Geometry of Integrable Hamiltonian Systems

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Origin & structure

& Non-Hamiltonian

Geometry of Integrable Hamiltonian Systems with symmetry

0. What is integrability?

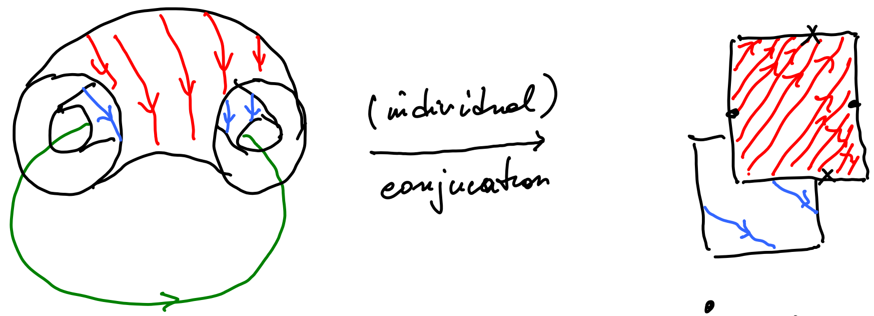
1. Hamiltonian case
   - Algebra of first integrals
     - Complete integrability
     - Non-commutative / Super integrability
   - Symplectic reduction

2. Non-Hamiltonian cases:
   - Algebra of first integrals & dynamical symmetries
   - Integrability via symmetry and reductions

Many things

Dynamical point of view: quasi-periodic dynamics

Vector field X on manifold M, $\dim M = d_M$



$\dot{\alpha} = \omega$

$\alpha \in \mathbb{T}^n, \omega \in \mathbb{R}^n$

1. Hamiltonian systems

(M symplectic, $d_M = 2n$, $X = X_h$ Hamiltonian)

A. First integrals point of view

2 "criteria"

- Complete integrability (Liouville-Mineur-Arnold)

First integrals $f = (f_1, \dots, f_n)$
- independent
- compact + connected fibers
- in involution: $\{f_i, f_j\} = 0$

$\Longrightarrow$ fibers of f are
- invariant
- $\simeq \mathbb{T}^n$

} and flow is quasi-periodic

- Lagrangian

↳ action-angle coords

$(a, \alpha) \in \mathbb{R}^n \times \mathbb{T}^n$

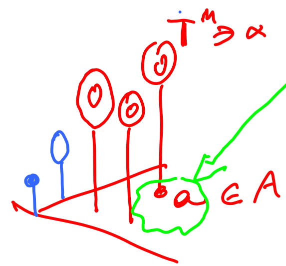
$\sigma = \sum_i da_i \wedge d\alpha_i$

$h = h(a)$

$$\begin{cases} \dot{a} = 0 \\ \dot{\alpha} = \dfrac{\partial h}{\partial a}(a) =: \omega(a) \end{cases}$$

Fibration $\quad M \xrightarrow{\quad} A \simeq f(M) \subseteq \mathbb{R}^n$

(Not necessarily a $\mathbb{T}^n$-principal bundle)



$\rightsquigarrow$

- action-angle coords are semi-locally defined
- A affine
- Obstructions to top. triviality (Nekhoroshev 1972, Duistermaat 1980)

Central & very important notion in Ham. mechanics, symplectic geometry, Physics (quantization, ...), Ham. PDE's, KAM theory, ...

- Non Commutative integrability (Nekhoroshev 1971, Mischenko–Fomenko 1978)

Completely integrable systems with additional first integrals

- Point in central force field ($M=\mathbb{R}^6$, 4 first integrals $\rightsquigarrow \mathbb{T}^2$)
- Kepler (like ↑ but 5 first integrals $\rightsquigarrow \mathbb{T}^1 \rightsquigarrow$ periodic flow)
- "Euler–Poinsot" rigid body

First integrals $f=(f_1,\dots,f_{2n-k})$ — independent; compact + connected fibers; $\{f_i,f_j\}=P_{ij}(f)$; rank $P = 2(n-k)$

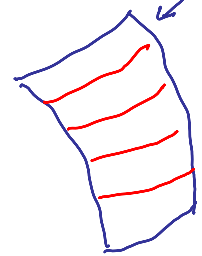
$(1\le k\le n)$

$\Longrightarrow$ fibers of f are

- invariant
- $\approx \mathbb{T}^k$

} and flow is quasi-periodic

- isotropic
- fibration by $f=\text{const}$ has (coisotropic) polar foliation



- $B := f(M)$ Poisson mfld
  $f: M\to B$ Poisson morphism

$\Rightarrow$ 2 foliations (fibrations) of M

- isotropic (the invariant tori)
- coisotropic (?)

$$\begin{tikzcd} & M \arrow[dl, "f"'] \arrow[dr] & \\ f(M)=B \arrow[rr] & & ? \end{tikzcd}$$

(dual pairs, bifoliations / bifibrations)

- The two fibrations (Dazord-Delzant 1987, Karasev-Maslov 1993)

Semi-local description : generalized action-angle coordinates $(a,\alpha,p,q)\in\mathbb{R}^k\times\mathbb{T}^k\times\mathbb{R}^{n-k}\times\mathbb{R}^{n-k}$

$$\sigma = \sum_i da_i\wedge d\alpha_i + \sum_j dp_j\wedge dq_j$$

$$h = h(a,\cancel{p},\cancel{q},\cancel{\alpha}) \qquad \dot a = 0,\ \dot p = 0,\ \dot q = 0,\ \dot\alpha = \frac{\partial h}{\partial a}(a) =: \omega(a)$$

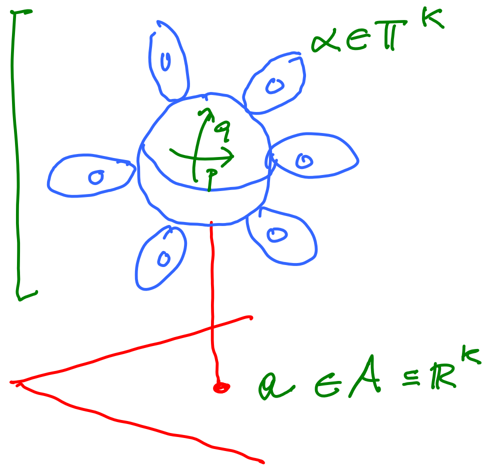
$$f \rightsquigarrow (a,p,q) \qquad \underset{\mathbb{T}^k}{(a,p,q)=\text{const}} \quad \perp^{\sigma} \quad a = \text{const}$$

coisotropic fibers of M
‖
symplectic leaves of B
$\Rightarrow a$: Casimirs of B

Global structure

$$\begin{tikzcd} & (a,\alpha,p,q)\in M \arrow[dl, "f"'] \arrow[dr, "c"] & \\ (a,p,q)\in B = f(M) \arrow[rr, "s"'] & & A \ni a \end{tikzcd}$$

$c^{-1}(a)$



- Flowers $c^{-1}(a)$: set of tori with = frequencies
- Petals : invariant tori
- Centers : symplectic leaves

- Complete integrability : special case of NCI (general theory of integrability for Hamiltonian systems)

$X=0$ → 0

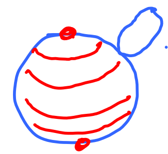
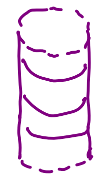
Periodic flow → 1
("Gordon's Period-Energy relation")

C.I. ("Liouville-Mineur-Arnold theorem") → n

- Frequent questions : But are NCI systems also CI ? Yes and No

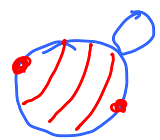
$f_1, \dots, f_{2n-k} \rightsquigarrow g_1, \dots, g_n$?

Yes : semi-locally, near individual petal / invariant torus

 $S^2 \times S^1 \approx$  $(-1,1) \times \mathbb{T}^2$

Put together the $\mathbb{T}^k$ so as to form $\mathbb{T}^n$:

local & not unique operation



No : semi-globally, near individual flowers

Describing them as CI looses information (and structure)

B. Integrability via Symplectic (MMW) Reduction

"Minority" point of view (Marsden & Weinstein 1983, Bloom 1996, ..., Modin & Viviani 2020)

Momentum map $J : M \to \mathfrak{g}^*$

- Equivariant

$$\begin{tikzcd} M \arrow[r, "J"] \arrow[d, "\Psi_g"'] & \mathfrak{g}^* \arrow[d, "Ad^*_g"] \\ M \arrow[r, "J"] & \mathfrak{g}^* \end{tikzcd}$$

- $J^{-1}(\mu)$ is G_μ-invariant
  ↳ isotropy subgroup of μ

- $M_\mu := J^{-1}(\mu)/G_\mu$

  $\dim M_\mu = \dim M - 2\dim(G_\mu)$

Action Ψ of G on M: free, proper, Hamiltonian

G-invariant Hamiltonian system

Criterion: Reduced symplectic spaces M_μ are:

- zero dimensional → Dynamics in relative equilibria of compact group is q-p.

- 2-dimensional → not fully analyzed in this context

(Zung, Bolsinov & Jovanovic: Poisson Reduction)

2. Non-Hamiltonian systems

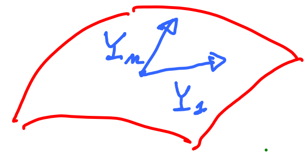
A. First integrals + Dynamical Symmetries point of view

(Bogoyavlenskij 1998, Fedorov 1999)

Criterion (almost necessary)

X has $(0 \le k \le d_M)$

- $d_M - k$ first integrals $f = (f_1, \dots, f_{d_M-k})$
  - independent
  - compact connected fibers
- k dynamical symmetries $Y_1, \dots, Y_k$, $[X, Y_i] = 0$
  - independent
  - $[Y_i, Y_j] = 0 \ \forall i,j$
  - $L_{Y_i} f_h = 0 \ \forall i, h$



$f = \text{const}$

$\Longrightarrow$ fibers of $f \approx \mathbb{T}^k$ + q.p.-flow

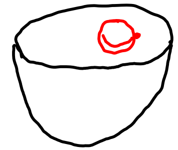
In Hamiltonian case : $(f_1, \dots, f_n) \rightsquigarrow (X_{f_1}, \dots, X_{f_n})$

$\{f_i, f_j\} = 0 \rightsquigarrow L_{X_{f_i}} f_j = 0 \ , \ [X_{f_i}, X_{f_j}] = 0$

- symplectic structure
- Noether theorem

B. Symmetry point of view

criterion detected in (nonholonomic) examples (Hermans 1995)



fig

$$\begin{tikzcd} M \arrow[r, "X"] \arrow[d, "\pi"'] & TM \arrow[d, "T\pi"] \\ M/G \arrow[r, "\overline{X}"] & T(M/G) \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\phi_t^X"] \arrow[d, "\pi"'] & M \arrow[d, "\pi"] \\ M/G \arrow[r, "\phi_t^{\overline{X}}"] & M/G \end{tikzcd}$$

- Free action of compact Lie group G on M
- X is G-invariant
- Reduced vector field $\overline{X}$ on M/G
  - $\overline{X} = 0$ $\Longrightarrow$ $\mathbb{T}^p$
  - has periodic dynamics $\Longrightarrow$ $\mathbb{T}^{p+1}$

$0 \leq p \leq \operatorname{rank}(G)$

Uses known facts on relative equilibria and relative periodic orbits of systems with symmetry (Krupa 1990, Field 1991, Hermans 1995)

Why ?

1. $\bar{X} = 0$:

$$\begin{array}{ccc} M & & \pi^{-1}(U) \simeq U \times G \\ \downarrow \pi & & \downarrow \pi \\ M/G & \supset & U \end{array}$$

G-bundle Local Trivialization

$$X|_{\pi^{-1}(U)} \rightsquigarrow \begin{cases} \dot{u} = 0 \\ \dot{g} = T_e L_g \cdot \underbrace{\xi(u)}_{\in \mathfrak{g}} \end{cases}$$

Flow of $X|_{\pi^{-1}(U)} \rightsquigarrow$ $t \mapsto (u, g\exp(t\xi))$ curve in $\{u\} \times g.T_\xi$ with

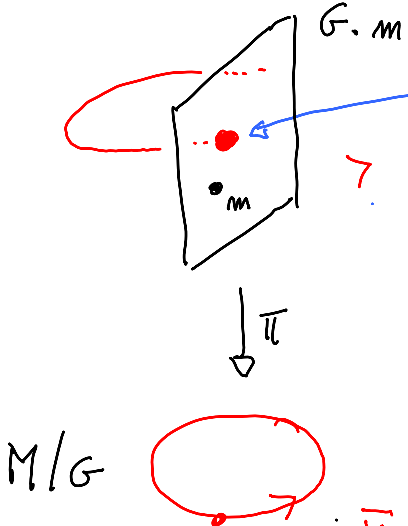
$T_\xi := \{\exp(t\xi) : t \in \mathbb{R}\}$ — closed abelian subgroup ["torus"] of G

$\dim T_\xi \leq \operatorname{rank}(G)$

$t \mapsto \exp(t\xi)$ quasi-periodic in T_ξ

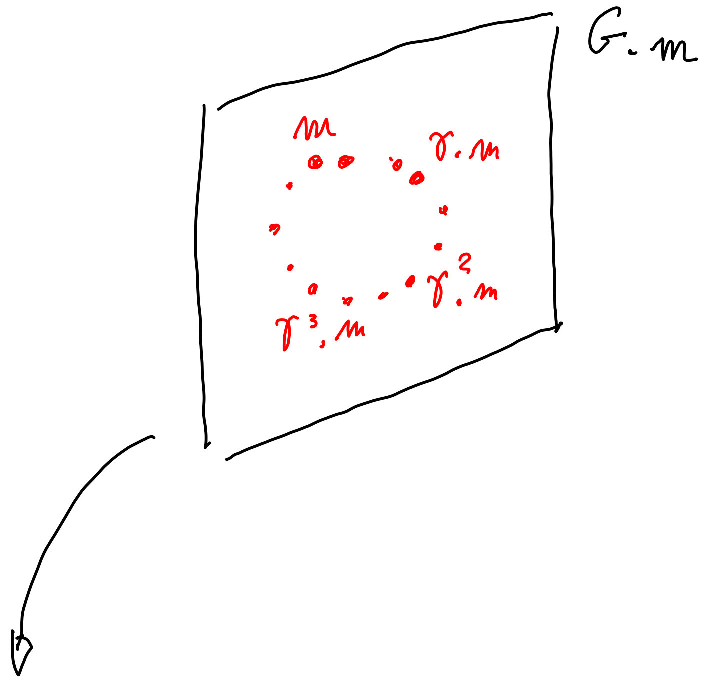
(Elementary and well known)

2. $\bar{X}$ has periodic flow



$$\phi^{X}_{T(m)}(m) = \underbrace{\gamma(m)}_{\in G \text{ "phase"}} . m$$

$$\bar{m} = \phi^{\bar{X}}_{T(m)}(\bar{m})$$



Flow takes place in submanifold $\approx \overline{\{\gamma^j(m).m : j \in \mathbb{Z}\}} \times S^1 \simeq \mathbb{T}^{p+1}$

$0 \leq p \leq \operatorname{rank}(G)$

References (to my works which contain references to the pertinent literature):

1. Noncommutatively integrable Hamiltonian systems and their symplectic geometry:

F. Fassò, *Superintegrable Hamiltonian systems: geometry and perturbations.* Acta Applicandae Mathematicae **87**, 93-121 (2005).

2. Global structure of (non Hamiltonian) systems integrable in Bogoyavlenskij's sense:

F. Fassò and A. Giacobbe, *Geometric structure of "broadly integrable" Hamiltonian systems.* Journal of Geometry and Physics **44**, 156-170 (2002)

3. Global structure of (non Hamiltonian) systems integrable via reconstruction from periodic dynamics:

F. Fassò and A. Giacobbe, *Geometry of invariant tori of certain integrable systems with symmetry and an application to a nonholonomic system*. Sigma-Symmetry, Integrability and Geometry: Methods and Applications 3, article 051 (2007)