The Calculus of Variations and Geometry: a Historical Perspective

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Introduction

- The Calculus of Variations has a very long history and has had, and continues to have, an immense impact on the development of Mathematics.
- This lecture discusses in particular its relation with Geometry.
- **A word of warning**: for a long time, mathematicians were called "geometers", and "Geometry" was referring to truly geometric constructions and drawings. In this lecture, we use this word in the modern sense it has taken in Mathematics since the end of the XIX$^{\text{th}}$ century, emphasising the various structures that shape objects without reference to actual drawings and its relation to Groups of transformations.
- A good reason for that is the consideration of spaces of dimensions larger than the usual 3-dimensional space, which has become standard in XX$^{\text{th}}$ century Mathematics, as well as critical for the use of geometric structures in Theoretical Physics.

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Outline of the Lecture

Several reasons justify why coupling the two topics is worthwhile:

- The Calculus of Variations has been a remarkable stimulus to develop "new geometric concepts";
- At the same time, in a number of situations, critical values in the Calculus of Variations were dictated by geometric considerations;
- We will give a few examples of such situations.

Here is the outline of the lecture:

1. The Birth of the Calculus of Variations
2. A Breakthrough by Joseph Louis de LAGRANGE
3. The Geometric Approach to the Calculus of Variations
4. The Calculus of Variations and Functional Analysis
5. Singularities and the Calculus of Variations
6. Concluding Remarks

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1. The Birth of the Calculus of Variations

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Problems of “optimising” a curve were discussed among geometers of the early XVIII$^{\text{th}}$ century:

- one which attracted a lot of attention was to find the **brachistochrone**, i.e. the curve allowing a ball to join two points at different altitudes in the shortest time under the only effect of gravity.
- Several solutions were proposed, in particular by Jean BERNOULLI and by Leonhard EULER, many using geometric arguments (the solution is an inverse **cycloid**).
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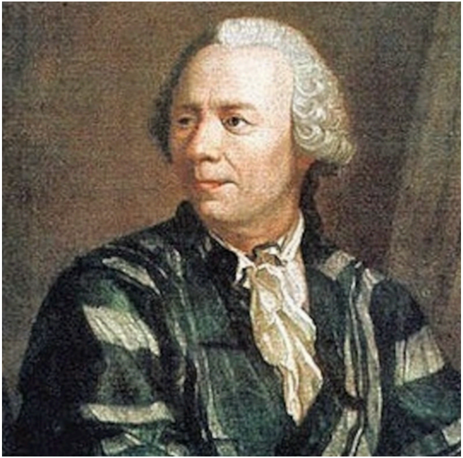

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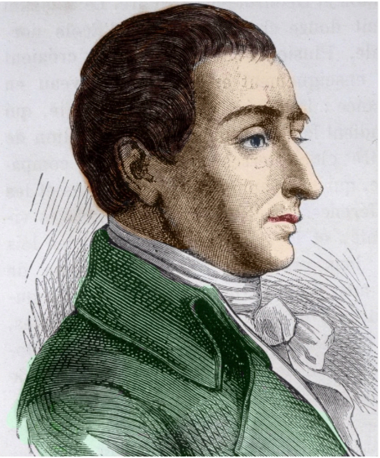

The Euler-Lagrange Equations

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- The curve c will be an *extremal* for the function $\mathcal{L}$ if, for all variations $C = (c_s)$ of c, $\frac{d}{ds}(\mathcal{L}(c_s))_{|s=0} = 0$, and one has:

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The Euler-Lagrange Equations (cont.)

Euler-Lagrange Equations are a system of second-order differential equations.

Here are some examples of **Lagrangians** of curves considered:

- The **length** for which $L(x, X) = ||X||$, where $||\ \ ||$ denotes the Euclidean norm;
- The **energy** for which $L(x, X) = \frac{1}{2}||X||^2$;
- Note that *the length is invariant under reparametrisation,* creating a degeneracy for its Euler-Lagrange Equations since, if c is an extremal, then for any diffeomorphism φ of the interval $[a, b]$, then $c \circ \varphi$ is also an extremal;
- *The correspondence between large groups of invariance and degeneracies is a fundamental phenomenon in Mathematics*;
- For the motion of a particle of mass m in a potential V, the usual equations of motion $m\ddot{c} = -Grad\ V$ are obtained from the **Action** Lagrangian $L(x, X) = \frac{1}{2}m||X||^2 - V(x)$.

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Euler-Lagrange Equations are a system of second-order differential equations.

Here are some examples of **Lagrangians** of curves considered:

- The **length** for which $L(x, X) = ||X||$, where $||\quad||$ denotes the Euclidean norm;
- The **energy** for which $L(x, X) = \frac{1}{2}\, ||X||^2$;
- Note that *the length is invariant under reparametrisation,* creating a degeneracy for its Euler-Lagrange Equations since, if c is an extremal, then for any diffeomorphism φ of the interval $[a, b]$, then $c \circ \varphi$ is also an extremal;
- *The correspondence between large groups of invariance and degeneracies is a fundamental phenomenon in Mathematics*;
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The Least Action Principle

The idea that a **Least Action Principle** was widely at work in Nature was already present early in the XVII$^{\mathrm{th}}$ century.

Here are a few instances:

- in 1662, his study of light rays led Pierre de FERMAT to state that their reflection on a surface and their diffraction through a surface between two media could be recovered applying a fastest path principle (the Snell-Descartes Laws of Optics);
- In 1746, this was widely generalised to mechanical systems by Pierre-Louis MOREAU de MAUPERTUIS, arguing in a philosophical manner about the economical behaviour of Nature, something challenged by his contemporaries;
- LAGRANGE made it a scientific statement in 1760.

This idea that *minimising an action among all possible paths joining two states of a system continued to inspire, and proved relevant, through the history of Physics*, stimulating at the same time broader and broader mathematical formulations.

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2. A Breakthrough by LAGRANGE

LAGRANGE's Further Developments

All along the XVIIIth century, Celestial Mechanics worked with great success on the basis of Newtonian Mechanics thanks to the Theory of Perturbations .

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LAGRANGE's 1788 Méchanique Analitique

MÉCHANIQUE

ANALITIQUE;

Par M. DE LA GRANGE, de l'Académie des Sciences de Paris, de celles de Berlin, de Pétersbourg, de Turin, &c.

A PARIS,

Chez LA VEUVE DESAINT, Libraire, rue du Foin S. Jacques.

M. DCC. LXXXVIII.

AVEC APPROBATION ET PRIVILEGE DU ROI.

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MÉMOIRE SUR LA THÉORIE

DES

VARIATIONS DES ÉLÉMENTS DES PLANÈTES

ET EN PARTICULIER

DES VARIATIONS DES GRANDS AXES DE LEURS ORBITES (*)

(Mémoires de la première classe de l'Institut de France, année 1808.)

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On entend, en Astronomie, par éléments d'une planète les quantités qui déterminent son orbite autour du Soleil, supposée elliptique, ainsi que le lieu de la planète dans un instant marqué, qu'on appelle l'*époque*. Ces quantités sont au nombre de cinq, dont deux, le grand axe ou la distance moyenne qui en est la moitié, et l'excentricité, déterminent la grandeur de l'ellipse dont le Soleil occupe l'un des foyers; les trois autres, la longitude de l'aphélie, celle des nœuds, et l'inclinaison, déterminent la position du grand axe sur le plan de l'ellipse et la position de ce plan sur un plan qu'on regarde comme fixe par rapport aux étoiles. Ces cinq quantités, jointes à l'époque, étant connues pour une planète, on peut trouver en tout temps son lieu dans le ciel par le moyen de ces deux lois, découvertes par Képler, que les aires décrites dans l'ellipse par le rayon vecteur croissent proportionnellement au temps, et que la durée de la révolution est proportionnelle à la racine carrée du cube du grand axe. Les Tables d'une planète, abstraction faite de ses perturba-

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LAGRANGE's New Geometry

L'objet de ce Mémoire est d'exposer les nouvelles formules que j'ai trouvées pour les variations des éléments des planètes, ainsi que leur application aux variations des grands axes, et de développer surtout l'Analyse qui m'y a conduit, et qui me parait mériter l'attention des Géomètres par son uniformité et par sa généralité, puisqu'elle est indépendante de la considération des orbites elliptiques, et qu'elle peut s'appliquer avec le même succès à toute autre hypothèse de gravitation dans laquelle les orbites ne seraient plus des sections coniques.

LAGRANGE's New Geometry

In the space of elliptic motions, the trajectory of a planet around the Sun under the influence of other planets is a motion which can be captured by a single *perturbation function* Ω which satisfies some very specific equations.

These show that the new space is naturally endowed with a geometric structure of a new type involving antisymmetric products as one can see below:

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On aura donc ainsi les quatre formules correspondantes

$$\frac{d\Omega}{dc}dt = -(a,c)\,da - (b,c)\,db + (c,f)\,df + (c,g)\,dg + (c,h)\,dh,$$

$$\frac{d\Omega}{df}dt = -(a,f)\,da - (b,f)\,db - (c,f)\,dc + (f,g)\,dg + (f,h)\,dh,$$

$$\frac{d\Omega}{dg}dt = -(a,g)\,da - (b,g)\,db - (c,g)\,dc - (f,g)\,df + (g,h)\,dh,$$

$$\frac{d\Omega}{dh}dt = -(a,h)\,da - (b,h)\,db - (c,h)\,dc - (f,h)\,df - (g,h)\,dg.$$

où

$$(f,g) = \left(\frac{dP}{df}\frac{dQ}{dg} - \frac{dP}{dg}\frac{dQ}{df}\right) na^2\sqrt{1-b^2},$$

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- The new structure LAGRANGE uncovered in dealing with a specific problem in Celestial Mechanics is based on an antisymmetric form, a type of objects which had not been given much consideration in Mathematics so far;
- Pierre-Simon de LAPLACE and Simon-Denis POISSON were also giving some attention to similar structures at this time;
- This is the birth of a true *new Geometry*, which will be at the heart of the geometric approach to the Calculus of Varations.

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3. The Geometric Approach to the Calculus of Variations

Going to the Cotangent Bundle

From now on, we work with the modern tools of Geometry as they were established in the first part of the XX$^{\text{th}}$ century using manifolds, vector bundles and more general bundles, the exterior differential. We will be using local coordinates when it makes the formulation easier to follow, even if our strong preference goes to intrinsic notions.

On a manifold M, the key new actor will the *cotangent bundle*, namely the collection T_p^*M of all 1-forms on tangent spaces at points $p \in M$:

- Let $(x^i)_{1\leq i\leq n}$ be a system of local coordinates around a point p; of critical importance is the fact that the exterior 2-form $\omega = \sum_{i=1}^n d\xi_i \wedge dx^i$, where (ξ_i) denote the linear coordinates on T_p^*M induced by the basis $(\partial/\partial x^i)$ of T_pM, is independent of the choice of coordinates, hence is universally defined;
- the 2-form ω is non degenerate, and closed $d\omega = 0$;
- ω is called the *Liouville symplectic form*.

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The Hamiltonian Approach

LAGRANGE's breakthrough in the Calculus of Variations we presented fits with the approach systematically developed by Sir William Rowan HAMILTON in 1834. Hence the name of *Hamiltonian Approach* given to it.

Here are the key features of this approach:

- To any function φ on T^*M, one can associate a vector field Ω_φ using the duality that ω determines, namely $i_{\Omega_\varphi}\omega = d\varphi$;
- Then, the integral curves of $-\Omega_\varphi$ satisfy *Hamilton's equations written, in a natural chart* (x^i, ξ_i), *in the classical form*:

$$\begin{cases} \dfrac{dx^i}{d\tau} = \dfrac{\partial \varphi}{\partial \xi_i} \\ \dfrac{d\xi_i}{d\tau} = -\dfrac{\partial \varphi}{\partial x^i} \end{cases} \quad 1 \le i \le n.$$

- The key point is the possibility to connect it with a problem of the Calculus of Variations in the Lagrangian setting.

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Here are the key features of this approach:

- To any function φ on T^*M, one can associate a vector field Ω_φ using the duality that ω determines, namely $i_{\Omega_\varphi}\omega = d\varphi$;
- Then, the integral curves of $-\Omega_\varphi$ satisfy *Hamilton's equations written*, in a natural chart (x^i, ξ_i), *in the classical form*

$$\begin{cases} \dfrac{dx^i}{d\tau} = \dfrac{\partial\varphi}{\partial\xi_i} \\ \dfrac{d\xi_i}{d\tau} = -\dfrac{\partial\varphi}{\partial x^i} \end{cases}, \quad 1 \leq i \leq n\ ;$$

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Mapping Lagrangian and Hamiltonian Approaches

The correspondence we are looking for is achieved, in the good situations, by the *Legendre Transform*, which goes as follows:

- For a Lagrangian $L : TM \to \mathbb{R}$, the *Legendre transform* associated to L is the map $\Lambda_L : TM \longrightarrow T^*M$ defined for $v \in T_qM$ by $\Lambda_L(v) = d(L_{|T_qM})(v)$ (where $T_v^*(T_qM)$ is identified with T_q^*M by translation by the vector $-v$);
- One has $(\Lambda_L)(x^i, X^i) = (x^i, \partial L/\partial X^i)$;
- When the matrix $(\partial^2 L/\partial X^i \partial X^j)$ is invertible, L is said to be *regular* and the Legendre Transform is a local diffeomorphism;
- This is in particular the case when L is the *energy of curves associated to a Riemannian metric g*;
- If L is regular, then Λ_L maps locally the integral curves of the symplectic gradient of the Hamiltonian $H = E_L \circ (\Lambda_L)^{-1}$, where E_L is the *energy* associated to L, to the curves of velocity-vectors of the extremals of the action functional $\mathcal{L}$ associated to L, a full picture of the Calculus of Variations.

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Symmetries and the Calculus of Variations

The presence of symmetries in a problem of the Calculus of Variations allows to simplify the search for solutions because of the existence of conserved quantities. A Hamiltonian φ is a conserved quantity along an integral curve of the vector field Ω_φ since $\partial_{\Omega_\varphi}\varphi = (d\varphi)(\Omega_\varphi) = -\omega(\Omega_\varphi, \Omega_\varphi) = 0$ because ω is antisymmetric.

The presence of the action of a symmetry group creates other conserved quantities:

- Let G be a Lie group acting on T^*M preserving ω (e.g. if it is the extension of an action on M); an observable μ on T^*M with values in $(T_eG)^*$ (where e is the identity element of the group) is called a *moment* if, for any vector $X \in T_eG$ and for any point $\lambda \in T_p^*M$, we have $d\langle\mu, X\rangle(\lambda) = (i_{X_p}\omega)(p)$.
- Then Emmy NOETHER stated the fundamental *conservation law: for any function left invariant by G, any moment map $p : T^*M \longrightarrow (T_eG)^*$ for this action is a first integral of the Hamiltonian motion defined by H.*

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4. The Calculus of Variations and Analysis

Broadening the Calculus of Variations

In its historic setting, the Calculus of Variations focused on optimising curves but it deals with many other geometric objects and functionals defined over them. This made it a central tool in the modelisation in Theoretical Physics, in Computer Science, in Engineering, etc. Here are a few examples of functionals:

- The *volume* of surfaces and, in higher dimensions of submanifolds; to find the surface of least area spanning a curve in 3-space is the famous *Plateau problem*;
- For Riemannian manifolds (M, g) and (N, h) and maps $f : M \to N$, *harmonic maps* are critical points of the *energy* $\mathcal{E}(f) = \int_M g^{-1}(f^*(h))\, vol_g = \int_M (g^{ij} \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j} h_{\alpha\beta}) vol_g$;
- Let G be a Lie group and $\pi : E \to M$ be a vector bundle equipped with a G-invariant structure over the Riemannian manifold (M, g); *Yang-Mills fields* are critical points of the *Yang-Mills functional* $\mathcal{YM}(\nabla) = \frac{1}{2} \int_M ||R^\nabla||^2\, vol_g$ where R^∇ is the curvature of the G-covariant derivative ∇.

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Broadening the Calculus of Variations (cont.)

Here are some more examples of questions which can be approached using the Calculus of Variations:

- In the theory of General Relativity, where the unknown is a Lorentzian metric g on a 4-dimensional space-time M, the *Hilbert-Einstein Lagrangian* $\mathcal{HE}(g) = \int_M Scal_g\, vol_g$, where $Scal_g$ denotes the scalar curvature of g, led to the *Einstein Equations* $Ric_g - \frac{1}{2} Scal_g\, g = T$, where T is the *stress-energy tensor* representing physical interactions outside gravity;
- Segmentation of an image into meaningful regions goes as follows: starting from an image whose grayscale is given by $f : [0,1] \times [0,1] \to [0,1]$, the segmentations of the image can be obtained by the level sets of a real-valued function u defined on $[0,1] \times [0,1]$ focusing attention on $u^{-1}(0)$ and by forcing some simplification by penalising its length;
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Spaces of Functions

Spaces of functions play a decisive role in modern Analysis. They are often infinite dimensional Hilbert or Banach spaces with topologies defined by norms appropriate for the problem studied, often linked to the Calculus of Variations. The idea is often to take the largest possible space where the functional makes sense.

Various collections of spaces have been considered:

- Classically, spaces of differentiable functions are natural to consider but to take advantage of weak methods spaces using generalised derivatives are most of the time used;
- On a Riemannian manifold M of dimension n, the *Sobolev spaces* $W^{k,p}(M)$ consist of functions on M whose weak partial derivatives up to order k are L^p-integrable; they are in general Banach spaces, but Hilbert spaces for $p = 2$; the key inclusion states: $W^{k,p} \subset W^{m,q}$ if $k \geq m$ and $k - \frac{n}{p} \geq m - \frac{n}{q}$;
- One also uses Hölder spaces $C^{k,\alpha}$ of functions with non integral derivatives.

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- One also uses Hölder spaces $C^{k,\alpha}$ of functions with non integral derivatives.

Spaces of Functions

Spaces of functions play a decisive role in modern Analysis. They are often infinite dimensional Hilbert or Banach spaces with topologies defined by norms appropriate for the problem studied, often linked to the Calculus of Variations. The idea is often to take the largest possible space where the functional makes sense.

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The Search for Critical Points

Solving a problem in the Calculus of Variations amounts to finding a critical point of a functional, typically a minimum or a maximum (but saddle points can also be interesting). For a functional bounded from below, a minimum is guaranteed if some kind of compactness holds. Compactness being rare in infinite dimensional spaces, the right notion is that of *weak compactness*.

Various tools allow to deal with the existence of critical points. Here are a few paths that have been followed:

- Several methods relate the topology of a space and the existence of critical points of functions, such as *Morse Theory*;
- To circumvent the action of a non compact group preserving the functional, constraints can be added, e.g. *gauge fixing*;
- The *Concentration-Compactness Principle*, introduced by Pierre-Louis LIONS, decomposes any sequence of functions failing to converge into a convergent part and a part exhibiting concentration by action of a non-compact group.

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Special Solutions and Moduli Spaces

All equations we have been considering so far are **second-order equations**. In a number of important geometric situations, some solutions of the Calculus of Variations problem are actually **solutions of a first-order equation**.

Here are some examples:

- For complex manifolds M and N, *holomorphic* maps between them, solutions of the first-order *Cauchy-Riemann Equations*, are automatically harmonic if the metrics are *Kählerian*;
- If $\dim M = 4$ and M is oriented, 2-forms over M split into *self-dual* and *anti-self-dual* parts; due to the Chern-Weil characteristic constraint, *self-dual* (or *anti-self-dual*) connections are minima of the Yang-Mills functional $\mathcal{YM}$, i.e. *solutions of a first order equation;*
- Simon DONALDSON turned the moduli of these solutions into a tool to show that $\mathbb{R}^4$ has many non-isomorphic differential structures, the only $\mathbb{R}^n$ to have this property.

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5. Singularities and the Calculus of Variations

The Euler-Lagrange Equation for Curves

The regularity theory is simpler for curves, when the Euler-Lagrange Equation are differential equations than when they are systems of partial differential equations; still, it seems that a rigorous solution for curves was only obtained by David HILBERT in 1900.

The theory of shortest paths between two points exhibit some general features in a simpler form:

- When two points are not too far apart, there is uniqueness of the geodesic joigning two points which is the shortest path;
- Along a geodesic γ, a singular situation appears when the second variation of the energy has some degenerate directions; the two endpoints are then said to be *conjugate* of each other along γ; beyond these points γ is not anymore minimising;
- Another important approach deals with closed curves, and the study of closed geodesics on a compact manifold has been a motivating problem in the Calculus of Variations.

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Regularity of Harmonic Maps

When dim $M > 1$, the energy functional of maps from M had a different scaling behaviour. In dimension 2, the energy functional is the same for two conformally related metrics: indeed, if one composes a map f with a conformal transformation φ, then $\mathcal{E}(f) = \mathcal{E}(f \circ \varphi)$; on S^2, the group of conformal transformations is non compact; hence, one can construct a sequence of maps with the same energy without convergent subsequence: in this dimension, the energy of a map does not control its continuity.

To overcome this difficulty more work is required:

- The approach taken by Jonathan SACKS and Karen UHLENBECK is to consider the energy to a power $\alpha > 1$, for which minima exist, then to pass to the limit;
- what may occur: there is good convergence towards a harmonic map but at some points, some bubbles may appear, giving rise to some harmonic maps from S^2 to the target;
- Small energy prevents this because of the cost of bubbles.

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Removable Singularities for Yang-Mills Fields

UHLENBECK contributed also in a fundamental way to Yang-Mils theory, in particular in the choice of a gauge controlled by the energy of the Yang-Mills field. Her most spectacular result there is her *removable singularity theorem*.

Here is the result:

- *Any Yang-Mills field* ∇ *over* $B^4 - \{0\}$ *with* $\mathcal{YM}(\nabla) < \infty$ *extends smoothly in a local trivialisation*;
- An important corollary is that *any finite energy Yang-Mills field over* $\mathbb{R}^4$ *extends smoothly to* S^4, taking advantage of the conformal invariance of $\mathcal{YM}$;
- This phenomenon leads, for Yang-Mills fields in 4 dimensions, to an understanding analogous to what had been achieved for harmonic maps with the possibility of another phenomenon of "bubbling", this time in the form of the appearance of $\pm$-self-dual connections over S^4.

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Singularities for Minimal Submanifolds

The question of the absence of singularities of minimal submanifolds has been an important driver of Geometric Analysis over the years, in particular for stable submanifolds.

Here is a brief outline:

- In 1961, Ennio DE GIORGI established the smooth embedding of a locally area minimising submanifold which bounds in a Riemannian manifold away from a singular set Σ of vanishing n-dimensional Hausdorff measure;
- Through stepwise progress, it was proved that Σ was empty if $n \leq 6$, discrete for $n = 7$ and has Hausdorff dimension less than $n - 7$ if $n \geq 8$;
- James SIMONS showed that the (singular) cone over $\frac{1}{\sqrt{2}}S^3 \times \frac{1}{\sqrt{2}}S^3$ in $\mathbb{R}^8$ was stable, hence that the bound on the Hausdorff dimension of the singular set is best possible;
- This transition from regularity to singularities in minimal submanifolds when $n \geq 7$ has been a key feature of the theory.

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The question of the absence of singularities of minimal submanifolds has been an important driver of Geometric Analysis over the years, in particular for stable submanifolds.

Here is a brief outline:

- In 1961, Ennio DE GIORGI established the smooth embedding of a locally area minimising submanifold which bounds in a Riemannian manifold away from a singular set Σ of vanishing n-dimensional Hausdorff measure;
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6. Concluding Remarks

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I am aware that what brings you all here are developments on a topic which is not directly related to the Calculus of Variations.

My objective in presenting this historic perspective on it was 3-fold:

- Challenging problems force to develop new techniques to solve them and, from that point of view, the Calculus of Variations is exemplary;
- More fundamentally, they can also lead to the development of radically new concepts:
  - the first example of this phenomenon I gave is the consideration of an abstract space in the form of the space of *elliptic motions* by LAGRANGE, and even more importantly the introduction of *symplectic geometry* in the same setting;
  - the second is the use by Simon DONALDSON of moduli spaces of 4-self-dual connections on bundles over a space as a new geometric invariant attached to a space;
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One more dimension that the wonderful mathematical adventure that the Calculus of Variations puts in perspective is the *unity of mathematics*, its various components coming together repeatedly and often unexpectedly.

Here are a few examples:

- A priori the Calculus of Variations points to Analysis although one of its first problem, the search for the brachistochrone, was solved by means of Geometry;
- Its later developments pointed to Geometry repeatedly, including leading to new fields as in Henri POINCARÉ's *"Nouvelles méthodes de la mécanique céleste"*, which has been the cradle of the Theory of Dynamical Systems;
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This can be done at various stages:

- Of course when mobilising various tools to solve problems;
- But also to have an in depth understanding of the mathematical structure under consideration (a very good example showing that one sometimes need to be patient is the unraveling of the geometric structure at the heart of the Euler-Lagrange Equations by Joseph KLEIN in 1962!);
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I thank you for your attention.

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