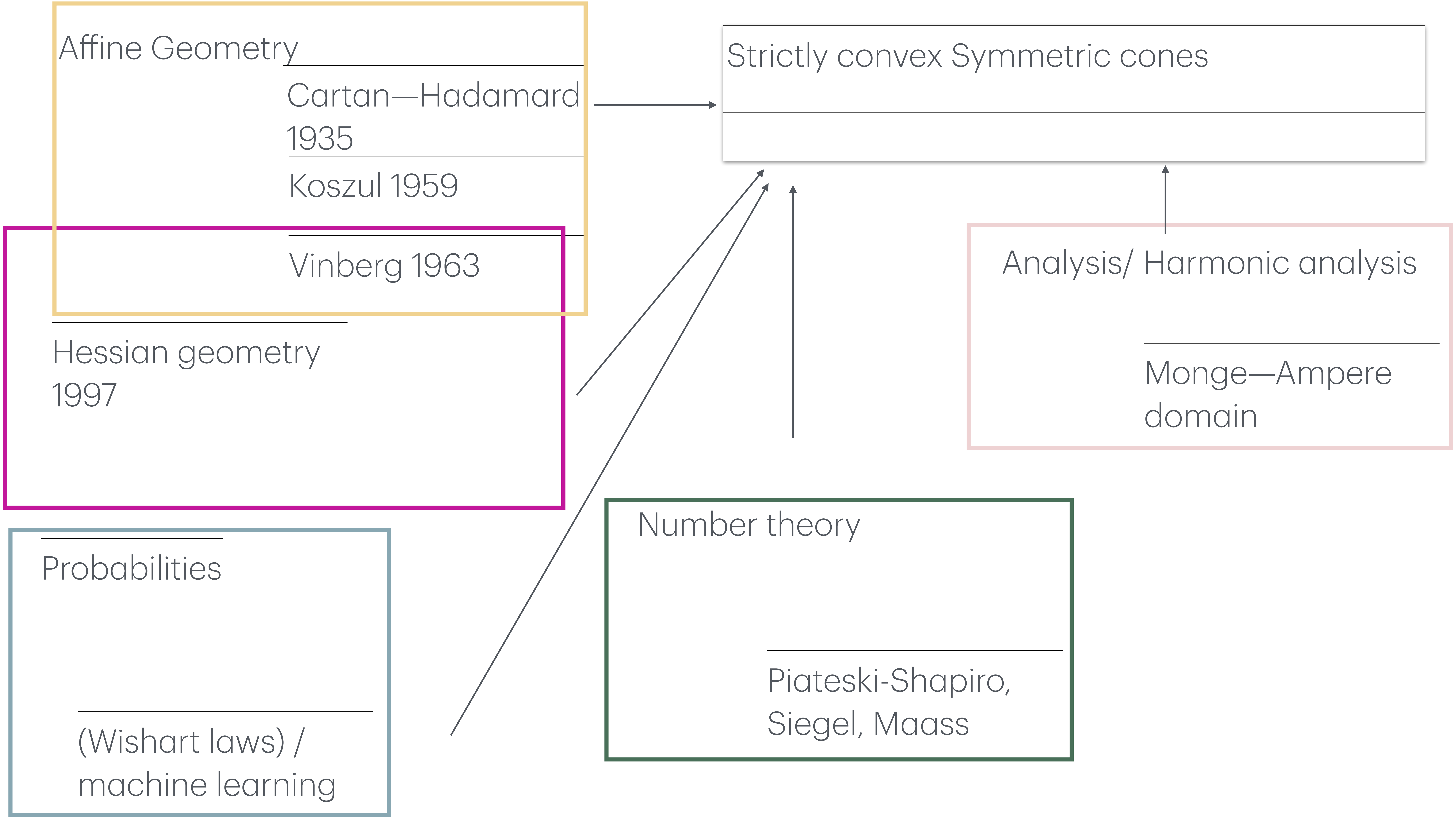
# Koszul domains & Souriau's thermodynamics

**Noemie Combe, Uniwersytet Warszawski**

# Part 1: Bounded convex domains.

Noemie Combe, 3 September 2024, University of Warsaw

# State of the art



Affine Geometry
Cartan—Hadamard 1935
Koszul 1959
Vinberg 1963
Strictly convex Symmetric cones
Hessian geometry 1997
Analysis/ Harmonic analysis
Monge—Ampere domain
Probabilities
(Wishart laws) / machine learning
Number theory
Piateski-Shapiro, Siegel, Maass

# Information Geometry side

# Density of a measure

- A measure $P$ is absolutely continuous w.r.t. $\lambda$ if for every measurable set $A$,

$$\lambda(A) = 0 \Longrightarrow P(A) = 0.$$

It implies the existence of a measurable function $\rho$ such that:

$$P(A) = \int_A \rho d\lambda, \quad \forall A \subset \mathcal{F}.$$

Here $\rho$ is called the density of the measure $P$ and $\rho = \frac{dP}{d\lambda}$ is called the Radon—Nikodym derivative.

# The manifold of probability distributions

Let $(\Omega, \mathscr{F})$, where $\mathscr{F}$ is a $\sigma$- algebra of $\Omega$ be a **measurable space**.

Consider a family of **parametrised probability distributions** on $(\Omega, \mathscr{F})$.

The set of all probability distributions over a finite set forms a **manifold**.

# 2D TFT

- **Combe—Manin 2020:**

**Flat statistical exponential manifolds are Frobenius (i.e. satisfy WDVV equation).**

- **WDVV PDE-equation**

$$\forall a, b, c, d : \sum_{ef} \Phi_{abe} g^{ef} \Phi_{fcd} = (-1)^{a(b+c)} \sum_{ef} \Phi_{bce} g^{ef} \Phi_{fad}$$

- **Geometrisation: Frobenius manifold (Manin)**
- **Hydrodynamical type (Novikov, Dubrovin)**

Source: Mike Feng for Quanta magazine

# Frobenius manifolds = geometrisation WDVV

mirror between two sources

of Frobenius manifolds

# Some new statements (2024)

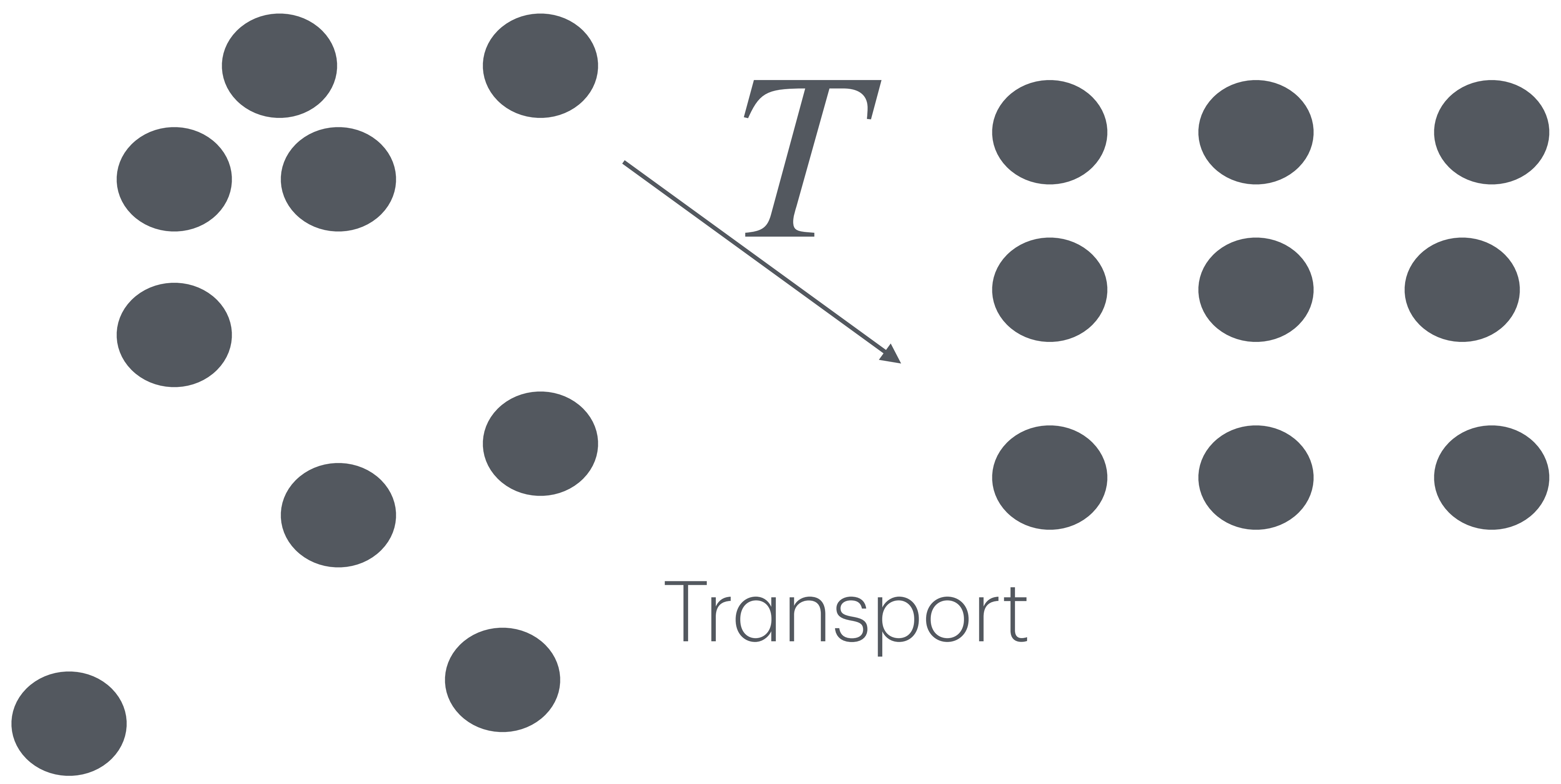
# But first … : the Monge problem

What is the most efficient way of transporting one distribution of mass into another?

$$\det(D^2\phi) = \frac{\rho}{\bar{\rho}}$$

Monge-Ampere equation

$$\iff$$




$x_i$ $q_j$

$$inf \sum_i |x_i - q_j|^2$$

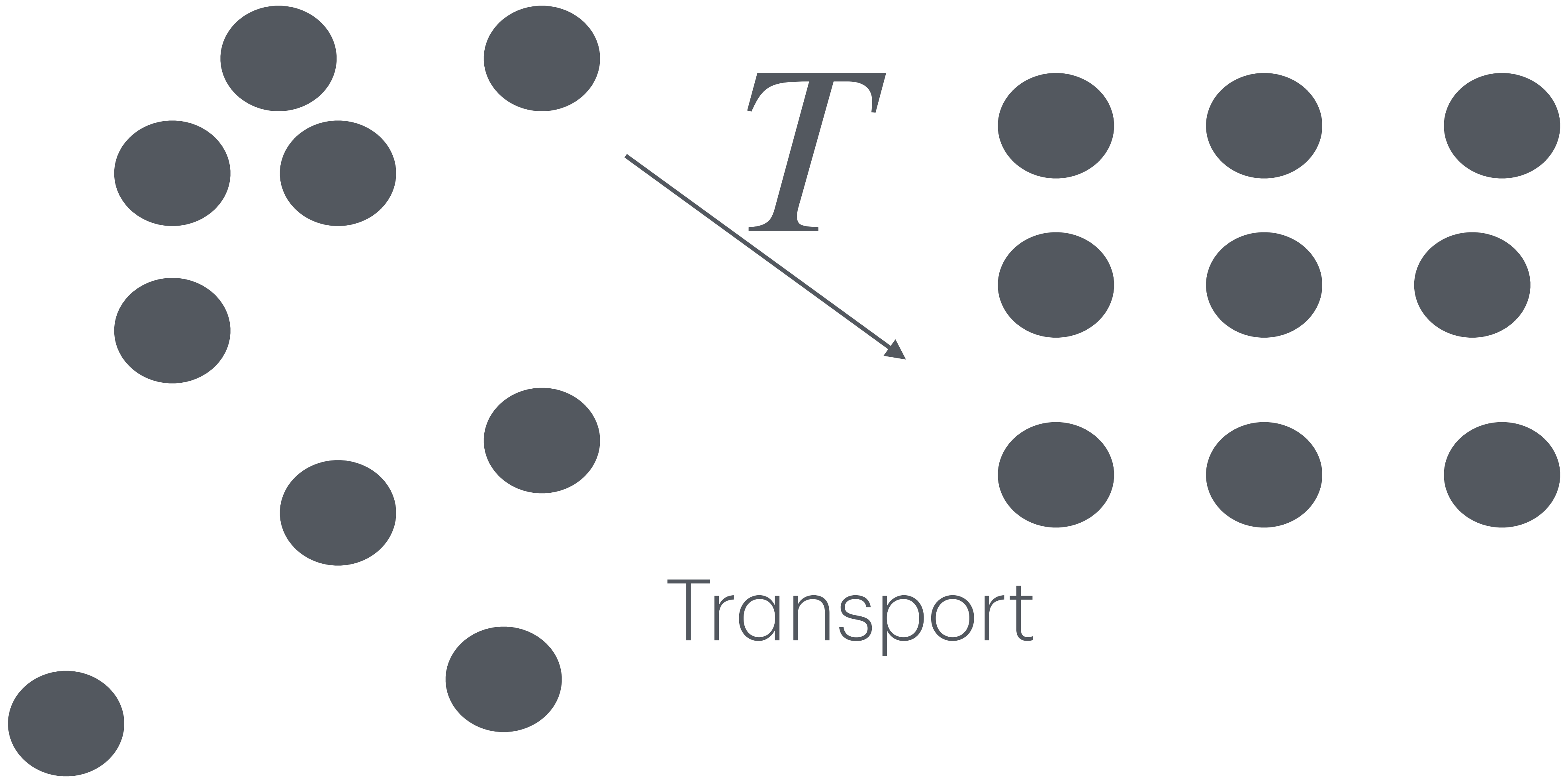
# Monge problem

What is the most efficient way of transporting one distribution of mass into another?

$$\det(D^2\phi) = \frac{\rho}{\bar{\rho}}$$

Monge-Ampere equation

$$\Longleftrightarrow$$




$$x_i \qquad q_j$$

Mass conservation gives

$$\bar{\rho} d^3 q = \rho(x) d^3 x$$

With change of variable $q \to x$

$$inf \sum_i |x_i - q_j|^2$$

# Monge problem

What is the most efficient way of transporting one distribution of mass into another?

$$\det(D^2\phi) = \frac{\rho}{\bar{\rho}}$$

Monge-Ampere equation

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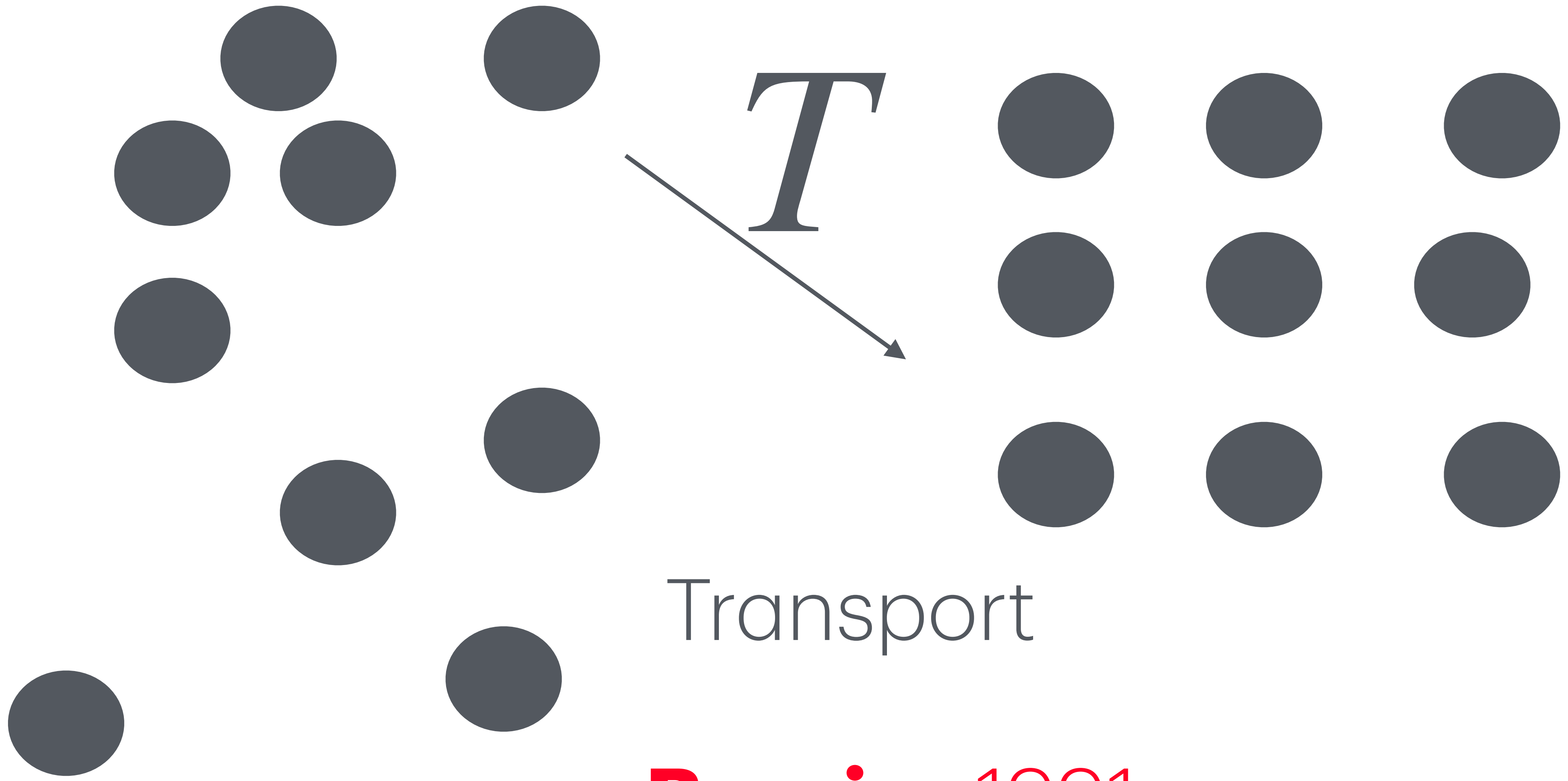



$x_i$

**Brenier** 1991

$$T = \nabla_x \phi(x)$$

$q_j$

Mass conservation gives

$$\bar{\rho} d^3 q = \rho(x) d^3 x$$

With change of

variable $q \rightarrow x$

$$inf \sum_i |x_i - q_j|^2$$

# Results

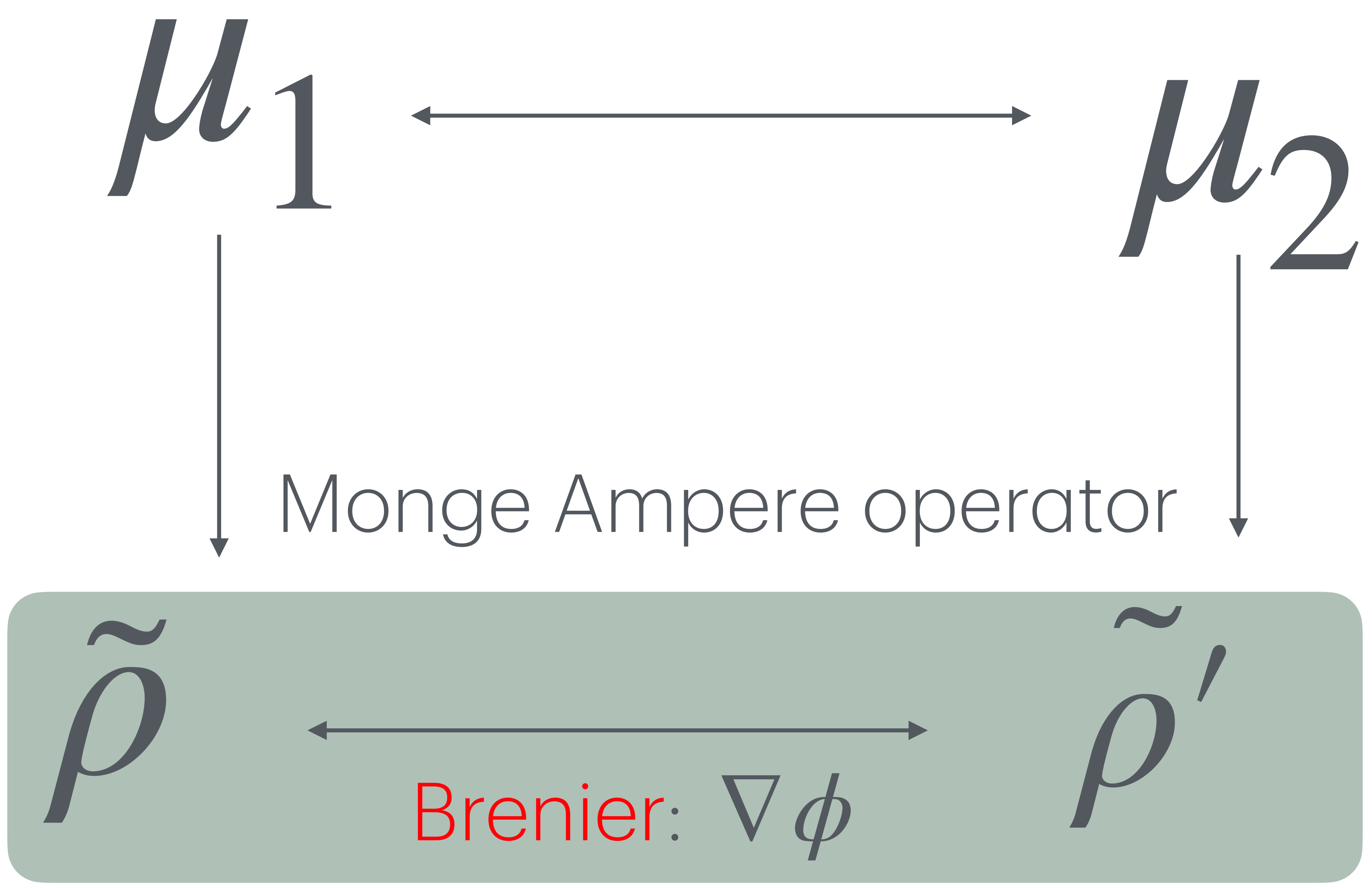
- COMBE 24: The cones $\mathscr{P}$ form a **Monge—Ampere domain**
- (i.e. everywhere locally there exists a unique function $\Phi$ on the domain $\mathscr{P}$ s.t. $\det \mathrm{Hess}\Phi = f$, where $f$ is a nonnegative real function on $\mathscr{P}$.

- COMBE 24: The cones $\mathscr{P}$ satisfy the axioms of a **(pre-)Frobenius manifold**

(a preliminary structure of an object in **Mirror symmetry**).

# Monge problem

- There exists an adaptation of the Monge-Ampere problem for density of measures*.




*Measures a.c wrt Lebesgue measure

# Classical results on KV domains

# Koszul—Vinberg domains

*In terms of matrix*

*representation*

| *Nb* | *Symbol* | *Irreducible symmetric cones* |
|---|---|---|
| *1.* | $\mathscr{P}_n(\mathbb{R})$ | *Cone of $n \times n$ positive definite symmetric real matrices.* |
| *2.* | $\mathscr{P}_n(\mathbb{C})$ | *Cone of $n \times n$ positive definite self-adjoint complex matrices.* |
| *3.* | $\mathscr{P}_n(\mathbb{H})$ | *Cone of $n \times n$ positive definite self-adjoint quaternionic matrices.* |
| *4.* | $\mathscr{P}_3(\mathbb{O})$ | *Cone of $3 \times 3$ positive definite self-adjoint octavic matrices.* |
| *5.* | $\Lambda_n$ | *Lorentz cone given by $x_0 > \sqrt{\sum_{i=1}^{n} x_i^2}$ (aka spherical cone).* |

One can add irreducible cones to obtain a new cone <== symmetric monoidal category

# Properties of strictly convex symmetric cones

- #1 An affine structure (flat torsionless affine connection)
- #2 Characteristic KV function
- #3 Hessian structure
- #4 rank 3 symmetric tensor
- #5 Jordan algebra structure of the tangent space
- #6 KV algebra on the tangent space

# #1 Affine structure

- An *affine structure* on $M$ is defined by a collection of coordinate charts $\{U_a, \phi_a\}$:
  - $\{U_a\}$ is an open cover and
  - $\phi_a : U_a \to \mathbb{R}^n$ is a local coordinate system, such that the coordinate change $\phi_b \circ \phi_a^{-1}$ is an affine transformation of $\phi_a(U_a \cap U_b)$ onto $\phi_b(U_a \cap U_b)$.

$\Longrightarrow$ **affine manifolds/affinely flat manifolds.**

- This implies the existence of a flat affine connection $\nabla$ on the tangent space, and reciprocally. Therefore, on $M$ there exists a flat (affine) torsionless connection $\nabla$.

# #2. KV - Characteristic function

Let $\mathcal{P} \subset V$ be a strictly convex homogeneous cone.

$\forall x \in \mathcal{P}$ (vector) the KV-characteristic function is:

$$\chi(x) = \int_{\mathcal{P}*} \exp\{-\langle x, a* \rangle\} da*$$

where $da*$ is a volume form invariant under translations in $\mathcal{P}*$ (dual cone).

# #3. Hessian structure

Let $\boldsymbol{\Phi} = \mathbf{ln}\,\chi$ be a potential function.

Then there exists a Hessian metric on the KV cone such that

Given affine coordinates $(x_i)$ (i.e. s.t. $\nabla(dx_i) = 0$) the metric tensor is

$$g_{ij} = \partial_i \partial_j \boldsymbol{\Phi},$$

where $\partial_i = \dfrac{\partial}{\partial x_i}$.

# #4. Rank 3 symmetric tensor

Assuming that $\Phi$ is smooth, it is easy to define a rank 3 symmetric tensor by putting:

$$A_{ijk} = \partial_i \partial_j \partial_k \Phi.$$

# #5. Jordan algebra.

## Symmetries of the tangent space

For each irreducible cone $\mathscr{P}$ we have the following structure on the tangent space at a point $\boldsymbol{e}$.

| *Formally real simpl Jordan algebras* |
|---|
| *Jordan algebra of* $n \times n$ *self-adjoint real matrices.* |
| *Jordan algebra of* $n \times n$ *self-adjoint complex matrices.* |
| *Jordan algebra of* $n \times n$ *self-adjoint quaternionic matrices.* |
| *Jordan algebra of* $3 \times 3$ *self-adjoint octonionic matrices: Albert algebra.* |
| *Spin factor algebra* $JSpin^{+}$ *on the space* $\mathbb{R}1 \oplus \mathbb{R}^{n}$ *for* $n \geq 2$. |

# #6. KV algebra

## Symmetries of the tangent space

Let $\mathscr{P}$ be a cone.

At any point of $\mathscr{P}$, the tangent sheaf $(T\mathscr{P}, \circ\,)$ forms a **pre-Lie algebra.** The multiplication is given in local affine coordinates as follows:

$$(X \circ Y)^i = -\sum_{j,k} \Gamma^i_{jk}(x) X^j Y^k \quad 1 \leq i \leq n,$$

where $\Gamma^i_{jk} = \frac{1}{2}\sum_l \partial_{jkl}\Phi g^{li}$ .

# Symmetric spaces
# (Lie algebras/ groups)

# Symmetric space classification

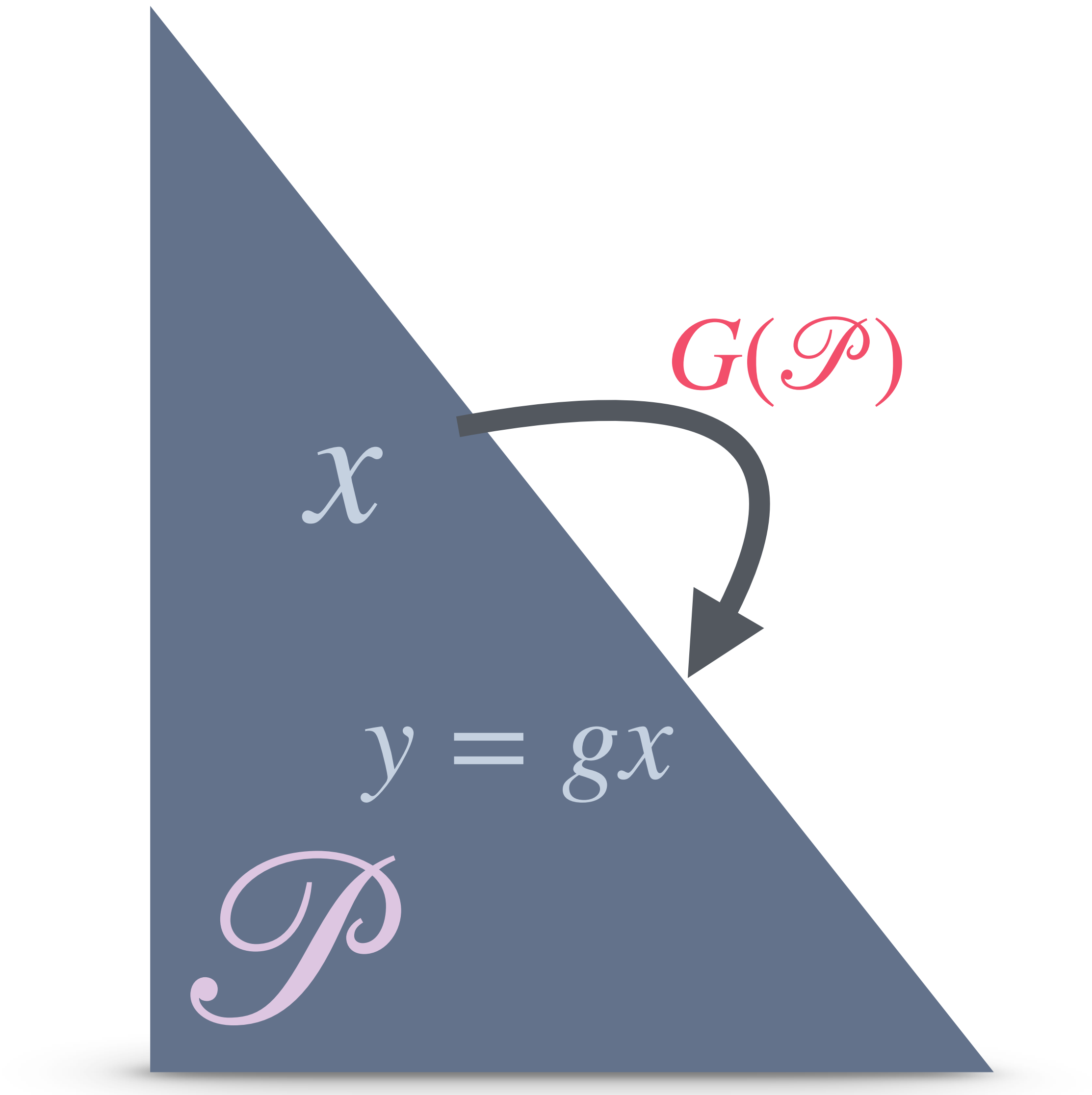
| Type | Property |
|---|---|
| Compact | Non negative sectional curvature |
| Non compact | Non positive sectional curvature |
| Euclidean | Vanishing curvature |

**Definition.**

A Riemannian symmetric space $M$ is diffeomorphic to a homogeneous space $G/K$, where: $G$ is a connected Lie group with an involutive automorphism, whose fixed point set is essentially the compact subgroup $K \subset G$

# Automorphism group

- Let $V$ be a real affine space.
- Let $\mathscr{P} \subset V$ be the convex cone.




$G(\mathscr{P})$ - group of all automorphisms;

$G_e = K(\mathscr{P})$ - stability subgroup for some point $e \in \mathscr{P}$

$T(\mathscr{P})$ - maximal connected triangular subgroup of $G(\mathscr{P})$

$\Longrightarrow$

- Then, we have $G(\mathscr{P}) = K(\mathscr{P}) \cdot T(\mathscr{P})$, where $K(\mathscr{P}) \cap T(\mathscr{P}) = e$ and the group $T$ acts simply transitively.

# Lie algebra version

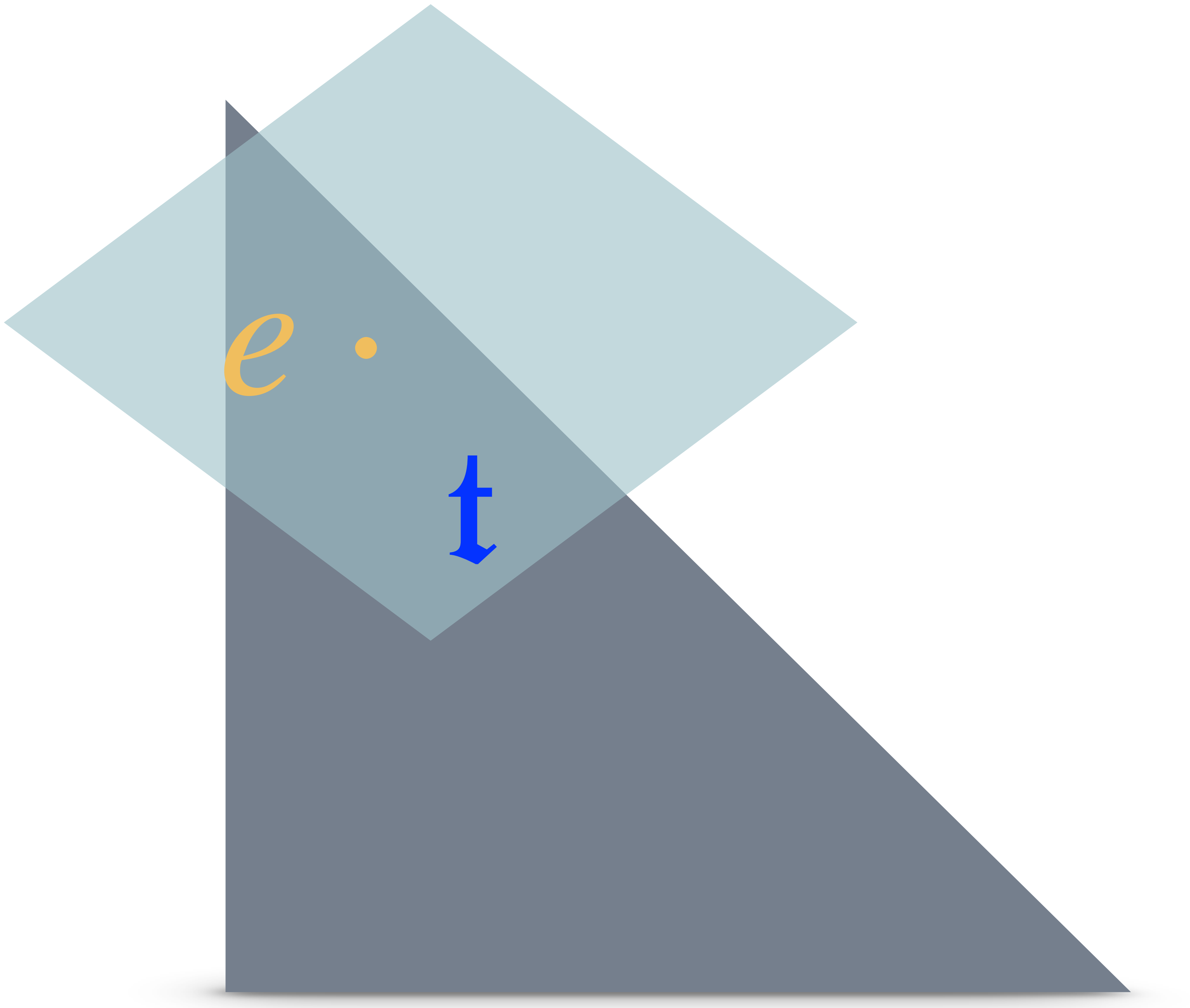
- We obtain $\mathfrak{g} = \mathfrak{K} \oplus \mathfrak{t}$, where:

- $\mathfrak{t}$ identified with the tangent space of $\mathscr{P}$ at $e$.

- $\mathfrak{K}$ is the Lie algebra associated to $K(\mathscr{P})$

$$[\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{K},$$

$$[\mathfrak{K}, \mathfrak{t}] \subset \mathfrak{t}\,.$$



# Classification

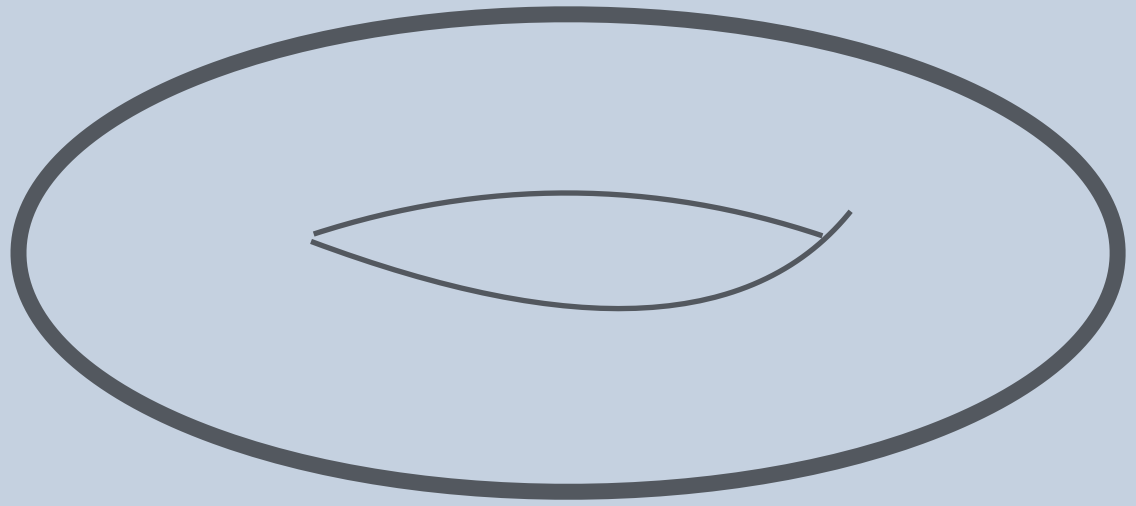
| $\mathcal{J}$ | $\Omega$ | $\mathfrak{g}$ | $\mathfrak{k}$ | $dim\mathcal{J}$ | rank $\mathcal{J}$ | $d$ |
|---|---|---|---|---|---|---|
| $Sym(n,\mathbb{R})$ | $\mathscr{P}_n(\mathbb{R})$ | $\mathfrak{sl}(n,\mathbb{R})\oplus\mathbb{R}$ | $\mathfrak{o}(\mathrm{n})$ | $\frac{1}{2}n(n+1)$ | $n$ | 1 |
| $Herm(n,\mathbb{C})$ | $\mathscr{P}_n(\mathbb{C})$ | $\mathfrak{sl}(n,\mathbb{C})\oplus\mathbb{R}$ | $\mathfrak{su}(\mathrm{n})$ | $n^2$ | $n$ | 2 |
| $Herm(n,\mathbb{H})$ | $\mathscr{P}_n(\mathbb{H})$ | $\mathfrak{sl}(m,\mathbb{H})\oplus\mathbb{R}$ | $\mathfrak{su}(n,\mathbb{H})$ | $n(2n-1)$ | $n$ | 4 |
| $\mathbb{R}\times\mathbb{R}^{n-1}$ | $\Lambda_n$ | $\mathfrak{o}(1,n-1)\oplus\mathbb{R}$ | $\mathfrak{o}(\mathrm{n}\text{-}1)$ | $n$ | 2 | $n-2$ |
| $Herm(3,\mathbb{O})$ | $\mathscr{P}_3(\mathbb{O})$ | $\mathfrak{e}_{(-26)}\oplus\mathbb{R}$ | $\mathfrak{f}_4$ | 27 | 3 | 8 |

# Some Lie groups

$$
\begin{aligned}
\mathrm{SL}_n(\mathbb{R}) &= \{x \in \mathrm{GL}_n(\mathbb{R}) \ : \ \det\ x = 1\} \\
\mathrm{SL}_n(\mathbb{C}) &= \{x \in \mathrm{GL}_n(\mathbb{C}) \ : \ \det\ x = 1\} \\
\mathrm{O}(n) &= \{x \in \mathrm{GL}_n(\mathbb{R}) \ : \ {}^t x\, x = I_n\} \\
\mathrm{SO}(n) &= \mathrm{O}(n) \cap \mathrm{SL}_n(\mathbb{R}) \\
\mathrm{O}(p,q) &= \{x \in \mathrm{GL}_{p+q}(\mathbb{R}) \ : \ {}^t x\, I_{p,q}\, x = I_{p,q}\} \\
\mathrm{SO}(p,q) &= \mathrm{O}(p,q) \cap \mathrm{SL}_{p+q}(\mathbb{R}) \\
\mathrm{U}(n) &= \{x \in \mathrm{GL}_n(\mathbb{C}) \ : \ x^*\, x = I_n\} \\
\mathrm{SU}(n) &= \mathrm{U}(n) \cap \mathrm{SL}_n(\mathbb{C})
\end{aligned}
$$

# The KV cones are non-compact symmetric domains.

— The sectional curvature is 0 on an **algebraic torus**



— The sectional curvature is <0 otherwise (=Cartan—Hadamard space)

# Koszul domains & Souriau's thermodynamics

## Part 2

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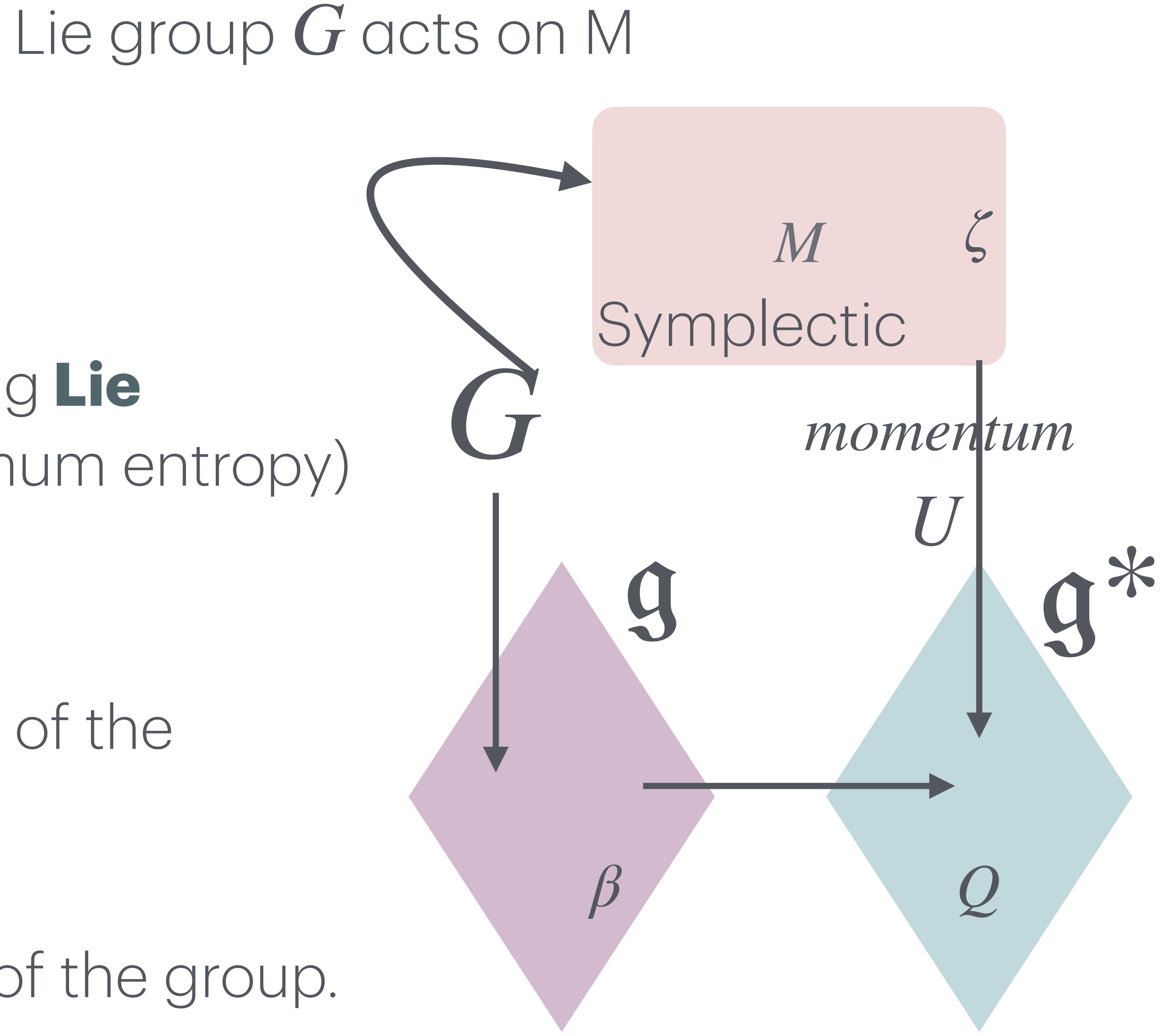
# Souriau model vs Probability theory

# Souriau's model

## (Applications of the KV cone)




Statistical mechanics - Souriau's model consists in considering **Lie groups thermodynamics** of dynamical systems. The (maximum entropy) Gibbs density is covariant wrt the action of the Lie group $G$.

- $\beta$ is a (geometric) temperature - element of a Lie algebra $\mathfrak{g}$ of the group.
- $Q$ is a (geometric) heat - element of a dual Lie algebra $\mathfrak{g}^*$ of the group.

# Some relations

- $I(\beta) = -\dfrac{\partial^2 \Phi}{\partial \beta^2}$ where $\Phi(\beta) = -\int_M \exp - \langle \beta, U(\zeta) \rangle d\lambda$ (KV function)
- $S(Q) = \langle \beta, Q \rangle - \Phi(\beta)$ where $Q = \dfrac{\partial \Phi(\beta)}{\partial \beta} \in \mathfrak{g}^*$ and $\beta = \dfrac{\partial S(Q)}{\partial Q} \in \mathfrak{g}$

**Souriau Entropy** — Heat — 1/ temperature

$$dS = \beta dQ, \quad \beta = \frac{1}{T}$$

Legendre transformation

$$Q = \frac{\partial \Phi(\beta)}{\partial \beta} \in \mathfrak{g}^* \longleftrightarrow \beta = \frac{\partial S(Q)}{\partial Q} \in \mathfrak{g}$$

# Adjoint representation of Lie groups

## Recollections

- $G$ - Lie group.
  - Let $\Psi : G \to Aut(G)$

$$g \mapsto \Psi_g(h) = ghg^{-1}$$

$$Ad_g : (d\Psi_g)_e : \mathfrak{g} \to \mathfrak{g}$$

$$X \mapsto Ad_g(X) = gXg^{-1}$$

$$ad = T_eAd : T_eG \to End(T_eG)$$

$$X, Y \in T_eG \mapsto ad_X(Y) = [X, Y]$$

# Coadjoint representation

Dual of the adjoint representation.

$$\forall g \in G,\ Y \in \mathfrak{g},\ F \in \mathfrak{g}^* \implies$$

$$\langle Ad_g^* F, Y \rangle = \langle F, Ad_{g^{-1}} Y \rangle$$

# Coadjoint orbits

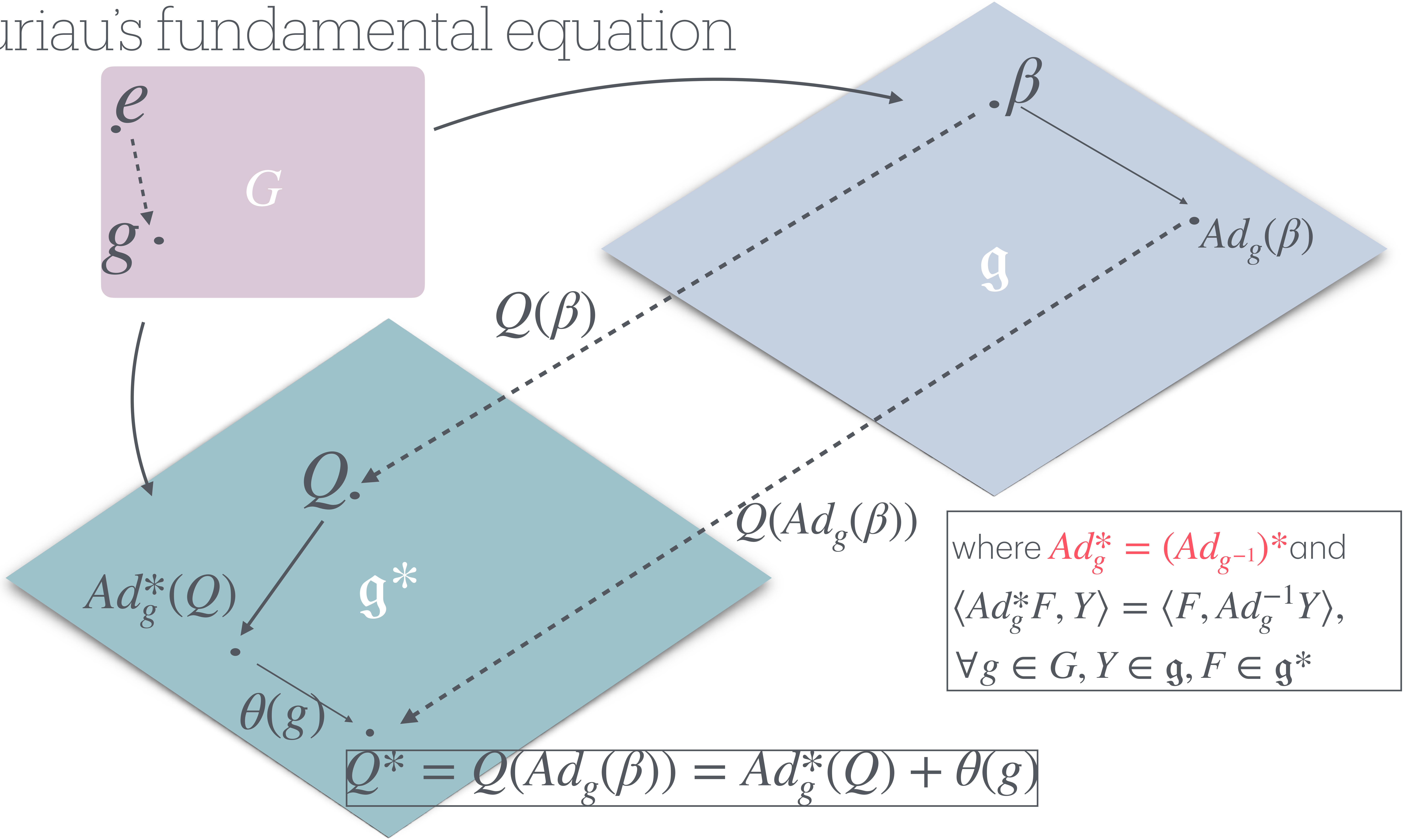
Given $F \in \mathfrak{g}^*$, a coadjoint orbit is given by $O_F = \{Ad_g^* F,\ g \in G\} \subset \mathfrak{g}^*$.

**REMARK:** Coadjoint orbits carry a symplectic structure

# Kostant—Kirillov—Souriau

A symplectic maniflold
(homogeneous under the action of a Lie group)
is isomorphic (up to covering) to a coadjoint orbit, possibly affine.

# Souriau's fundamental equation




where $Ad^*_g = (Ad_{g^{-1}})^*$ and

$\langle Ad^*_g F, Y \rangle = \langle F, Ad^{-1}_g Y \rangle$,

$\forall g \in G, Y \in \mathfrak{g}, F \in \mathfrak{g}^*$

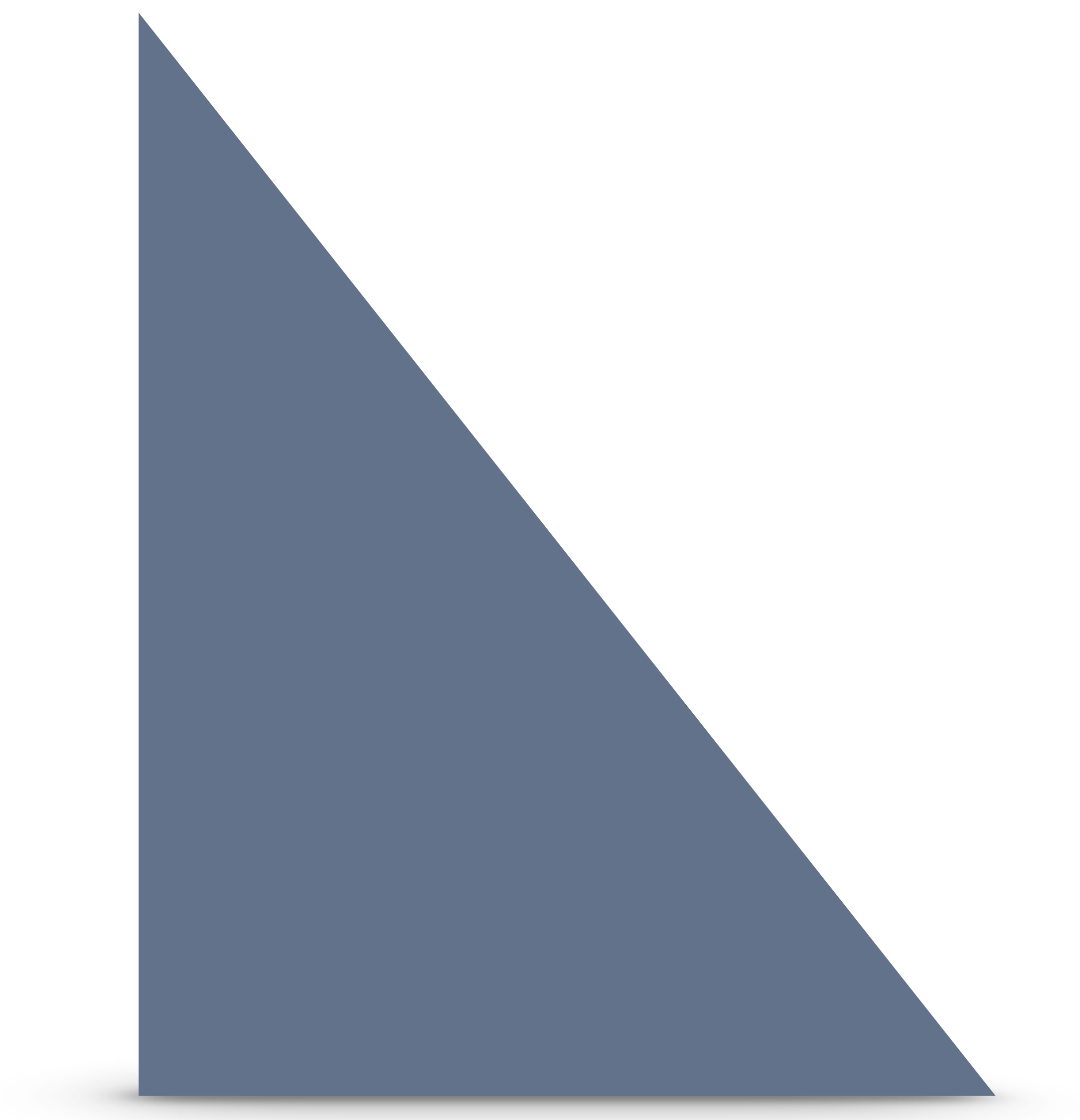
$$Q^* = Q(Ad_g(\beta)) = Ad^*_g(Q) + \theta(g)$$

# Information geometry / Gibbs density

# From KV cone to information geometry.

The (Gibbs) density is given by:

$$P(\zeta) = \frac{\exp\langle -U(\zeta), \beta\rangle}{\int_M \exp\langle -U(\zeta), \beta\rangle d\lambda_\omega},$$



$\mathscr{P}$

Where:

$$U : M \to \mathfrak{g}^*$$

$$Q = \frac{\partial \Phi(\beta)}{\partial \beta} = \int_M U(\zeta) p(\zeta) d\lambda_\omega$$

$$\Phi = -\log \int_M \exp - \langle U(\zeta), \beta\rangle d\lambda$$ ⟷ KV potential !

# Coadjoint representation

Let $S : \mathfrak{g}^* \to , \mathbb{R} \quad Q \mapsto S(Q)$.

Souriau entropy $S(Q)$ has a property of invariance:

$$S(Q(Ad_g(\beta))) = S(Q),$$

where

$$Q(Ad_g(\beta)) = Ad_g^*(Q) + \boxed{\theta(g)} \longleftrightarrow \text{Cocyle}$$

# Entropy

- $H$ - hamiltonian $\forall H : \mathfrak{g}^* \rightarrow \mathbb{R}$ ; $S : \mathfrak{g}^* \rightarrow \mathbb{R}$ - Souriau entropy. Then, $\{S, H\}(Q) = 0$

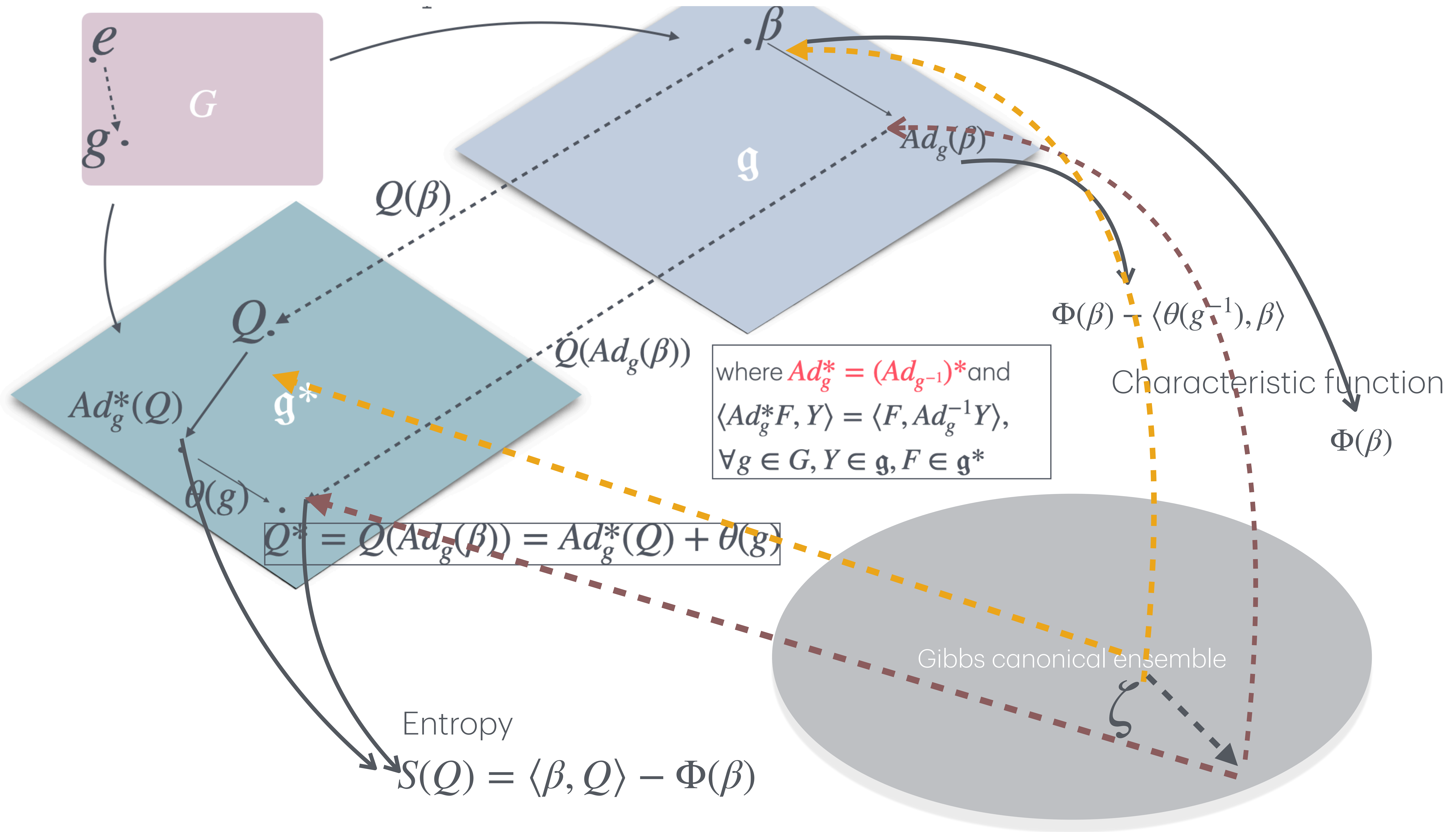
Souriau equation => $ad^*_{\frac{\partial S}{\partial Q}} Q + \Theta\left(\frac{\partial S}{\partial Q}\right) = 0$

Constant structures

$$\{S, H\}(Q) = \left\langle Q, \left[\frac{\partial S}{\partial Q}, \frac{\partial H}{\partial Q}\right]\right\rangle = -C_{ij}^{k} Q_k \frac{\partial S}{\partial Q_i} \frac{\partial H}{\partial Q_j}$$

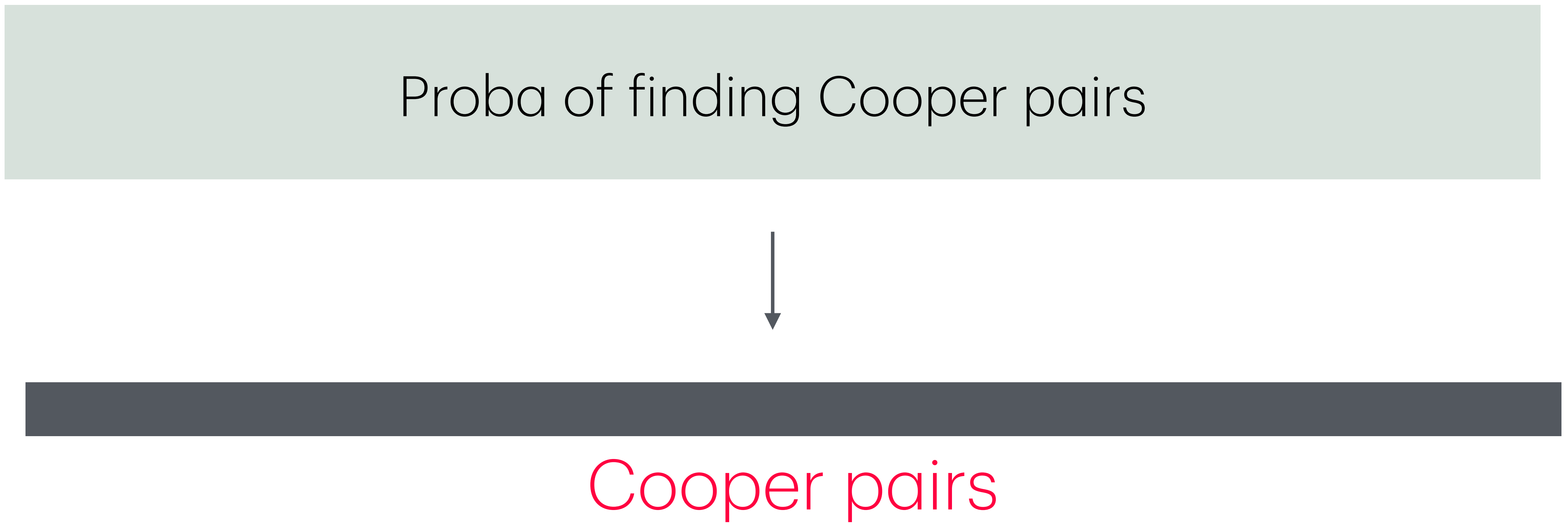
By putting $\tilde{\Theta}(X, Y) = \langle \Theta(X), Y \rangle$ where $\Theta(\beta) = \dfrac{\partial \Phi(\beta)}{\partial \beta}$ we get:

$$\{S, H\}_{\tilde{\Theta}} = \left\langle Q, \left(\frac{\partial S}{\partial Q}, \frac{\partial H}{\partial Q}\right)\right\rangle + \Theta\left(\frac{\partial S}{\partial Q}, \frac{\partial H}{\partial Q}\right) = 0$$


$e$
$g$
$G$
$\beta$
$\mathfrak{g}$
$Ad_g(\beta)$
$Q(\beta)$
$Q$
$Ad_g^*(Q)$
$\mathfrak{g}^*$
$\theta(g)$
$Q(Ad_g(\beta))$
$Q^* = Q(Ad_g(\beta)) = Ad_g^*(Q) + \theta(g)$
where $Ad_g^* = (Ad_{g^{-1}})^*$ and
$\langle Ad_g^* F, Y \rangle = \langle F, Ad_g^{-1} Y \rangle$,
$\forall g \in G, Y \in \mathfrak{g}, F \in \mathfrak{g}^*$
$\Phi(\beta) - \langle \theta(g^{-1}), \beta \rangle$
Characteristic function
$\Phi(\beta)$
Gibbs canonical ensemble
$\zeta$
Entropy
$S(Q) = \langle \beta, Q \rangle - \Phi(\beta)$

# Landau—Ginzburg theory & Souriau's thermodynamics

## Part 3

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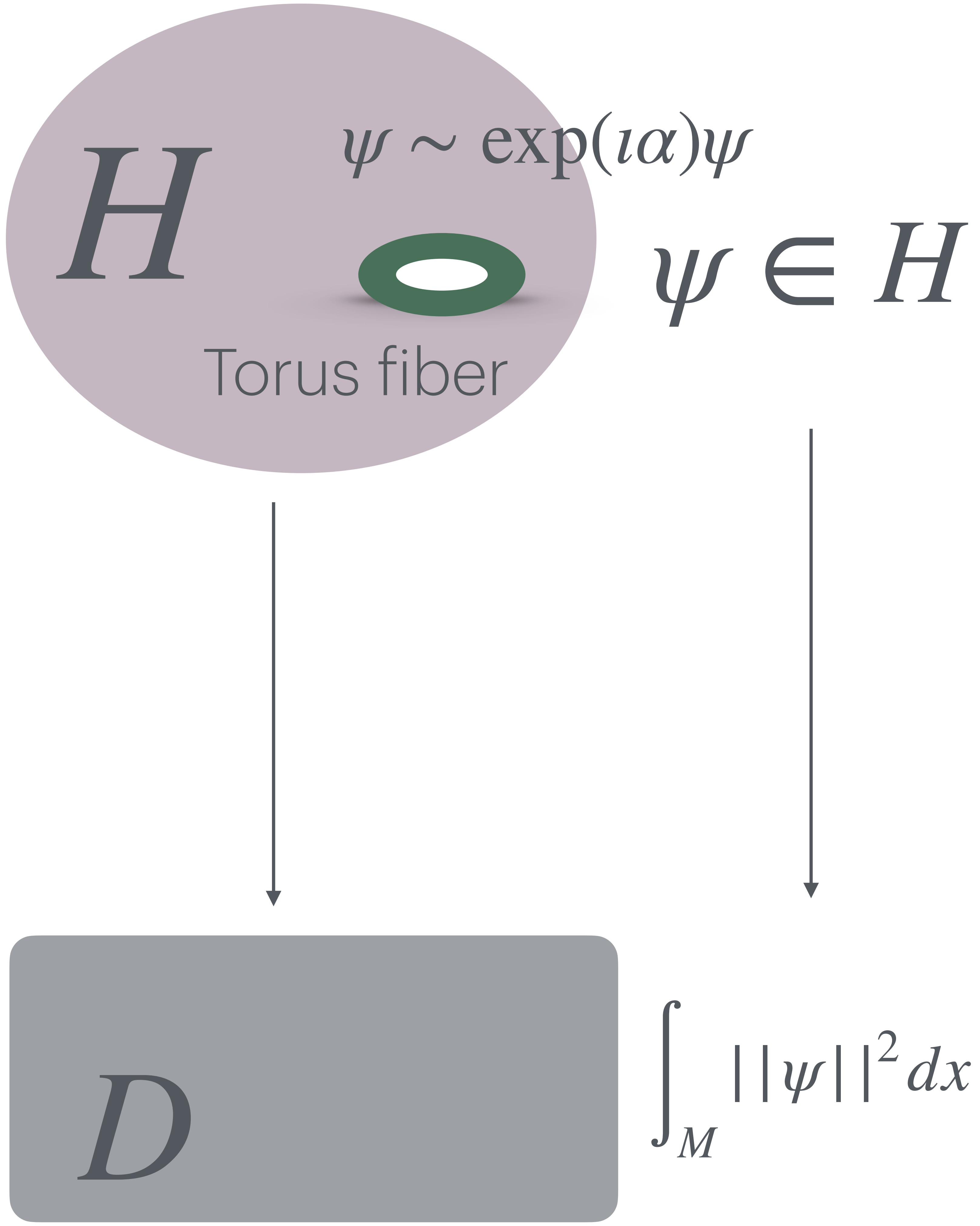
# LG theory à la Koopman-von Neuman

- Hilbert space $H$ parameterised by a real domain $D$ (space of density of probabilities).
- $\pi : H \to D$
- $\psi \in H$ is a wave function (complex valued $L^2$ integrable function.



# LG free energy + thermo-?

- LG free energy can be represented in a symplectic manifold with the phase space content. This is achieved via a thermo group, Lie group structure associated with the thermofield dynamics and C* algebras.

- The Euclidean group is a subgroup of the Galilei group.

# Euclidean Lie algebra on $H$

**Euclidean Lie group** = semi-direct product of the group of rotations and group of translations, preserving the norm.

**Eulcidean Lie algebra:** Generators of rotations $l_i$; generators of space translations $p_j$

Relations:

$$[l_i, l_j] = \imath\hbar\delta_{ijk}l_k$$

$$[l_i, p_j] = \imath\hbar\delta_{ijk}p_k$$

**Prop**. *The vector space of self-adjoint linear operators acting on the Hilbert space $H$ forms an* ***Eulcidean-Lie algebra.***

# LG - Free energy

$$F_s(r) = F_0 - \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m}|-\imath\hbar\nabla - q\frac{A}{c}\psi|^2 - \int_0^{B_a} M \cdot dB_a$$

- $\alpha, \beta, m$ positive constants;
- $F_0$ | free energy density of the normal state;
- $-\alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4$ is a Landau form for the expansion of free energy vanishing at a second-order phase transition;
- the term in $|\nabla|\psi|^2$ represents an increase in energy caused by a spatial variation of the order parameter.
- the term $\int_0^{B_a} M \cdot dB_a$ represents the increase in the superconducting free energy.

# LG equation

$$[\frac{1}{2m}(-\imath\hbar\nabla - q\frac{A}{c})^2 - \alpha\psi + \beta|\psi|^2]\psi = 0$$

**Prop.** Stationary solutions to the LG equation are invariant under the symmetry group of Galilei.

# Theorem#

- The space of density of probabilities $D$ is a Monge—Ampere domain i.e. everywhere locally the Elliptic Monge—Ampere operator (EMA) is satisfied:

- e.w locally there exists a unique function $\Phi$ such that $\det \mathrm{Hess}\Phi = f$, where $f$ is positive or null real valued function.

**Ex**: The KV cone is a EMA domain

# State space for n-quantum system

- One can represent the state space of an n-dimension quantum system by the set of $n \times n$ positive (semi)definite **complex** matrices of trace 1. These matrices are known as density matrices.

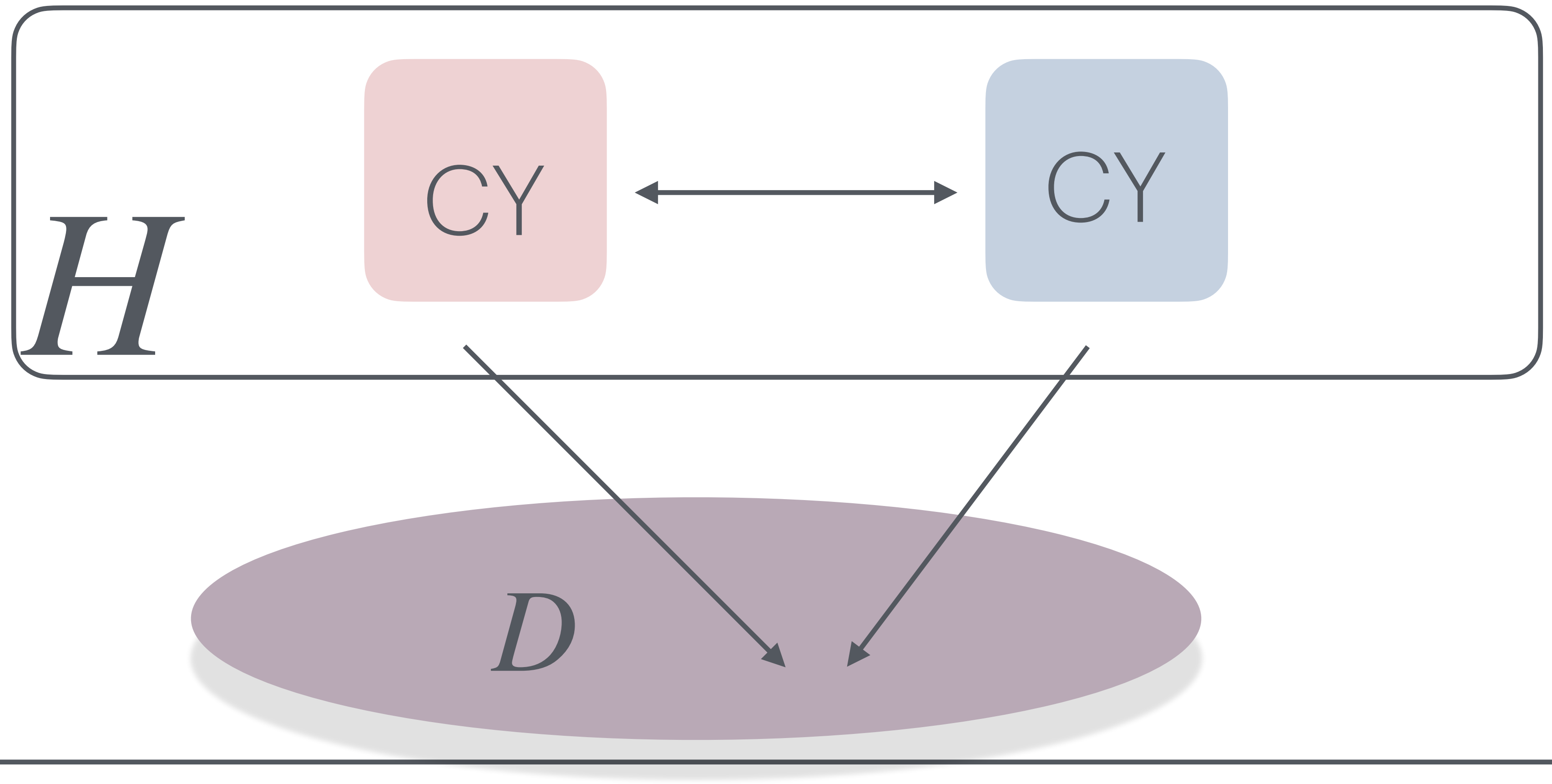
- Relaxing the condition of Tr=1 gives the complex KV cone.

# Theorem #

- There exists an equivalence of categories  of the  complex KV cones and the von Neumann algebras.

# Last Theorem#

- LG theory plays an important role in Mirror symmetry.




**Theoreme** Combe 24.

There exists a Monge—Ampere domain parametrising a pair of mirror dual

Calabi—Yau manifolds . The Monge—Ampere domain is <u>a space of densities of probabilities</u>. The construction forms a torus fibration.

As an example one can take the real KV cone. It parametrises a complex Torus

( simplest type of Calabi Yau manifold)

THANKS