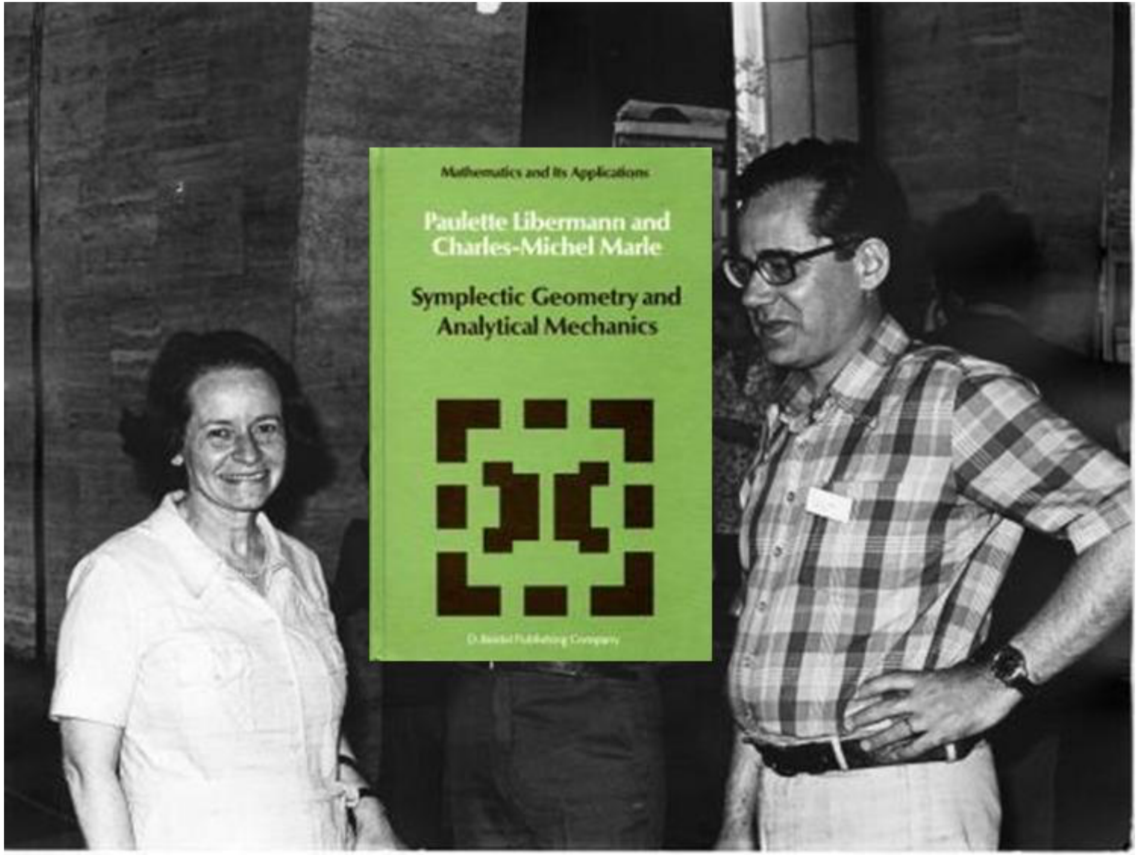

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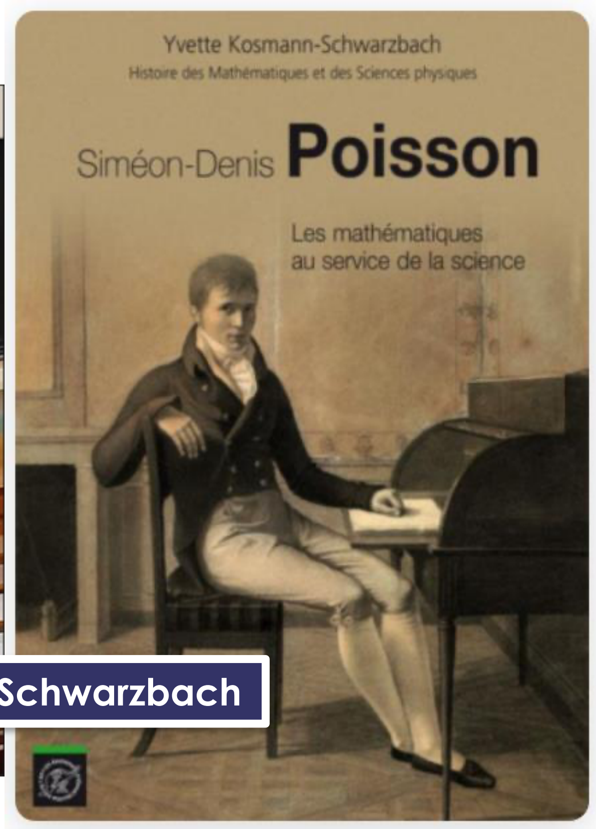
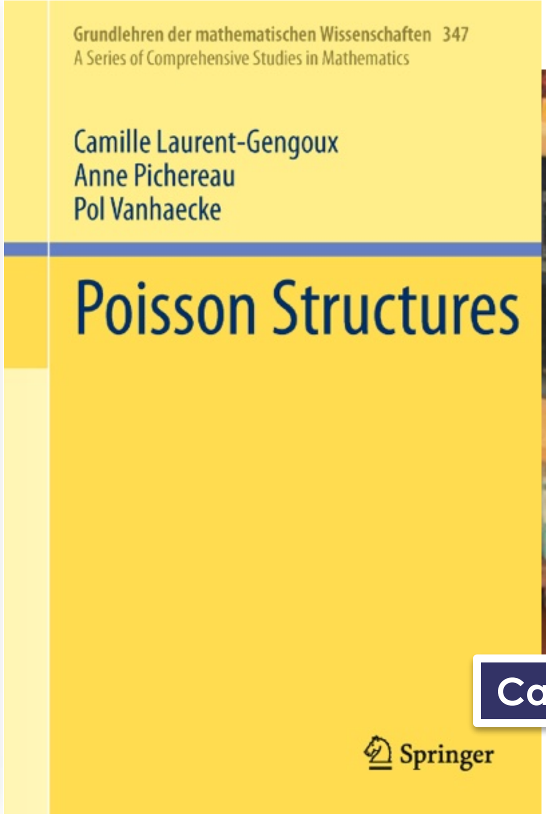
THALES

Poisson Geometry

Yvette Kosmann-Schwarzbach





Camille Laurent-Gengoux

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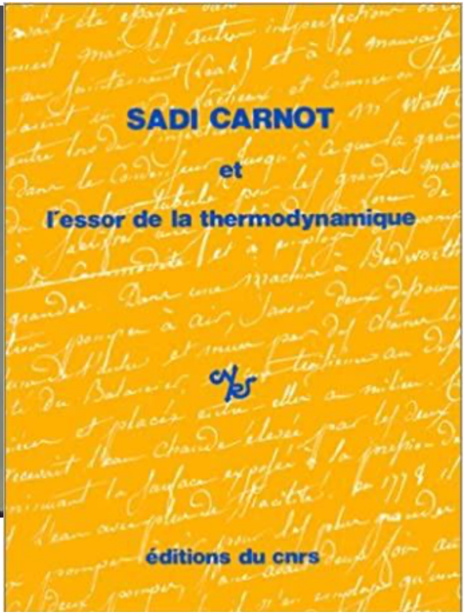
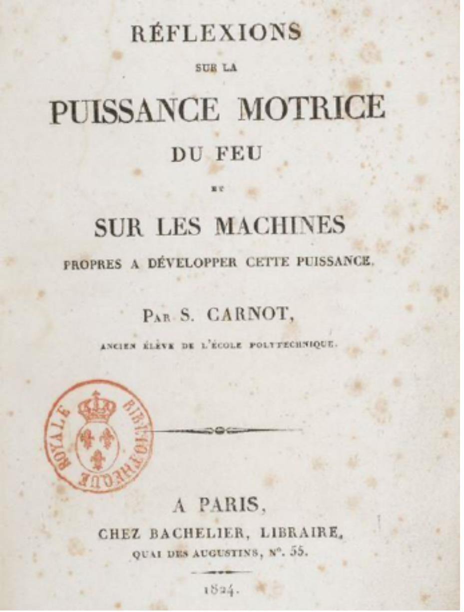
150th birthday of Thermodynamics – Ecole Polytechnique



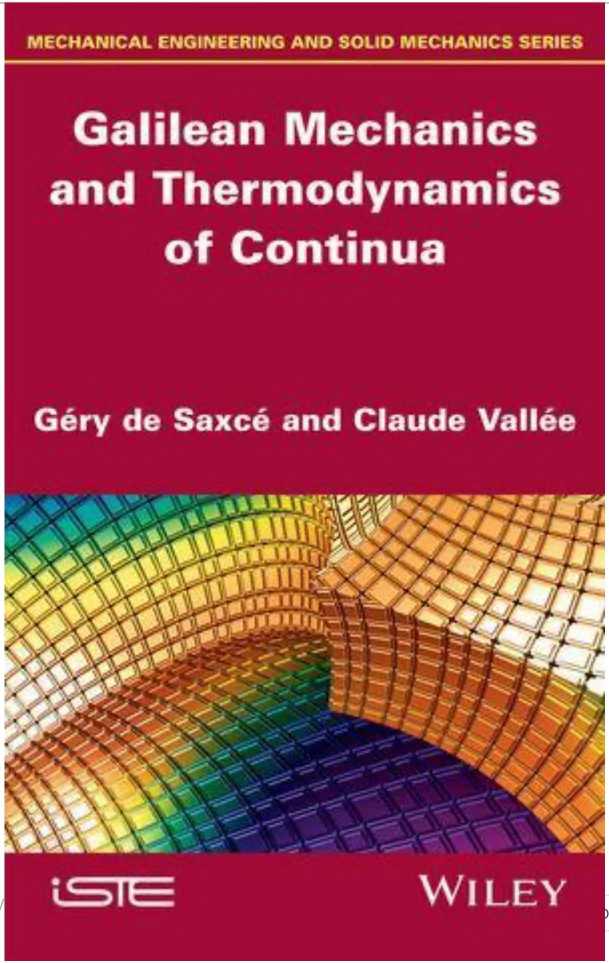

RÉFLEXIONS
SUR LA
PUISSANCE MOTRICE
DU FEU
ET
SUR LES MACHINES
PROPRES A DÉVELOPPER CETTE PUISSANCE.
PAR S. CARNOT,
ANCIEN ÉLÈVE DE L'ÉCOLE POLYTECHNIQUE.
A PARIS,
CHEZ BACHELIER, LIBRAIRE,
QUAI DES AUGUSTINS, N°. 55.
1824.

1st & 2nd Thermodynamics Principles

1824-2024: 200 years publication of Carnot's Book

THALES

Gery de Saxcé & Claude Vallée

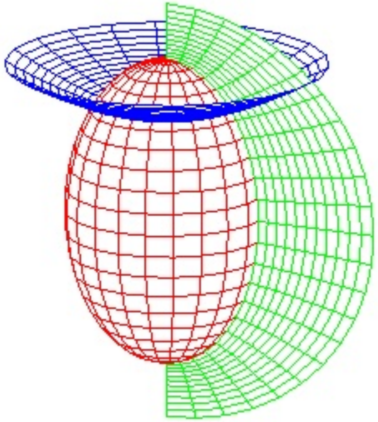


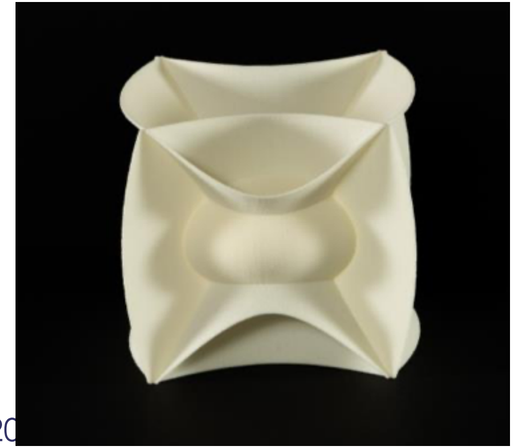
This title proposes a unified approach to continuum mechanics which is consistent with Galilean relativity. Based on the notion of affine tensors, a simple generalization of the classical tensors, this approach allows gathering the usual mechanical entities — mass, energy, force, moment, stresses, linear and angular momentum — in a single tensor.

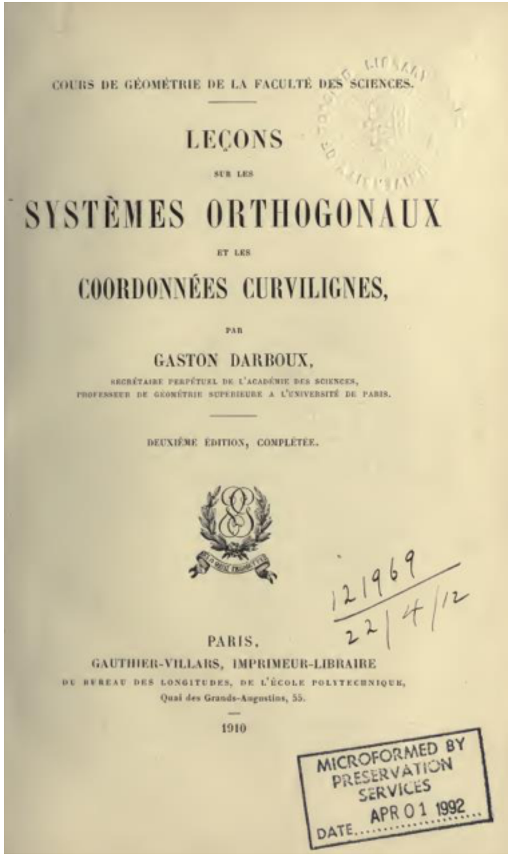
Starting with the basic subjects, and continuing through to the most advanced topics, the authors' presentation is progressive, inductive and bottom-up. They begin with the concept of an affine tensor, a natural extension of the classical tensors. The simplest types of affine tensors are the points of an affine space and the affine functions on this space, but there are more complex ones which are relevant for mechanics − torsors and momenta. The essential point is to derive the balance equations of a continuum from a unique principle which claims that these tensors are affine-divergence free.

https://www.wiley.com/en-us/Galilean+Mechanics+and+Thermodynamics+of+Continua-p-9781848216426

Triple Orthogonal Systems : Darboux and Lamé







COURS DE GÉOMÉTRIE DE LA FACULTÉ DES SCIENCES.

LEÇONS

SUR LES

SYSTÈMES ORTHOGONAUX

ET LES

COORDONNÉES CURVILIGNES,

PAR

GASTON DARBOUX,

SECRÉTAIRE PERPÉTUEL DE L'ACADÉMIE DES SCIENCES,
PROFESSEUR DE GÉOMÉTRIE SUPÉRIEURE A L'UNIVERSITÉ DE PARIS.

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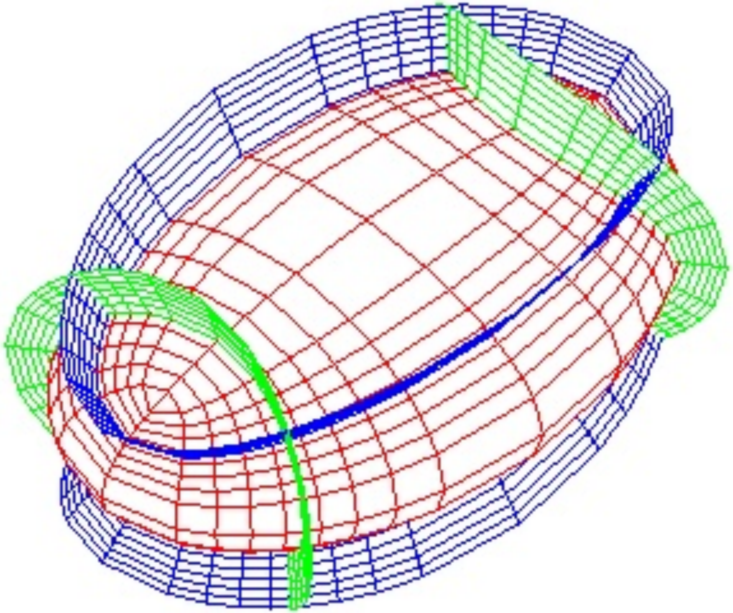
1910



Representation Theory & (Co-adjoint) Orbits Method: A. Kirillov

Grundlehren der mathematischen Wissenschaften 220
A Series of Comprehensive Studies in Mathematics

A. A. Kirillov

Elements of the Theory of Representations

MERITS VERSUS DEMERITS

1. Universality: the method works for Lie groups of any type over any field.	1. The recipes are not accurately and precisely formulated.
2. The rules are visual, easy to memorize and illustrate by a picture.	2. Sometimes they are wrong and need corrections or modifications.
3. The method explains some facts which otherwise look mysterious.	3. It could be difficult to transform this explanation into a rigorous proof.
4. It provides a great number of symplectic manifolds and Poisson commuting families of functions.	4. Most completely integrable dynamical systems were discovered earlier by other methods.
5. The method introduces two new fundamental notions: coadjoint orbits and moment maps.	5. The description of coadjoint orbits and their structure is sometimes not an easy problem.

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§ 15. The Method of Orbits

At the basis of the method of orbits lies the following "experimental fact": the theory of infinite-dimensional representations of every Lie group is closely connected with a certain special finite-dimensional representation of this group. This representation acts in the dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of the group under study. We will call it a *co-adjoint* or briefly a *K-representation*.[1]

Orbits of a Lie group in the space of a K-representation are symplectic manifolds. They can be interpreted as phase spaces of a Hamiltonian mechanical system for which the given Lie group is the group of symmetries. In 15.2, we shall give a classification of all homogeneous symplectic manifolds with a given group of symmetries.

It turns out that unitary irreducible representations of the group G are connected with orbits of this group in the K-representation. The construction of the representation in an orbit is given in 15.3.

This is a generalization of the procedure of quantization that is used in quantum mechanics. This point of view is explained in more detail in 15.4.

The author sees the significance of the method of orbits not only in the specific theorems obtained by this method, but also in the great collection of simple and intuitive heuristic rules that give the solution of the basic questions of the theory of representations. With the passage of time, these rules will be elevated to the level of strict theorems, but already now their value is indisputable.

We shall show in 15.5 how the operations of restriction to a subgroup and induction from this subgroup can be described with the aid of the natural projection $p: \mathfrak{g}^* \to \mathfrak{h}^*$, where $\mathfrak{h}$ is the Lie algebra of the subgroup H.

As we shall see in 15.6, generalized characters of irreducible unitary representations admit a simple expression in the form of an integral over the corresponding orbit. In many cases, this allows us to write an explicit expression for the Plancherel measure.

Finally, in 15.7 we show that infinitesimal characters of irreducible unitary representations of a group G can be computed as values of G-invariant polynomials on the corresponding orbits.

15.1. The Co-Adjoint Representation of a Lie Group

Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{g}^*$ the dual space to $\mathfrak{g}$. The group G acts in $\mathfrak{g}$ with the aid of the adjoint representation Ad (see 6.3) and in $\mathfrak{g}^*$ with the aid of the co-adjoint representation, or, briefly, the K-representation. If the Lie algebra $\mathfrak{g}$ is realized in the form of the algebra of left-invariant vector-fields on G, then it is natural to realize $\mathfrak{g}^*$ in the form of the space of left-invariant differential forms of the first order on G. The K-representation of the group G acts in the space of 1-forms by right translations.

A. KIRILLOV

ÉLÉMENTS DE LA THÉORIE DES REPRÉSENTATIONS

ÉDITIONS MIR • MOSCOU

THALES

Kirillov by Dixmier

Brèves remarques sur l'œuvre de A. A. Kirillov

Jacques Dixmier

La thèse de Kirillov, parue en 1962, a suscité immédiatement beaucoup d'intérêt.

Soit G un groupe de Lie nilpotent simplement connexe. Soient $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{g}^*$ l'espace vectoriel dual de $\mathfrak{g}$, dans lequel G opère par la représentation coadjointe. Soit $\mathfrak{g}^*/G$ l'ensemble des orbites coadjointes. Soit $\widehat{G}$ l'ensemble des représentations unitaires irréductibles de G (la plupart sont de dimension infinie). Alors Kirillov construit une bijection canonique $\mathfrak{g}^*/G \to \widehat{G}$; ainsi, les représentations unitaires irréductibles de G sont paramétrées par les orbites coadjointes. En outre, quantité de notions naturelles concernant les représentations s'interprètent géométriquement en termes d'orbites coadjointes: restriction à un sous-groupe, induction unitaire, produit tensoriel, mesure de Plancherel, topologie de $\widehat{G}$. Détaillons seulement un résultat, d'une élégance extraordinaire: soient $\pi \in \widehat{G}$ et Ω l'orbite associée ; soit μ la mesure G-invariante canonique sur Ω (c'est une distribution tempérée sur $\mathfrak{g}^*$) ; soit μ' sa transformée de Fourier, distribution tempérée sur $\mathfrak{g}$; transportant μ' sur G par l'application exponentielle, on obtient le caractère-distribution de π !

Kirillov s'est vite convaincu, et il a convaincu la communauté mathématique, que cette "méthode des orbites" devait être applicable à des groupes bien plus généraux que les groupes nilpotents.

Il n'a pas hésité à aborder le cas des groupes de Lie connexes quelconques. Evidemment, des difficultés considérables ont surgi immédiatement. Néanmoins, Kirillov a indiqué une voie d'accès, qui ensuite a été largement utilisée.

Depuis plus de trente ans, Kirillov applique la méthode des orbites aux groupes de Lie–Cartan de dimension infinie. Par exemple, soit M une variété C^∞ compacte. Soit G le groupe des difféomorphismes de M, ou l'un des sous-groupes obtenus en prenant les éléments de G qui conservent une forme volume, ou une structure symplectique, ou une structure de contact. On considère l'algèbre de Lie $\mathfrak{g}$ et son dual $\mathfrak{g}^*$, tous deux définis en tenant compte de la topologie. Aux orbites coadjointes sont associées des représentations unitaires irréductibles de G, du moins dans certains cas, par exemple si l'orbite est de dimension finie ou de codimension finie. Le groupe fondamental de ces orbites peut être non commutatif, par exemple être un groupe symétrique. Kirillov envisage parfois des groupes encore plus généraux (par exemple, le groupe $\mathrm{Diff}_H(E)$ où E est un fibré principal de groupe structural H), et parfois au contraire étudie de manière très approfondie des cas particuliers importants; par exemple, soient S le cercle, $\mathrm{Diff}_+(S)$ le groupe des difféomorphismes de S conservant l'orientation; alors les orbites coadjointes pour ce groupe sont liées aux fonctions holomorphes univalentes dans le disque unité, et à certaines équations différentielles linéaires; Kirillov définit sur ces orbites des structures complexes kähleriennes invariantes, d'où des représentations de l'algèbre de Virasoro et du groupe de Virasoro–Bott.

Les démonstrations précédentes ont amené Kirillov à s'intéresser, dans plusieurs articles, à des questions de pure géométrie différentielle. Soient M une variété C^∞, E un fibré en droites sur M, $\Gamma(E)$ l'ensemble des sections C^∞ de E. Un crochet de Lie $[\ ,\ ]$ sur $\Gamma(E)$ est dit local si $[s_1, s_2]$ est continue en (s_1, s_2) et si $\mathrm{supp}[s_1, s_2] \subset \mathrm{supp}\, s_1 \bigcap \mathrm{supp}\, s_2$. Kirillov classifie ces crochets. Il détermine aussi, dans des cas généraux, quels sont les opérateurs multidifférentiels invariants par difféomorphismes (tels que la dérivation extérieure, ou le crochet ordinaire de deux champs de vecteurs).

Plus récemment, Kirillov a utilisé avec succès la méthode des orbites pour des groupes très différents, les $G_n(k)$ ($n = 1, 2, \ldots$; k corps commutatif). (On note $G_n(k)$ le groupe des matrices $n \times n$ unipotentes triangulaires supérieures à éléments dans k.) La théorie est particulièrement poussée lorsque k est un corps fini. Elle amène à introduire une suite remarquable des polynômes en une variable.

Voici un autre thème longuement étudié par Kirillov en collaboration avec Gelfand. Soient G un groupe algébrique complexe, $\mathfrak{g}$ son algèbre de Lie, U l'algèbre enveloppante de $\mathfrak{g}$, D le corps enveloppant de U. Soit $D_{n,k}$ le corps engendré par des indéterminées $x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_k$, où tous les crochets sont nuls sauf $[x_i, y_i] = 1$. Alors Gelfand et Kirillov conjecturent en 1966 que D est isomorphe à un corps $D_{n,k}$. Dans les années suivantes, ils démontrent des cas particuliers de plus en plus nombreux de cette conjecture. Bien après, la conjecture sera reconnue comme fausse, mais elle aura suscité un grand nombre de travaux intéressants. Surtout, dès l'article de 1966, Gelfand et Kirillov introduisent diverses notions de dimensions non commutatives; l'une d'entre elles, promise à un grand avenir, sera appelée dimension de Gelfand–Kirillov.

Il n'est pas surprenant que Kirillov ait étudié, pour elles-mêmes, les algèbres de Lie de dimension infinie. Dans un série d'articles, il a mis en évidence des cas où la "croissance" de ces algèbres est strictement intermédiaire entre la croissance polynomiale et la croissance exponentielle et il a construit des identités remarquables vérifiées dans l'algèbre de Lie des champs de vecteurs sur $\mathbb{R}$, ou des champs hamiltoniens sur $\mathbb{R}^2$.

La place me manque pour analyser les nombreux articles isolés abordant des thèmes sur lesquels Kirillov n'est pas revenu par la suite. Mentionnons tout de même 3 de ces articles: 1) il a complété sur un point très important l'étude (par Gelfand–Graev) des représentations unitaires irréductibles de $\mathrm{SL}_2(k)$ (k corps localement compact non connexe). 2) Soient $\mathcal{H}$ un espace hilbertien, G le groupe des opérateurs unitaires u dans $\mathcal{H}$ tels que $1 - u$ soit compact. Alors Kirillov a déterminé les représentations unitaires irréductibles de G. 3) Kirillov a résolu un problème essentiel de géométrie intégrale posé par Gelfand. Ces études n'utilisent pas la méthode des orbites.

Kirillov a écrit plusieurs livres. Deux d'entre eux ont été traduits en français et en anglais: 1) *Eléments de la théorie des représentations*, 2) *Théorèmes et problèmes d'analyse fonctionnelle*. Il a d'autre part rédigé beaucoup d'articles d'exposition: chacun d'eux est en réalité un court livre. La clarté des exposés fait que ces écrits ont été largement lus, et notamment, avec profit, par des chercheurs débutants.

C'est l'occasion de signaler que Kirillov a eu de nombreux élèves de thèse. Les organisateurs de ce colloque ont eu la bonne idée de faire imprimer sur un T-shirt (offert à Kirillov) la liste de ces élèves: 57 noms, un record !

Kirillov a participé, en 1967, à la fondation d'un journal célèbre: *Functional Analysis and its Applications*. Il a, pendant 4 ans, été vice-président de la Société Mathématique de Moscou. Il est membre de comités éditoriaux de plusieurs journaux. Il a suscité la traduction en russe de nombreux livres et articles édités à l'ouest.

Dans sa conversation, Kirillov mentionne le soutien constant qu'il a reçu de son épouse (qui travaille en informatique). Il est fier de l'œuvre mathématique très connue de son fils.

Ce colloque honore donc un grand mathématicien: grand mathématicien par les voies qu'il a ouvertes, grand mathématicien par l'influence qu'il exerce depuis des années.

Jacques Dixmier
11 bis rue du Val de Grâce
F-75005 Paris, France

SOURIAU'S MOMENT MAP = GEOMETRIZATION OF NOETHER THEOREM

3.5 Noether type theorems

Consider now the "levels" of the moment map[13] $\mu : W \to \mathfrak{g}^\star$ of a hamiltonian G-action on the symplectic manifold (W,ω). The most classical form of E. Noether's theorem seems to be stated nowadays as:

[13] I shall call *level* ξ the inverse image $\mu^{-1}(\xi)$ even if ξ is not a number.

Theorem 3.5.1 *Let H be a function on W which is invariant by the G-action. Then μ is constant on the trajectories of the hamiltonian vector field X_H.*

Proof. Indeed, if $\gamma(t)$ is a trajectory of X_H, one can write, for any $X \in \mathfrak{g}$:

$$\begin{aligned}
\frac{d}{dt} < \mu \circ \gamma(t), X > &= < T_{\gamma(t)}\mu(X_H(\gamma(t))), X > \\
&= < X_H(\gamma(t)), {}^tT_{\gamma(t)}\mu(\underline{X}) > \\
&= < X_H(\gamma(t)), (i_X\omega)_{\gamma(t)} > \\
&= \omega(\underline{X}, X_H)_{\gamma(t)} \\
&= -dH_{\gamma(t)}(\underline{X})
\end{aligned}$$

But H is invariant and $\underline{X}$ a fundamental vector field of the action, and thus:

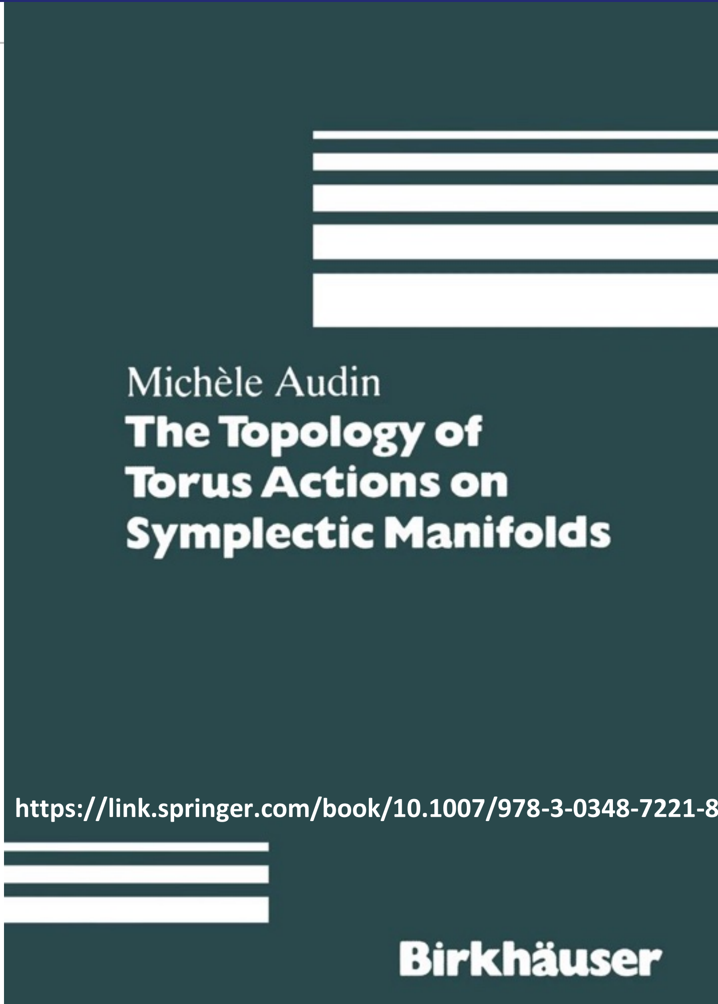
$$H(\exp(sX)\cdot\gamma(t)) = H(\gamma(t))$$

which, when differentiated at $s = 0$, is:

$$dH(\underline{X}(\gamma(t))) = 0.$$

□

The field X_H is thus tangent to the levels $\mu^{-1}(\xi)$.



https://link.springer.com/book/10.1007/978-3-0348-7221-8

SOURIAU'S MOMENT MAP By VLADIMIR ARNOLD

A Poisson action of a group G on a symplectic manifold M defines a mapping of M into the dual space of the Lie algebra of the group

$$P: M \to \mathfrak{g}^*.$$

That is, we fix a point x in M and consider the function on the Lie algebra which associates to an element a of the Lie algebra the value of the Hamiltonian H_a at the fixed point x:

$$p_x(a) = H_a(x).$$

This p_x is a linear function on the Lie algebra and is the element of the dual space to the algebra associated to x:

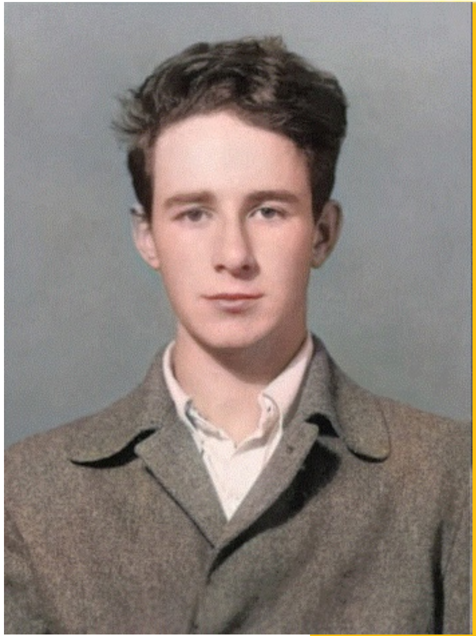
$$P(x) = p_x.$$

Following Souriau (*Structure des systèmes dynamiques*, Dunod, 1970), we will call the mapping P the *momentum*. Note that the value of the momentum is always a vector in the space $\mathfrak{g}^*$.

Theorem. *Under the momentum mapping P, a Poisson action of a connected Lie group G is taken to the co-adjoint action of G on the dual space $\mathfrak{g}^*$ of its Lie algebra* (*cf.* Appendix 2), *i.e., the following diagram commutes:*

$$\begin{array}{ccc} M & \xrightarrow{g} & M \\ \downarrow P & & \downarrow P \\ \mathfrak{g}^* & \xrightarrow{\mathrm{Ad}^*_{g^{-1}}} & \mathfrak{g}^* \end{array}$$

Corollary. *Suppose that a hamiltonian function $H: M \to \mathbb{R}$ is invariant under the Poisson action of a group G on M. Then the momentum is a first integral of the system with hamiltonian function H.*



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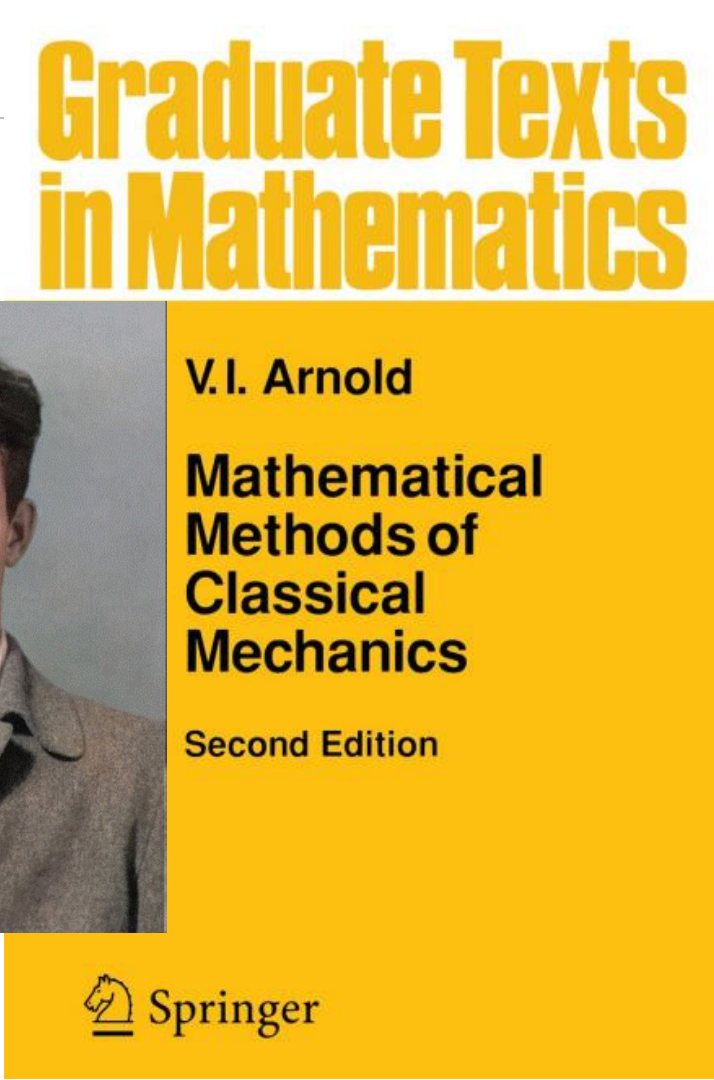

Moment Map Geometry & Libermann Foliation: Condevaux, Dazord, Molino

Séminaire Sud-Rhodanien

CHAPITRE V

GEOMETRIE DU MOMENT

M. Condevaux, P. Dazord, P. Molino

Introduction

Le but de cet exposé est double : d'une part définir le cadre naturel dans lequel se généralise l'étude du moment d'une action hamiltonienne ; d'autre part donner une démonstration nouvelle, plus directement inspirée par l'intuition géométrique que les démonstrations connues, des théorèmes de convexité du moment d'Atiyah-Guillemin-Sternberg-Kirwan [A] $[G\text{-}S]_1$ $[G\text{-}S]_2$ [K].

L'idée de base de la première partie est la suivante : la dualité symplectique permet à la fois de faire correspondre à un feuilletage symplectiquement complet au sens de P. Libermann $[Lib]_1$ un feuilletage orthogonal, et d'associer à l'action hamiltonienne d'un groupe de Lie un moment au sens de J.M. Souriau [So]. Une théorie générale du moment, englobant ces deux situations comme cas particuliers, devra rendre compte de la correspondance établie par dualité symplectique entre certains types de feuilletages singuliers de P. Stefan [St] et H.. Süssmann [Su] et certaines Γ-structures de A. Haefliger [H]. C'est ce qui nous amène à introduire la notion de Γ-structure Poissonienne ; une telle structure est le moment généralisé d'un feuilletage de P. Stefan "symplectiquement complet". On montre que les deux cas particuliers évoqués rentrent bien dans ce cadre ; en fait, plus généralement, toute action symplectique d'un groupe de Lie admet en ce sens un moment généralisé. Par un procédé élémentaire de "développement de Darboux", on définit pour une telle action symplectique un moment réduit, à valeurs dans un cylindre muni d'une structure de Poisson affine ; il semble que l'étude de ces différentes notions puisse conduire à quelques problèmes intéressants.

En ce qui concerne les théorèmes de convexité du moment, on traite d'abord le cas des tores à partir de l'idée suivante : la description élémentaire des modèles locaux conduit immédiatement à un résultat de convexité locale ; si T^k agit de manière hamiltonienne sur (M,σ), on en déduit facilement que l'espace $\hat{M}$ des composantes connexes des préimages des points par le moment $J : M \to \mathbb{R}^k$ a une structure naturelle de variété affine plate à bords et coins affines localement convexe au voisinage du bord. L'appli-

CHAPITRE II

Γ-STRUCTURES POISSONNIENNES ET FEUILLETAGES DE LIBERMANN

P. DAZORD & P. MOLINO

INTRODUCTION -

Les feuilletages de Libermann sont la généralisation, dans le cadre des feuilletages de Stefan, des feuilletages étudiés dans le cas régulier par P. Libermann [8] sous le nom de feuilletages symplectiquement complets et connus des mécaniciens sous le nom de "système complet d'intégrales premières". Ceci justifie le nom donné à cette catégorie de feuilletages

L'étude de ces feuilletages a été clarifiée par l'introduction de la notion de Γ-structure poissonnienne qui est l'extension maximale de la notion de moment au sens de J.M. Souriau [10]. (cf. Chapitre V "Géométrie du Moment"). La notion de feuilletage de Libermann apparaît alors comme duale de celle de Γ-structure poissonnienne. C'est le point de vue adopté ici.

Dans un dernier paragraphe on utilise la théorie des feuilletages de Libermann pour donner une preuve du théorème de structure locale de A. Weinstein [11] pour les variété de Poisson.

DUALITE SYMPLECTIQUE, FEUILLETAGES ET GEOMETRIE DU MOMENT

P. MOLINO

Abstract

One gives the links between th notions of singular foliations, Γ–structures and momentum mapping in the context of symplectic geometry.

Introduction

Dans le premier paragraphe de ce papier, après quelques rappels élémentaires de géométrie symplectique, on introduit la notion de *moment généralisé* mise au point en collaboration avec M. Condevaux et P. Dazord [3] dans le cadre du *Seminaire Sud-Rhodanien de Geometrie*. Essentiellement, c'est la traduction en termes de dualité symplectique entre Γ–structures de Haefliger (ici, les *Γ–structures Poissoniennes*) et feuilletages singuliers de Stefan (ici les *feuilletages de Stefan–Libermann*), de la notion de moment d'une action hamiltonienne due à J. M. Souriau [16].

Le second paragraphe est une contribution à la méthode de réduction symplectique [13]: (M, ω, H) étant un système hamiltonien, on considère une sous-variété Σ de M, qui est une sous-variété de niveau d'une famille d'integrales premières. On suppose que ω induit sur Σ une forme ω_Σ de rang constant; mais au lieu de supposer le feuilletage caracteristique $\mathcal{C}_\Sigma$ simple, on suppose seulement qu'il admet une metrique *bundle-like* [15] complète. On peut alors introduire le *moment généralisé transverse* du feuilletage singulier défini par les adhérences des feuilles, et poursuivre la réduction en munissant l'espace $\Sigma/\bar{\mathcal{C}}_\Sigma$ des adherences de feuilles d'une structure de Poisson.

Un cas particulier interessant est celui où la varieté Σ est simplement connexe; on peut alors définir un vrai *moment transverse structural* $J_\Sigma : \Sigma \to \mathbf{R}^p$, où p est la dimension de l'algèbre de Lie structurale [14]. On peut d'ailleurs prouver, si Σ est compacte, un théorème de convexité du moment structural du type du théorème de convexsité du moment pour les actions hamiltoniennes de tores [2] [7].

Dans toute la suite, les structures considérées sent de classe C^∞.

Large Dimensions: Mathematics and Applications.(Unfinished and Unedited)

Misha Gromov

December 14, 2018

This is a guide to my lectures in the Fall 2018 at CIMS.

SYMMETRY AND ENTROPY: ARCHIMEDES → BOLTZMANN → FISHER.

By the above (iv), the quotient space $\mathbb{C}^{N+1}/\mathbb{T}^{N+1}$ can be identified with the positive cone $\mathbb{R}_+^{N+1}$, and the complex Archimedean map $A : \mathbb{C}^{N+1} : \mathbb{C}^{N+1} \to \mathbb{R}_+^{N+1}$ factors through the real one,

$$\mathbb{R}A : \mathbb{R}_+^{N+1} \to \mathbb{R}_+^{N+1} \text{ for } (x_0, x_1, ..., x_N) \mapsto (x_1^2, x_2^2, ..., x_N^2).$$

Since the quotient metric in $\mathbb{C}^{N+1}/\mathbb{T}^{N+1}$ is equal to the standard Euclidean metric in $\mathbb{R}_+^{N+1} = \mathbb{C}^{N+1}/\mathbb{T}^{N+1}$, the Riemannian metric in the receiving positive cone $\mathbb{R}_+^{N+1}$ which is induced from $\mathbb{C}^{N+1}$ equals the transport of the Euclidean (regarded as Riemannian) metric $\sum_i dx_i^2$ from $\mathbb{R}_+^{N+1} = \mathbb{C}^{N+1}/\mathbb{T}^{N+1}$ to $\mathbb{R}_+^{N+1} = \mathbb{R}A(\mathbb{R}_+^{N+1})$ by the map $\mathbb{R}A$, where this metric is

$$\mathbb{R}A_*(g_{Eu}) = \sum_{i=0}^{N} \frac{1}{2x_i} dx_i^2,$$

since the differential of the map $\mathbb{R}A$ is given by the diagonal matrix with the entries $2x_i = \frac{dx_i^2}{dx_i}$.

Remarkably – only exceptional quadratic forms have this property – *there exists a smooth function, say Φ on $\mathbb{R}_+^{N+1}$, the Hessian of which is equal to the (differential) quadratic form $\mathbb{R}A_*(g_{Eu})$,*

$$\frac{\partial_{ij}\Phi}{dx_i dx_j} dx_i dx_j = \mathbb{R}A_*(g_{Eu}).$$

In fact, the function $\frac{1}{2}\sum_i x_i \log x_i$can be taken for Φ, since $\frac{d^2}{dx^2}(x \log x) = \frac{1}{x}$. It follows that
the entropy function on the simplex $\blacktriangle^N \subset \mathbb{R}_+^{N+1}$,

$$ent(x_0, x_1, ...x_N) = -(x_0 \log x_0 + x_1 \log x_1 + ... + x_N \log x_N)$$

is concave, where, moreover - this, apparently, goes back to Ronald. Fisher–

$$Hess(ent)) = -2\mathbb{R}A_*(g_{Eu}).$$

In particular, the Riemannin metric defined by $-Hess(ent)$ has constant positive curvature.[55]

It is also clear that the entropy is *the only* function which satisfies this equality and vanishes at the vertices $(1,0,...0)$, $(0,1,...0)$,,,, $(0,0,...1)$ of the simplex $\blacktriangle^N$. Therefore one can take the equality $Hess(ent)) = -2\mathbb{R}A_*(g_{Eu})$ for the definition of entropy[56] entropy by Boltzmann-Shannon *computational*(!) formula $-\sum_i p_i \log p_i$ is as inappropriate as *defining* (rather than *computing*) the area of the disk as the limit of the areas of the inscribed regular n-gons. (The original Boltzmann's definition, translated to the 21st century mathematical language, is described in "*search for a structure-entropy*" on my page https://cims.nyu.edu/ gromov/).

Conversely, by taking the Hessian of $\sum_i x_i \log x_i$, one arrives from the clumsy simplex to the beautifully round sphere with the huge symmetry group.

(This, possibly, may explain the "unreasonable effectiveness" of entropy in mathematical physics and in math generated by physics, and which points toward "quantum nature" of entropy. But exactly this beautiful hidden symmetry makes one wary of transplanting the idea of entropy from physics to mathematical models of Life.)

Bernoully Lecture, March 27, 2018: Alternative Probabilities.

Misha Gromov,

April 8, 2018

I. What is probability?
II. What is it good for?
III. What are limitations of the concept of probability?
IV. What alternatives are desirable and what is available?

Fisher Metric. Recall (Archimedes, 287-212 BCE) the *real moment map* from the unit sphere $S^n \subset \mathbb{R}^{n+1}$ to the probability simplex $\triangle^n \subset \mathbb{R}^{n+1}$ for

$$(x_0, ..., x_n) \mapsto (p_0 = x_0^2, ..., p_n = x_n^2)$$

and observe following R. Fisher that the spherical metric (with constant curvature +1) thus transported to $\triangle^n$, call it ds^2 on $\triangle^n$, is equal, up to a scalar multiple, to the *Hessian of the entropy*

$$ent\{p_0, ..., p_n\} = -\sum_i p_i \log p_i.$$

$$ds^2 = const\frac{\partial^2 ent(p_i)}{dp_i dp_j}.$$

If, accordingly, we take the "inverse Hessian" – a kind of double integral "$\int\int ds^2$" for the *definition* of entropy – we arrive at

Question 2. Are there *interesting* "entropies" associated to (real and complex) moment maps of general toric varieties? Is there a *meaningful* concept of "generalised probability" grounded in positivity encountered in algebraic geometry?

Gromov question: Are there « entropies » associated to moment maps

Bernoulli Lecture - What is Probability?

- 27 March 2018 - CIB - EPFL - Switzerland
- Lecturer: Mikhail Gromov
- https://bernoulli.epfl.ch/images/website/What_is_Probability_v2(2).mp4
- http://forum.cs-dc.org/uploads/files/1525172771489-alternative-probabilities-2018.pdf

Fisher Metric. Recall (Archimedes, 287-212 BCE) the *real moment map* from the unit sphere $S^n \subset \mathbb{R}^{n+1}$ to the probability simplex $\Delta^n \subset \mathbb{R}^{n+1}$ for

$$(x_0, ..., x_n) \mapsto (p_0 = x_0^2, ..., p_n = x_n^2)$$

and observe following R. Fisher that the spherical metric (with constant curvature +1) thus transported to Δ^n, call it ds^2 on Δ^n, is equal, up to a scalar multiple, to the *Hessian of the entropy*

$$ent\{p_0, ..., p_n\} = -\sum_i p_i \log p_i.$$

$$ds^2 = const \frac{\partial^2 ent(p_i)}{dp_i dp_j}.$$

If, accordingly, we take the "inverse Hessian" – a kind of double integral "$\int\int ds^2$" for the *definition* of entropy – we arrive at

Question 2. Are there *interesting* "entropies" associated to (real and complex) moment maps of general toric varieties? Is there a *meaningful* concept of "generalised probability" grounded in positivity encountered in algebraic geometry?

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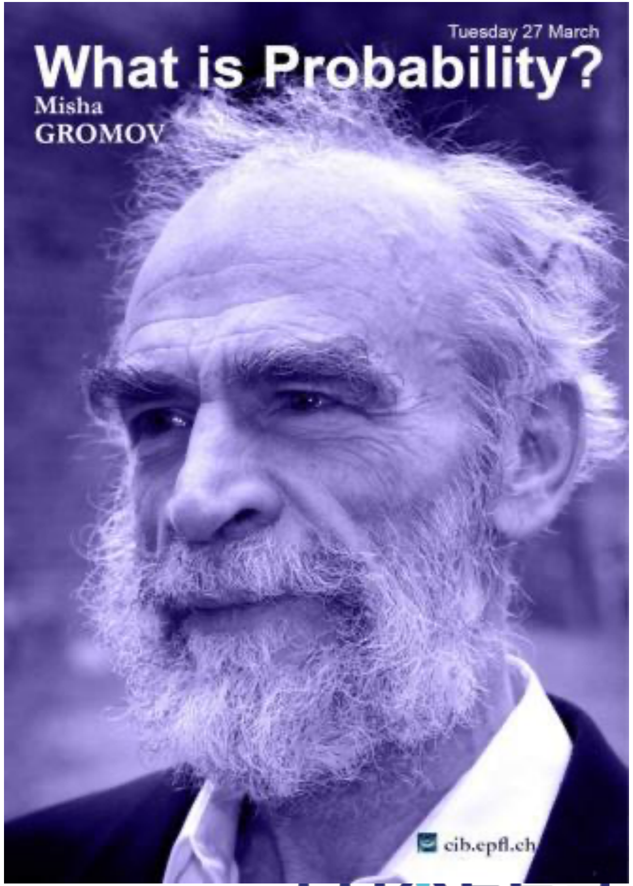

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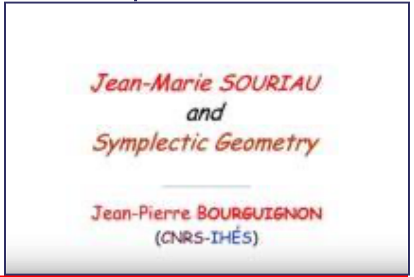
Building a future we can all trust

SOURIAU 2019

SOURIAU 2019

- Internet website : http://souriau2019.fr
- In 1969, 50 years ago, Jean-Marie Souriau published the book "**Structure des système dynamiques**", in which using the ideas of J.L. Lagrange, he formalized the "**Geometric Mechanics**" in its modern form based on **Symplectic Geometry**
- Chapter IV was dedicated to "Thermodynamics of Lie groups" (ref André Blanc-Lapierre)
- Testimony of **Jean-Pierre Bourguignon** at Souriau'19 (IHES, director of the European ERC)



https://www.youtube.com/watch?v=93hFolIBo0Q&t=3s

EUROPEAN ACTIONS : COST CaLISTA & MSCA CaLIGOLA

Nous partîmes cinq cents; mais par un prompt renfort, nous nous vîmes trois mille en arrivant au port, tant, à nous voir marcher avec un tel visage, les plus épouvantés reprenaient leur courage !

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Cartan geometry, Lie and representation theory, Integrable Systems, quantum Groups and quantum computing towards the understanding of the geometry of deep Learning and its Applications

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Horizon Europe Framework Programme MSCA Staff Exchanges 2021 (HORIZON-MSCA-2021-SE-01) Project ID: 101086123

https://site.unibo.it/caligola/en

WP5 The geometry of Deep Learning

Main research themes:

- **Lie groups thermodynamics**
- **Persistent Homology and Geneos**
- **The geometry of (Geometric) Deep Learning**

ALMA MATER STUDIORUM UNIVERSITÀ DI BOLOGNA — COST ACTION CALISTA

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Cartan geometry, Lie, Integrable Systems, quantum group Theories for Applications - CaLISTA aims to advance cutting-edge research in mathematics and physics through a systematic application of the ideas and philosophy of Cartan geometry, a thorough Lie theoretic approach to differential geometry.

cost — EUROPEAN COOPERATION IN SCIENCE & TECHNOLOGY

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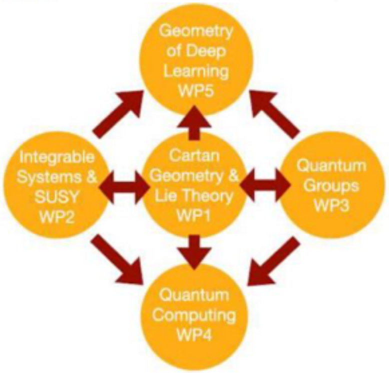

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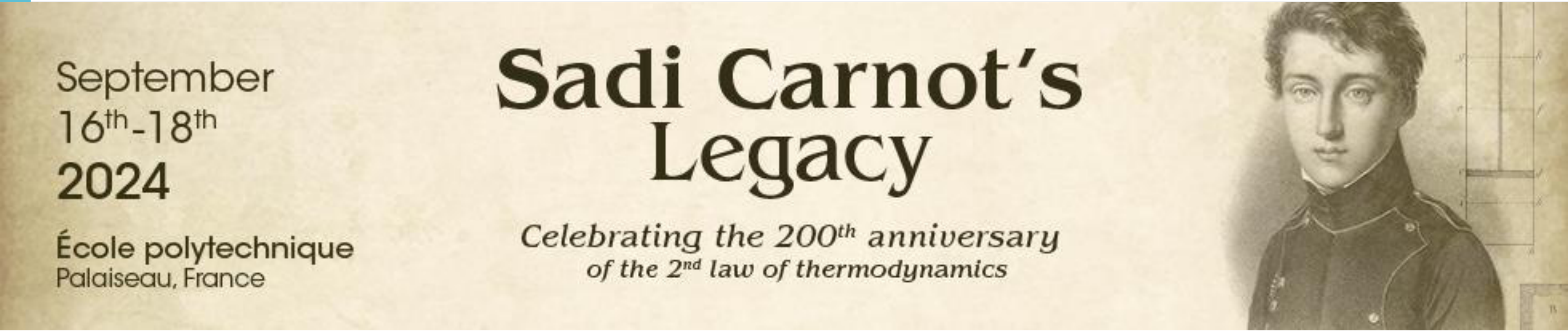
WG4 VISION: T4.3: Interpret SGD and the metric structure of the model space with Souriau Lie Thermodynamics. Interpret the coadjoint orbits of the symmetry group action as level set of entropy; exploit their symplectic structure to construct further symmetries (group equivariant GDL).

Ecole Polytechnique: Sadi Carnot's Legacy



SADI CARNOT'S LEGACY - "CELEBRATING THE 200TH ANNIVERSARY OF THE 2ND LAW OF THERMODYNAMICS"

When Sadi Carnot published his "Reflections on the motive power of fire" in 1824, there was no sign that one of the greatest scientific revolutions was about to take place, in a world then dominated by mechanics and optics. Yet, by bringing a conceptual analysis to the practical problem of the steam engine, Sadi Carnot wrote the birth certificate of thermodynamics, and, in particular, its second principle.

Today, thermodynamics has branched out into a multitude of fields and applications, from industrial processes to microscopic systems, and continues to renew our view of science.

Since its origins, thermodynamics has raised as many questions as it has answered.

To celebrate the bicentenary of the "Réflexions", this colloquium aims to bring together members of the thermodynamics community and to invite them to take a critical look at modern thermodynamics and the open questions it raises. The colloquium will be structured around pedagogical presentations introducing the various fields of the discipline. Poster sessions will allow participants to share their work.

https://carnot-legacy.sciencesconf.org/

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10. European HORIZON-MSCA-2021-SE-01-01 project CaLIGOLA - Cartan geometry, Lie and representation theory, Integrable Systems, quantum Groups and quantum computing towards the understanding of the geometry of deep Learning and its Applications; https://ec.europa.eu/info/funding-tenders/opportunities/portal/screen/opportunities/topic-details/horizon-msca-2021-se-01-01