# Tropical convexity and games: regression, separation, and beyond

Stephane.Gaubert@inria.fr

INRIA and CMAP, École polytechnique, IPP, CNRS

CALISTA workshop, ENSMP
September 2024

Based on a series of works with
Akian and Guterman (tropical geometry and games)
Allamigeon, Cohen, Katz, Quadrat, Sergeev, Singer, Skomra (tropical convexity)
Especially: Akian, Qi, Saadi (tropical linear regression), current work with Allamigeon, Boité, Molfessis (tropical SVM)

# This talk

- Interplay between tropical geometry = polyhedral geometry of algebraic or semialgebraic sets over nonarchimedean fields, zero-sum repeated games and operator methods (nonexpansive mappings).

# This talk

- Interplay between tropical geometry = polyhedral geometry of algebraic or semialgebraic sets over nonarchimedean fields, zero-sum repeated games and operator methods (nonexpansive mappings).
- Applications: solving tropical linear regression and tropical SVM problems;

# This talk

- Interplay between tropical geometry = polyhedral geometry of algebraic or semialgebraic sets over nonarchimedean fields, zero-sum repeated games and operator methods (nonexpansive mappings).
- Applications: solving tropical linear regression and tropical SVM problems;
- Main tool: tropical convexity, and in particular tropical polyhedra.

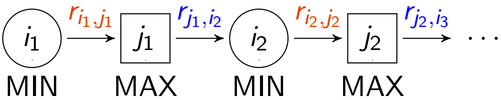
# Part I.
# Mean-payoff games

# Mean payoff games

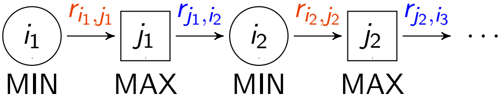
$G = (V, E)$ bipartite graph. $V = \{$ states of the game $\}$, $E = \{$ moves $\}$. $r_{ij} \in \mathbb{Z}$ price of the move $i \to j$.

Players MAX and MIN move a token, alternatively. $n$ MIN nodes, $m$ MAX nodes.

MIN always pays to MAX the price of the move (having a negative fortune is allowed)



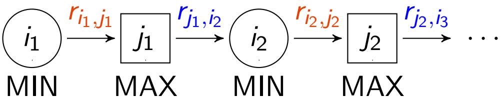
$i_1$
$r_{i_1,j_1}$
$j_1$
$r_{j_1,i_2}$
$i_2$
$r_{i_2,j_2}$
$j_2$
$r_{j_2,i_3}$
$\cdots$
MIN
MAX
MIN
MAX




Initial position $i_1 := \imath$ given. Player MAX wants to maximize his *mean payoff*, $\liminf$ of:

$$\frac{r_{i_1,j_1} + r_{j_1,i_2} + r_{i_2,j_2} + \cdots + r_{j_N,i_{N+1}}}{N} \qquad \text{when } N \to +\infty$$

while Player MIN wants to minimize her *mean loss*, the $\limsup$.




Initial position $i_1 := \imath$ given. Player MAX wants to maximize his *mean payoff*, lim inf of:

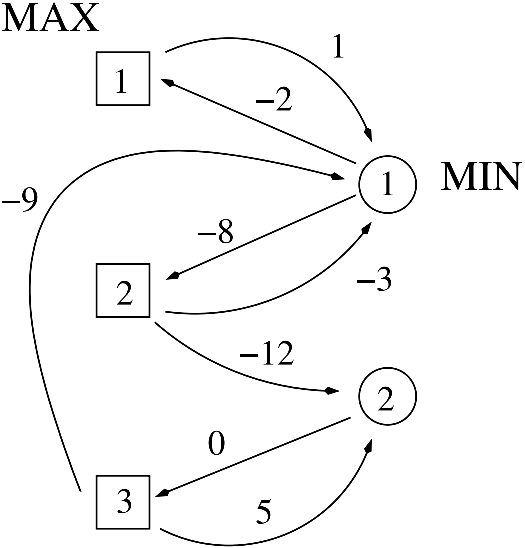
$$\frac{r_{i_1,j_1} + r_{j_1,i_2} + r_{i_2,j_2} + \cdots + r_{j_N,i_{N+1}}}{N} \qquad \text{when } N \to +\infty$$

while Player MIN wants to minimize her *mean loss*, the lim sup.

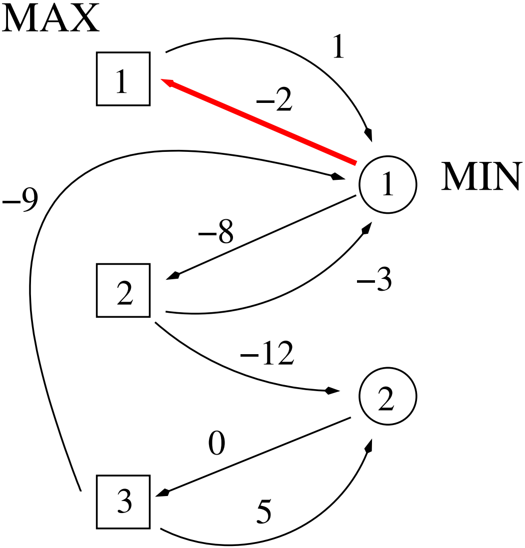
**Theorem (Ehrenfeucht and Mycielski, 1979)**

*There exists a value $\chi_\imath \in \mathbb{R}$, and positional strategies $\sigma$ and $\tau$ of Players MAX and MIN such that:*

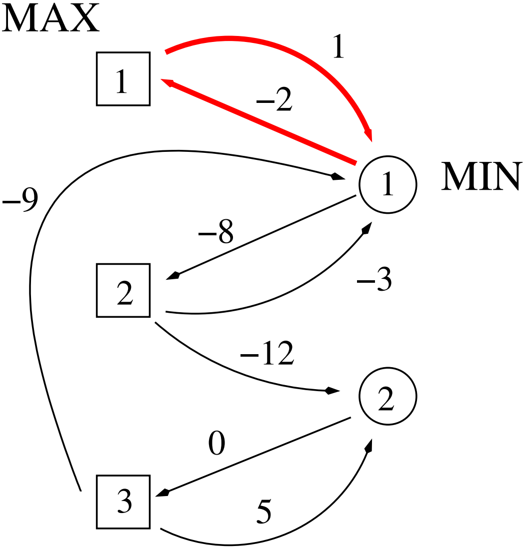
- *with strategy $\sigma$, the mean payoff of Player MAX is at least equal to $\chi_\imath$,*
- *with strategy $\tau$, the mean loss of Player MIN does not exceed $\chi_\imath$.*



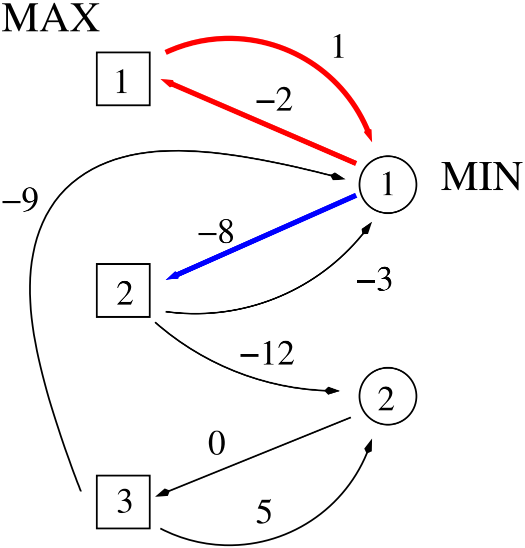

$$(\chi_1, \chi_2) = (-1, 5)$$



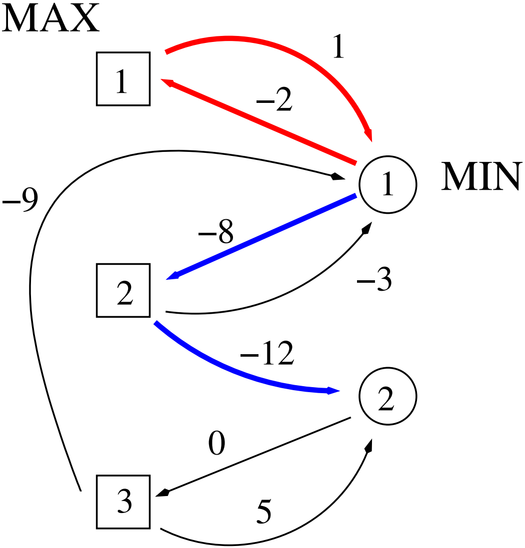

$(\chi_1, \chi_2) = (-1, 5)$



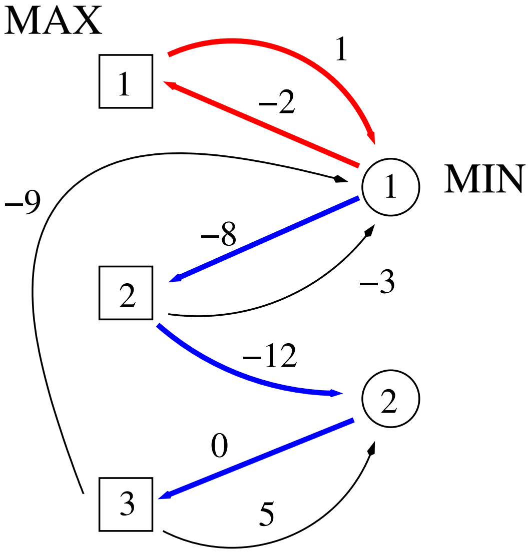

$$(\chi_1, \chi_2) = (-1, 5)$$



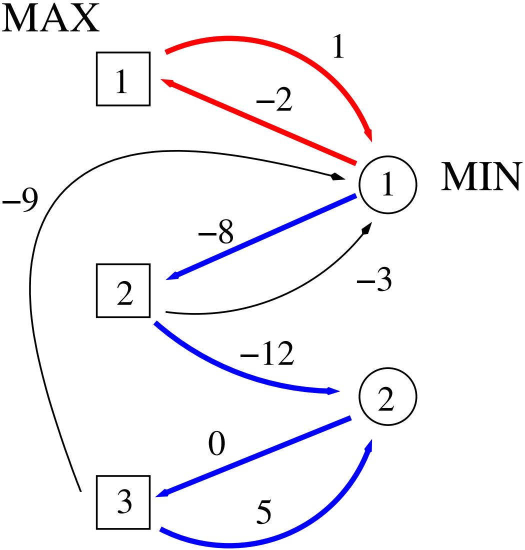

$$(\chi_1, \chi_2) = (-1, 5)$$




$$(\chi_1, \chi_2) = (-1, 5)$$




$$(\chi_1, \chi_2) = (-1, 5)$$




$$(\chi_1, \chi_2) = (-1, 5)$$

**Problem (Gurvich, Karzanov, Khachyan 88)**

*Can we solve mean payoff games in polynomial time?*

**Problem (Gurvich, Karzanov, Khachyan 88)**
*Can we solve mean payoff games in polynomial time?*

I.e., time $\leqslant$ poly($L$)? where $L$ is the bitlength of the input

$$L = \sum_{ij} \log_2(1 + |r_{ij}|)$$

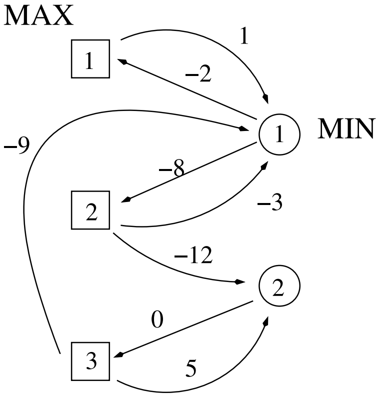
Mean payoff games in NP $\cap$ coNP Zwick and Paterson [1996], still not known to be in P.

A restricted subclass (parity games) can be solved in quasi-polynomial time, i.e., exp(poly(log($n + m$))), Calude, Jain, Khoussainov, Li, and Stephan [2017]

# Part II.
# Operator approach to mean payoff games

$v_i^k$ value of the game in horizon $k$ and initial state $(i, MIN)$.

$$v_1^k = \min(-2 + 1 + v_1^{k-1}, -8 + \max(-3 + v_1^{k-1}, -12 + v_2^{k-1}))$$
$$v_2^k = 0 + \max(-9 + v_1^{k-1}, 5 + v_2^{k-1})$$




$v^1 = (0, 0)$
$v^2 = (-11, 5)$
$v^3 = (-15, 10)$
$v^4 = (-16, 15)$
$\chi = \lim_{k\to\infty} v^k/k = (-1, 5)$

## Proposition

*The value vector $v^k$ of the game in horizon $k$ satisfies*

$$v^k = T(v^{k-1}), \qquad v^0 = 0$$

*where $T : \mathbb{R}^n \to \mathbb{R}^n$ is the Shapley operator:*

$$[T(x)]_j = \min_{i \in [m],\, j \to i} \left( r_{ji} + \max_{k \in [n],\, i \to k} (r_{ik} + x_k) \right)$$

An abstract Shapley operator is a map $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T$ is monotone (or order preserving)

$$(M): \qquad x \leqslant y \implies T(x) \leqslant T(y)$$

and additively homogeneous

$$(AH): \qquad T(se + x) = se + T(x), \qquad \forall s \in \mathbb{R}$$

where $e = (1, \ldots, 1)$ is the $n$-dimensional unit vector.

### Proposition

*$T$ is monotone and additively homogeneous iff it is nonexpansive:*

$$\mathrm{top}(T(y) - T(x)) \leqslant \mathrm{top}(y - x)$$

*where* $\mathrm{top}(z) := \max_i z_i$. *A fortiori,* $\|T(y) - T(x)\|_\infty \leqslant \|y - x\|_\infty$.

Known axioms in non-linear potential theory / game theory / PDE viscosity solutions theory, e.g. Crandall and Tartar, PAMS 80, also Kolokoltsov, Gunawardena and Keane.

General example of Shapley operator $T : \mathbb{R}^n \to \mathbb{R}^n$,

$$T_i(x) = \inf_{a \in A} \sup_{b \in B} \left( r_i^{ab} + \sum_{j \in [n]} P_{ij}^{ab} x_j \right)$$

where $P_{ij}^{ab} \geqslant 0$, $\sum_j P_{ij}^{ab} = 1$.

$T$ is the one day operator of a stochastic repeated game, in which MIN selects $a$, MAX selects $b$, MIN pays $r_i^{ab}$ in state $i$, and next state becomes $j$ with probability $P_{ij}^{ab}$.

$[T^k(0)]_i$ is the value of the standard game in horizon $k$, starting from state $i$.

$[T^k(u)]_i$ is the value of a modified game, in which MAX receives an additional payment of $u_j$ in the terminal state $j$.

We allow the inf and sup not to commute, this is the 'turn based' situation, MIN plays first, MAX plays next, and each player is informed of the previous action of the other player. In the original example of Shapley (1953), $T_i(x) = \inf_{\mu \in \Delta(A)} \sup_{\nu \in \Delta(B)} \int d\mu(a) d\nu(b) (r_i^{ab} + \sum_{j \in [n]} P_{ij}^{ab} x_j)$, where $\Delta(\cdot)$ denotes the set of probability measures on a space, i.e. players choose measures on actions rather than actions. This models the situations in which MAX and MIN play simultaneously. This reduces to the general example, replacing $A$ by $\Delta(A)$ and $B$ by $\Delta(B)$. More generally, every Shapley operator can be written as in the general example (Kolokoltsov 92), even with deterministic transitions, allowing infinite $A$ (Rubinov, Singer 01, Sparrow, and Gunawardena 04).

Theorem (Bewley, Kohlberg 76; Mertens, Neyman 01; Neyman 03; Bolte, SG, Vigeral 14)

*Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is nonexpansive in an arbitrary norm. Then, the escape rate vector*

$$\lim_{k\to\infty} T^k(0)/k$$

*does exist if $T$ is semi-algebraic (or more generally, definable in an o-minimal structure).*

Theorem (Bewley, Kohlberg 76; Mertens, Neyman 01; Neyman 03; Bolte, SG, Vigeral 14)

*Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is nonexpansive in an arbitrary norm. Then, the escape rate vector*

$$\lim_{k\to\infty} T^k(0)/k$$

*does exist if $T$ is semi-algebraic (or more generally, definable in an o-minimal structure).*

*When $T$ is a Shapley operator, this is $\lim_k v^k/k$ (limit of the normalized value vectors of the finite horizon games) and this coincides with the value vector of the mean-payoff game.*

**Theorem (Bewley, Kohlberg 76; Mertens, Neyman 01; Neyman 03; Bolte, SG, Vigeral 14)**

*Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is nonexpansive in an arbitrary norm. Then, the escape rate vector*

$$\lim_{k\to\infty} T^k(0)/k$$

*does exist if $T$ is semi-algebraic (or more generally, definable in an o-minimal structure).*

*When $T$ is a Shapley operator, this is $\lim_k v^k/k$ (limit of the normalized value vectors of the finite horizon games) and this coincides with the value vector of the mean-payoff game.*

Eg., for the above deterministic games, $T$ is piecewise linear $\implies$ trivially semialgebraic.

Some rigidity (o-minimality) assumption is indispensable, Vigeral 13.

Semi-algebraic is needed when players play simultaneously in randomized strategies (incomplete information) - Shapley's original example.

Let $T$ Shapley $\mathbb{R}^n \to \mathbb{R}^n$. Consider $F := \exp \circ T \circ \log, \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$, extends continuously to $\mathbb{R}^n_{\geqslant 0}$ (Burbanks, Nussbaum, Sparrow)

**Theorem (non-linear Collatz-Wielandt, Nussbaum 88)**

*Suppose $C$ is a closed pointed reproducing cone in a finite dimensional normed space, equipped with $x \leqslant y \stackrel{def}{\iff} y - x \in C$, $F : C \to C$, continuous, monotone and positively homogeneous, $e \in \operatorname{int} C$;*

$$\begin{aligned}
\rho(F) &:= \lim_{k\to\infty} \|F^k(e)\|^{1/k} \\
&= \max\{\lambda \in \mathbb{R}_{\geqslant 0} \mid \exists u \in C \setminus \{0\},\ F(u) \geqslant \lambda u\} \\
&= \max\{\lambda \in \mathbb{R}_{\geqslant 0} \mid \exists u \in C \setminus \{0\},\ F(u) = \lambda u\} \\
&= \inf\{\mu > 0 \mid \exists v \in \operatorname{int} C, F(v) \leqslant \mu v\}
\end{aligned}$$

Related with the Donsker-Varadhan characterization of the dominant eigenvalue. Extensions in Lemmens, Lins, Nussbaum, Wortel (2018). This is a Denjoy-Wolff type theorem. Remarkably, the nonpositive curvature condition is not needed in the case of cones. More information in the book Lemmens and Nussbaum (CUP).

# Winning certificates

Theorem ("subharmonic vectors" Akian, SG, Guterman, IJAC 2012)

*Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a Shapley operator. The following are equivalent.*

- *there exists one winning initial state $j$, meaning that*

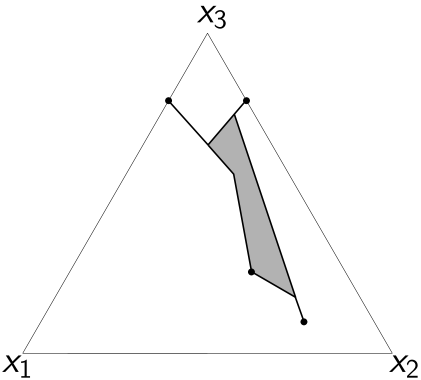
$$0 \leqslant \lim_{k\to\infty} [T^k(0)]_j/k$$

- *there exists $u \in (\mathbb{R} \cup \{-\infty\})^n$, $u \not\equiv -\infty$, and*

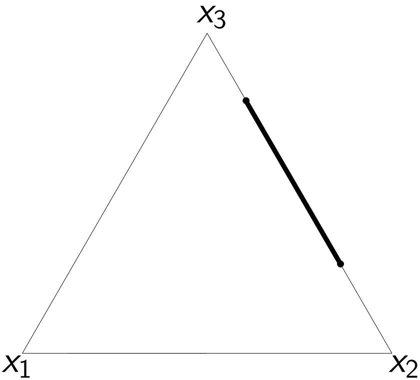
$$u \leqslant T(u)$$

*If the game is deterministic and the actions spaces are finite, the winning states are exactly the $j \in [n]$ such that there exists $u$ such that $u_j \neq -\infty$ and $u \leqslant T(u)$.*

# Space of subharmonic vectors




states 1,2,3 winning




states 2,3 winning

# Part III.
# Tropical modules / convex cones

Tropical semifield $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$, equipped with

$$\text{“}a + b\text{”} = \max(a, b) \qquad \text{“}a \times b\text{”} = a + b$$

$$\text{“}0\text{”} = -\infty, \qquad \text{“}1\text{”} = 0$$

Tropical semifield $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$, equipped with

$$\text{“}a + b\text{”} = \max(a, b) \qquad \text{“}a \times b\text{”} = a + b$$

$$\text{“}0\text{”} = -\infty, \qquad \text{“}1\text{”} = 0$$

Dual semifield: $\mathbb{R}_{\min} := \mathbb{R} \cup \{+\infty\}$, equipped with min as addition, instead of max.

The Shapley operator of a mean-payoff game can be written as

$$[T(v)]_j = \min_{i\in[m],\, j\to i} \left( -A_{ij} + \max_{k\in[n],\, i\to k} (B_{ik} + v_k) \right)$$

$$\begin{aligned} v \leqslant T(v) \iff & \max_{j\in[n]}(A_{ij} + v_j) \leqslant \max_{j\in[n]}(B_{ij} + v_j),\ i \in [m] \\ \iff & Av \leqslant Bv \end{aligned}$$

The Shapley operator of a mean-payoff game can be written as

$$[T(v)]_j = \min_{i\in[m],\ j\to i} \left( -A_{ij} + \max_{k\in[n],\ i\to k} (B_{ik} + v_k) \right)$$

$$v \leqslant T(v) \iff \max_{j\in[n]}(A_{ij} + v_j) \leqslant \max_{j\in[n]}(B_{ij} + v_j),\ i \in [m]$$

$$\iff Av \leqslant Bv$$

$T(v) = A^\sharp Bv$ where

$$(Bv)_i = \text{“}\sum_k B_{ik} v_k\text{”} = \max_k (B_{ij} + v_k) \qquad \text{tropically linear}$$

$$(A^\sharp y)_j = \text{“}\sum_i \bar{A}_{ij} y_i\text{”} = \min_i (-A_{ij} + y_i) \qquad \text{tropical adjoint} \ .$$

The Shapley operator of a mean-payoff game can be written as

$$[T(v)]_j = \min_{i\in[m],\, j\to i} \left( -A_{ij} + \max_{k\in[n],\, i\to k} (B_{ik} + v_k) \right)$$

$$v \leqslant T(v) \iff \max_{j\in[n]}(A_{ij} + v_j) \leqslant \max_{j\in[n]}(B_{ij} + v_j),\ i \in [m]$$

$$\iff Av \leqslant Bv$$

$T(v) = A^\sharp Bv$ where

$$(Bv)_i = \text{``}\sum_k B_{ik} v_k\text{''} = \max_k (B_{ij} + v_k) \qquad \text{tropically linear}$$

$$(A^\sharp y)_j = \text{``}\sum_i \bar{A}_{ij} y_i\text{''} = \min_i (-A_{ij} + y_i) \qquad \text{tropical adjoint} \ .$$

The sets of subharmonic certificates $\{v \mid Av \leqslant Bv\}$ is a tropical polyhedral cone – intersection of finitely many tropical half-spaces.

# Tropical half-spaces

Given $a, b \in \mathbb{R}^n_{\max}$, $a, b \not\equiv -\infty$,

$$H^{\leqslant} := \{x \in \mathbb{R}^n_{\max} \mid \text{“}ax \leqslant bx\text{”}\}$$

union of sectors separated by

$$H^{=} := \{x \in \mathbb{R}^n_{\max} \mid \max_{1 \leqslant i \leqslant n} c_i + x_i \text{ achieved twice}\}, \; c_i = \max(a_i, b_i)$$

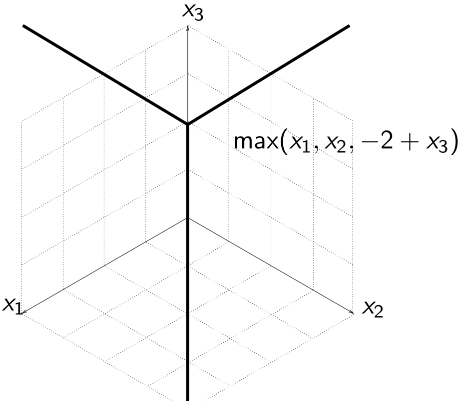
# Tropical half-spaces

Given $a, b \in \mathbb{R}^n_{\max}$, $a, b \not\equiv -\infty$,

$$H^{\leqslant} := \{x \in \mathbb{R}^n_{\max} \mid \max_{1 \leqslant i \leqslant n} a_i + x_i \leqslant \max_{1 \leqslant i \leqslant n} b_i + x_i\}$$

union of sectors separated by

$$H^{=} := \{x \in \mathbb{R}^n_{\max} \mid \max_{1 \leqslant i \leqslant n} c_i + x_i \text{ achieved twice}\},\ c_i = \max(a_i, b_i)$$



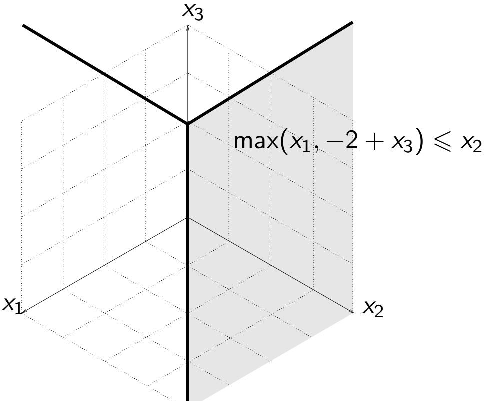
# Tropical half-spaces

Given $a, b \in \mathbb{R}^n_{\max}$, $a, b \not\equiv -\infty$,

$$H^{\leqslant} := \{x \in \mathbb{R}^n_{\max} \mid \max_{1 \leqslant i \leqslant n} a_i + x_i \leqslant \max_{1 \leqslant i \leqslant n} b_i + x_i\}$$

union of sectors separated by

$$H^{=} := \{x \in \mathbb{R}^n_{\max} \mid \max_{1 \leqslant i \leqslant n} c_i + x_i \text{ achieved twice}\},\ c_i = \max(a_i, b_i)$$



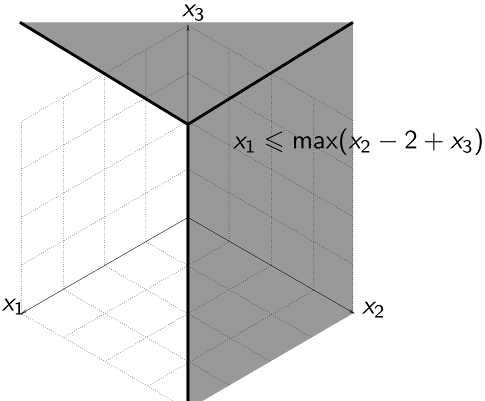
# Tropical half-spaces

Given $a, b \in \mathbb{R}^n_{\max}$, $a, b \not\equiv -\infty$,

$$H^{\leqslant} := \{x \in \mathbb{R}^n_{\max} \mid \max_{1 \leqslant i \leqslant n} a_i + x_i \leqslant \max_{1 \leqslant i \leqslant n} b_i + x_i\}$$

union of sectors separated by

$$H^{=} := \{x \in \mathbb{R}^n_{\max} \mid \max_{1 \leqslant i \leqslant n} c_i + x_i \text{ achieved twice}\}, \; c_i = \max(a_i, b_i)$$



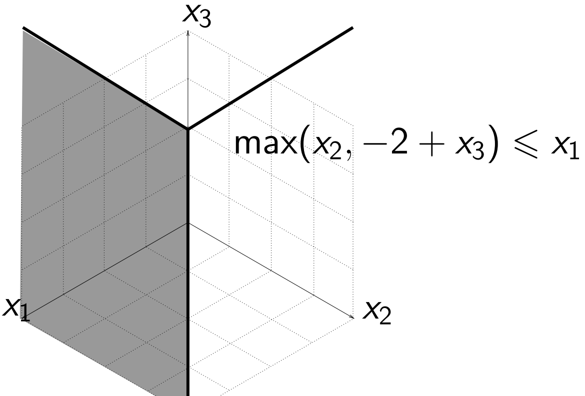
# Tropical half-spaces

Given $a, b \in \mathbb{R}^n_{\max}$, $a, b \not\equiv -\infty$,

$$H^{\leqslant} := \{x \in \mathbb{R}^n_{\max} \mid \max_{1 \leqslant i \leqslant n} a_i + x_i \leqslant \max_{1 \leqslant i \leqslant n} b_i + x_i\}$$

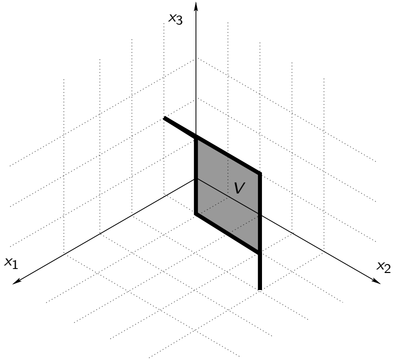
union of sectors separated by

$$H^{=} := \{x \in \mathbb{R}^n_{\max} \mid \max_{1 \leqslant i \leqslant n} c_i + x_i \text{ achieved twice}\},\ c_i = \max(a_i, b_i)$$



# Tropical polyhedral cones

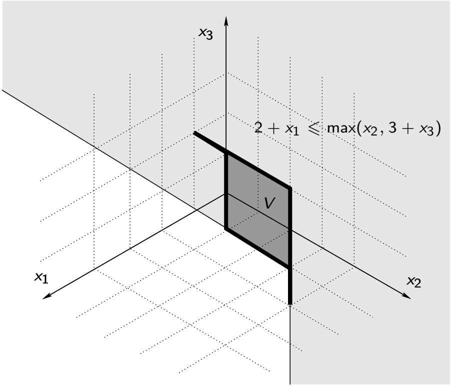
can be defined equivalently either as intersections of finitely many half-spaces or as finitely generated submodules of $\mathbb{R}^n_{\max}$.




More on external representations: Gaubert and Katz, 2011

# Tropical polyhedral cones

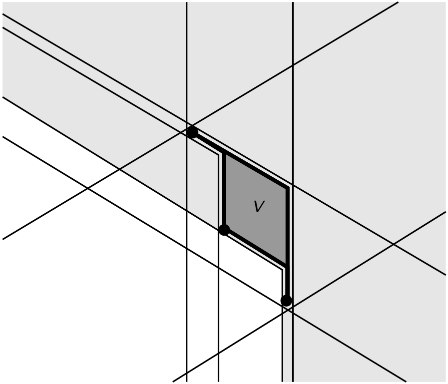
can be defined equivalently either as intersections of finitely many half-spaces or as finitely generated submodules of $\mathbb{R}^n_{\max}$.




More on external representations: Gaubert and Katz, 2011

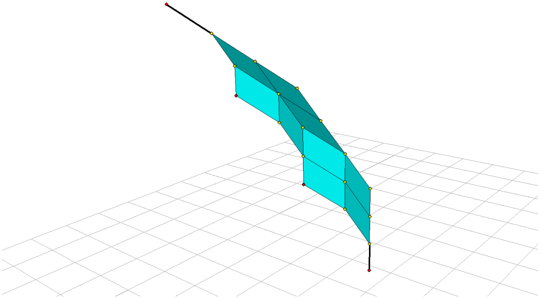
# Tropical polyhedral cones

can be defined equivalently either as intersections of finitely many half-spaces or as finitely generated submodules of $\mathbb{R}^n_{\max}$.




More on external representations: Gaubert and Katz, 2011

# A tropical polytope with four vertices



Structure of a polyhedral complex (Develin, Sturmfels) whose cells $C$ are alcoved polyhedra of $A_n$ type (Lam, Postnikov):

$$C := \{x \in \mathbb{R}^n \mid x_i - x_j \leqslant a_{ij}, \forall i, j\} \text{ , for some } a_{ij} \in \mathbb{R} \cup \{+\infty\}$$

# Part IV.
# Link between nonarchimedean and tropical convexity

Let $\mathbb{K}$ be an algebraically closed field with a nonarchimedean valuation having $\mathbb{R}$ as the value group.

E.g., generalized Puiseux series:

$$x = x(t) = \sum_{i=1}^{\infty} c_i t^{\alpha_i} ,$$

where the sequence $(\alpha_i)_i \subset \mathbb{R}$ is strictly decreasing and either finite or unbounded and $c_i$ are complex numbers.

Can take either formal series (Markwig), or rather the subfield series absolutely converging for $t$ large enough (van den Dries and Speissegger), so that:

$$\mathrm{val}(x) = \lim_{t \to \infty} \frac{\log |x(t)|}{\log t} = \alpha_1 \qquad (\text{and } \mathrm{val}(0) = -\infty) .$$

### Theorem (Kapranov, non archimedean amoebas of hypersurfaces)

*Given* $p = \sum_\alpha p_\alpha x^\alpha \in \mathbb{K}[x_1, \dots, x_n]$, *and* $X \in \mathbb{R}^n$,

$$(\exists x \in (\mathbb{K}^*)^n, \quad p(x) = 0, \quad X = \operatorname{val} x) \iff X \in H^{trop}(p)$$

$$H^{trop}(p) = \{X \in \mathbb{R}^n \mid \max_\alpha(\operatorname{val} p_\alpha + \langle \alpha, X\rangle) \textit{ attained twice}\} \ .$$

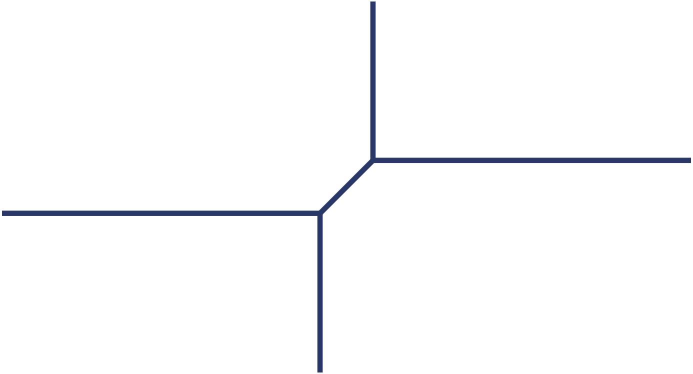
### Theorem ("Fundamental theorem of tropical geometry", see McLagan, Sturmfels)

*Let* $\mathcal{I}$ *be an ideal of* $\mathbb{K}[x_1, \dots, x_n]$. *Then, the image by the valuation of* $\{x \in (\mathbb{K}^*)^n \mid p(x) = 0, \forall p \in \mathcal{I}\}$ *coincides with*

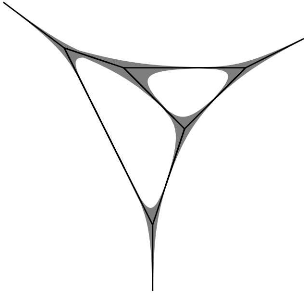
$$\bigcap_{p \in \mathcal{I}} H^{trop}(p) \ .$$

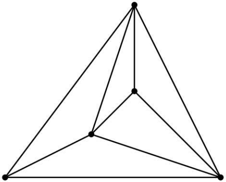
*(The intersection is achieved by a choice of finitely many p.)*

Example : The image by the valuation of $\boldsymbol{f}_1 = 0$ where $\boldsymbol{f}_1(x) = t + t^2\boldsymbol{x}_1 + t\boldsymbol{x}_2 + t\boldsymbol{x}_1\boldsymbol{x}_2$ is the tropical hypersurface associated to the polynomial $f_1 = \max(1, 2 + x_1, 1 + x_2, 1 + x_1 + x_2)$.



Fix $t = 1$ (archimedean case), so that $p = \sum_\alpha p_\alpha x^\alpha \in \mathbb{C}[x_1, \ldots, x_n]$.
Gelfand, Kapranov and Zelevinski defined the *amoeba* and $f$ to be $\{\log|x| \mid x \in (\mathbb{C}^*)^n, ; p(x) = 0\}$.
the tropical hypersurface approximates the amoeba, Passare, Rüllgard, metric estimates in Aveñado, Kogan, Nisse, Rojas

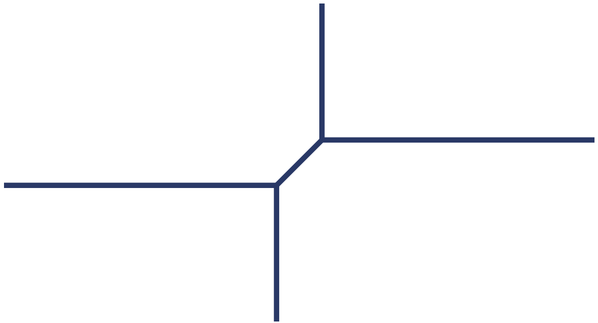




Tropical hypersurfaces appear in auction theory, cf. Baldwin and Klemperer

$$H^{\mathrm{trop}}(p) = \{X \in \mathbb{R}^n \mid \max_{\alpha}(\operatorname{val} p_\alpha + \langle \alpha, X \rangle) \text{ attained twice}\} \ .$$

An agent must choose between different bundles of elementary objects: $\alpha \in \mathbb{N}^n$, $\alpha_i =$ number of object $i$ in bundle $\alpha$, $X_i$ is price of object $i$, and $p_\alpha$ is the utility.
The tropical hypersurface is the indifference locus
$f_1 = \max(1, 2 + x_1, 1 + x_2, 1 + x_1 + x_2)$.



# Part V.
# Tropical Regression and best approximation

# What is a tropical linear space?

Answer 1. (In Optimization and Control) A tropical linear space is a tropical module, set $\mathcal{V}$ of vectors, or of functions, such that

$$\forall v, w \in \mathcal{V}, \lambda, \mu \in \mathbb{R} \cup \{-\infty\}, \sup(\lambda + v, \mu + w) \in \mathcal{V} \ .$$

Complexity = cardinality of a generating family of $\mathcal{V}$

E.g., McEneaney's maxplus method, the solution $v(t,x)$ of a HJ PDE is approximated by a supremum

$$v(t,x) \simeq \sup_{i \in I} \lambda_i(t) + v_i(x),$$

and one looks for a the "best set" of functions $v_i(x), i \in I$ with a prescribed cardinality, possibly taken in a restricted class (e.g. quadratic forms).

Continuous space version of facility location (NP-hard) - SG, McEneaney, Qu.

Includes the problem of best approximation of a convex function by a polyhedral convex function with $N$ facets, or of a convex body by a polytope with $N$ facets. See e.g. Grüber.

Finite dimensional version: tropical low rank approximation, given a $n \times m$ tropical matrix $V$, find an approximate factorization $V \simeq AB$ where $A : n \times r$ and $B : r \times m$, $(AB)_{ij} = \max_k A_{ik} + B_{kj}$, also NP-hard.

Analogous to nonnegative matrix factorization.

Answer 2. [In tropical (nonarchimedean) geometry, Speyer, Sturmfels], more restrictive.
A tropical linear space is a point of the tropical Grassmannian, which can be identified to the image by the non-archimedean valuation of a linear space over Puiseux series.
E.g. tropical hyperplanes.

# Hilbert's projective metric

$$d(x,y) = \inf\{\lambda - \mu \mid \lambda, \mu \in \mathbb{R},\ \mu + y_i \leqslant x_i \leqslant \lambda + y_i\ \forall i \in [n]\} \ .$$

Its restriction to $\mathbb{R}^n$ is induced by the Hilbert's seminorm

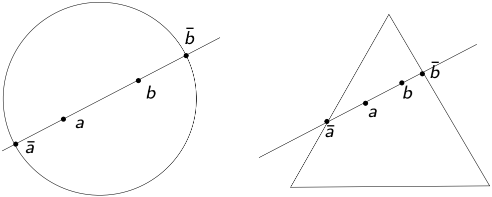
$$\|x\|_H := \max_{i\in[n]} x_i - \min_{i\in[n]} x_i \ .$$

It is a metric on the tropical projective space $\mathbb{P}(\mathbb{R}_{\max})^n$ (mod out $(\mathbb{R}_{\max})^n$ by the action of additive constants).
The one-sided Hausdorff distance from a set $A \subset \mathbb{P}(\mathbb{R}_{\max})^n$ to a set $B \subset \mathbb{P}(\mathbb{R}_{\max})^n$ is :

$$\mathrm{dist}_H(A,B) := \sup_{a\in A} \mathrm{dist}_H(a,B) \ , \quad \text{with } \mathrm{dist}_H(a,B) := \inf_{b\in B} d(a,b) \ .$$

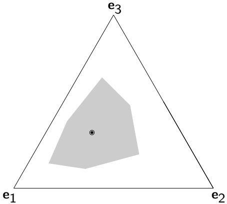
Hilbert's projective metric is a canonical choice in ntropical geometry.

# Hilbert's metric on an open convex set




$$d_H(a,b) = \log \frac{|b-\bar{a}||a-\bar{b}|}{|a-\bar{a}||b-\bar{b}|} \quad .$$

disc: Klein model of the hyperbolic space;
simplex: $d_H$ conjugate to the metric of Hilbert's seminorm, $d_H(x,y) = \|\log x - \log y\|_H$.



# Tropical linear regression

Given a collection of points $\mathcal{V} \subset \mathbb{P}(\mathbb{R}_{\max})^n$, find a hyperplane

$$\mathcal{H}_a := \{x \in (\mathbb{R}_{\max})^n \mid \max_{i \in [n]}(a_i + x_i) \text{ is achieved at least twice}\}$$

minimizing

$$\min_{a \in \mathbb{P}(\mathbb{R}_{\max})^n} \operatorname{dist}_H(\mathcal{V}, \mathcal{H}_a)$$

# Motivation: Repeated invitations to tenders (ITT)

- A public decision maker chooses the best offer given by $n$ local firms.
- Secret evaluation/preference $f_i > 0$ (**technical quality**) of each firm $i$
- For invitation $j \in [q]$, firm $i \in [n]$ asks for the price $p_{ij}$

# Motivation: Repeated invitations to tenders (ITT)

- A public decision maker chooses the best offer given by $n$ local firms.
- Secret evaluation/preference $f_i > 0$ (**technical quality**) of each firm $i$
- For invitation $j \in [q]$, firm $i \in [n]$ asks for the price $p_{ij}$
- The decision maker minimizes the weighted cost:

$$\min_{i \in [n]} p_{ij} f_i^{-1}$$

# Motivation: Repeated invitations to tenders (ITT)

- A public decision maker chooses the best offer given by $n$ local firms.
- Secret evaluation/preference $f_i > 0$ (**technical quality**) of each firm $i$
- For invitation $j \in [q]$, firm $i \in [n]$ asks for the price $p_{ij}$
- The decision maker minimizes the weighted cost:

$$\min_{i \in [n]} p_{ij} f_i^{-1}$$

- Other interpretation:

$$f_i^{-1} = 1 - \alpha_i \beta$$

  may represent a proportional **bribe**: firm $i$ promises to secretly give back $\alpha_i p_{ij}$ to the decision maker.
- This is a variant of first-price sealed-bid auction or blind auction.

# Equilibrium in invitation to tenders

The prices yield an equilibrium if :

$$\min_{i \in [q]} p_{ij} f_i^{-1} \quad \text{is achieved twice at least}$$

# Equilibrium in invitation to tenders

The prices yield an equilibrium if :

$$\min_{i \in [q]} p_{ij} f_i^{-1} \text{ is achieved twice at least}$$

Indeed, if $p_{ij} f_i^{-1} < p_{kj} f_k^{-1}, \forall k \neq i$, then firm $i$ may raise its price and still wins the offer.

# Equilibrium in invitation to tenders

The prices yield an equilibrium if :

$$\min_{i \in [q]} p_{ij} f_i^{-1} \text{ is achieved twice at least}$$

Indeed, if $p_{ij} f_i^{-1} < p_{kj} f_k^{-1}, \forall k \neq i$, then firm $i$ may raise its price and still wins the offer.

Let $V_{ij} = -\log(p_{ij})$ and $a_i = \log(f_i)$, the equilibrium is:

$$\max_{i \in [n]} (V_{ij} + a_i) \text{ is achieved twice at least}$$

$$\text{i.e.} \qquad \forall j \in [q], \quad V_{\cdot j} \in \mathcal{H}_a$$

# Equilibrium in invitation to tenders

The prices yield an equilibrium if :

$$\min_{i \in [q]} p_{ij} f_i^{-1} \text{ is achieved twice at least}$$

Indeed, if $p_{ij} f_i^{-1} < p_{kj} f_k^{-1}, \forall k \neq i$, then firm $i$ may raise its price and still wins the offer.

Let $V_{ij} = -\log(p_{ij})$ and $a_i = \log(f_i)$, the equilibrium is:

$$\max_{i \in [n]} (V_{ij} + a_i) \text{ is achieved twice at least}$$

$$\text{i.e.} \qquad \forall j \in [q], \quad V_{\cdot j} \in \mathcal{H}_a$$

# Inferring hidden information

Finding the secrete preferences $f_i, i \in [n]$ (or bribes) reduces to solving the tropical linear regression problem:

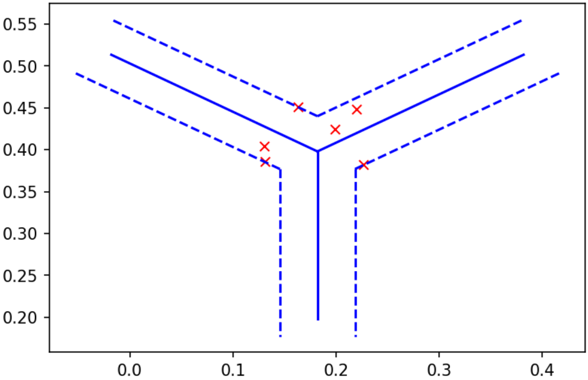
$$\inf_{b \in \mathbb{P}(\mathbb{R}_{\max})^n} \mathrm{dist}_H(\mathcal{V}, H_b) \ , \tag{1}$$

where $\mathcal{V} := (V_{\cdot j})_{j \in [q]} \subset (\mathbb{R}_{\max})^n$, with $V_{ij} = -\log(p_{ij})$.

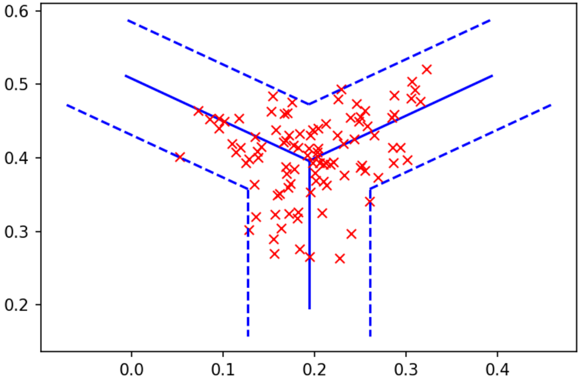
The value of (1) measures the "distance to equilibrium", and a minimizing vector $b^*$ is the hidden information $a = \log(f)$.

# Example

| | ind. houses | social housing | school | road | stadium | bridge | $f$ | $f^{\mathrm{reg}}$ |
|---|---|---|---|---|---|---|---|---|
| Firm 1 | 1.02 | 3.21 | <u>8.72</u> | 26.2 | 69.8 | <u>123</u> | 1 | 1 |
| Firm 2 | 0.81 | 2.65 | 7.49 | 20.3 | <u>53.8</u> | 106 | 0.8 | 0.81 |
| Firm 3 | <u>0.6</u> | <u>1.86</u> | 5.5 | <u>14.7</u> | 41.8 | 76 | 0.6 | 0.605 |




The same example.




A random example with 100 invitations to tenders.

**Theorem (Akian, SG, Qi, Saadi)**

*Solving the regression problem for tropical linear hyperplanes is equivalent to solving a (deterministic) mean payoff game.*

# Tit for tat game

Given a matrix $V \in (\mathbb{R}_{\max})^{n\times p}$, want to solve

$$\min_a \max_{k\in[p]} \mathrm{dist}_H(V_{\cdot k}, \mathcal{H}_a) \qquad \text{where}$$

$$\mathcal{H}_a := \{x \in (\mathbb{R}_{\max})^n \mid \max_{i\in[n]}(a_i + x_i) \text{ is achieved (at least) twice}\}$$

Associate to $V$ the *Shapley operator* $T : (\mathbb{R}_{\max})^n \to (\mathbb{R}_{\max})^n$,

$$T_i(x) = \min_{k\in[p], V_{ik}\neq -\infty} \Big[ - V_{ik} + \max_{j\in[n], j\neq i} (V_{jk} + x_j)\Big], \;\; i \in [n] \;\; ,$$

of the zero-sum two-player deterministic game:

# Tit for tat game

Given a matrix $V \in (\mathbb{R}_{\max})^{n\times p}$, want to solve

$$\min_a \max_{k\in[p]} \operatorname{dist}_H(V_{\cdot k}, \mathcal{H}_a) \qquad \text{where}$$

$$\mathcal{H}_a := \{x \in (\mathbb{R}_{\max})^n \mid \max_{i\in[n]}(a_i + x_i) \text{ is achieved (at least) twice}\}$$

Associate to $V$ the *Shapley operator* $T : (\mathbb{R}_{\max})^n \to (\mathbb{R}_{\max})^n$,

$$T_i(x) = \min_{k\in[p], V_{ik}\neq -\infty} \Big[ - V_{ik} + \max_{j\in[n], j\neq i} (V_{jk} + x_j) \Big], \;\; i \in [n] \;\; ,$$

of the zero-sum two-player deterministic game:

- There are two players Min and Max
- Starting from a state $i$, Min chooses $k \in [q]$ s.t. $V_{ik} \neq -\infty$
- Then Max chooses the next state $j \neq i$
- $-V_{ik} + V_{jk}$ is the instantaneous payment made by Min to Max

# Tit for tat game

Given a matrix $V \in (\mathbb{R}_{\max})^{n\times p}$, want to solve

$$\min_{a} \max_{k\in[p]} \mathrm{dist}_H(V_{\cdot k}, \mathcal{H}_a) \qquad \text{where}$$

$$\mathcal{H}_a := \{x \in (\mathbb{R}_{\max})^n \mid \max_{i\in[n]}(a_i + x_i) \text{ is achieved (at least) twice}\}$$

Associate to $V$ the *Shapley operator* $T : (\mathbb{R}_{\max})^n \to (\mathbb{R}_{\max})^n$,

$$T_i(x) = \min_{k\in[p], V_{ik}\neq -\infty} \Big[ - V_{ik} + \max_{j\in[n], j\neq i} (V_{jk} + x_j)\Big], \;\; i \in [n] \;\; ,$$

of the zero-sum two-player deterministic game:

- There are two players Min and Max
- Starting from a state $i$, Min chooses $k \in [q]$ s.t. $V_{ik} \neq -\infty$
- Then Max chooses the next state $j \neq i$
- $-V_{ik} + V_{jk}$ is the instantaneous payment made by Min to Max
- Note the asymetry: Min can play tit for tat but Max cannot!

Proposition (Akian, SG, Guterman IJAC 2012)

*The columns $V_{\cdot,k}, k \in [p]$ belong to the tropical hyperplane*

$$\mathcal{H}_a := \{x \in (\mathbb{R}_{\max})^n \mid \max_{i\in[n]}(a_i + x_i) \text{ is achieved (at least) twice}\}$$

*iff*

$$a \leqslant T(a)$$

*where*

$$T_i(x) = \min_{k\in[p], V_{ik}\neq -\infty} \Big[ - V_{ik} + \max_{j\in[n], j\neq i} (V_{jk} + x_j) \Big]$$

$$a \leqslant T(a) \iff V_{ik} + a_i \leqslant \max_{j\neq i} V_{jk} + a_j, \forall k$$

Let $\mathrm{Sp}(V) = \{\sup_k \lambda_k + V_{\cdot k}, \lambda_k \in \mathbb{R}_{\max}\}$ denote the set of tropical linear combinations of the columns of $V$.

**Theorem (Strong duality)**

$$\min_{a \in \mathbb{P}(\mathbb{R}_{\max})^n} \max_k \mathrm{dist}_H(V_{\cdot k}, \mathcal{H}_a) = -\rho(T)$$
$$= \sup\{r \geqslant 0 \mid \exists b \in \mathbb{R}^n, B(b,r) \subset \mathrm{Sp}(\mathcal{V})\}.$$

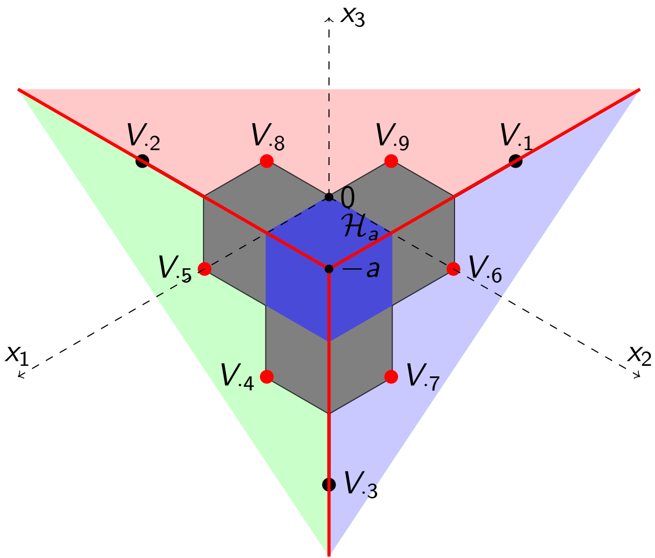
*Moreover,*

- *if $T(a) \geqslant \rho(T) + a$, then $\mathcal{H}_a$ is optimal*
- *if $\rho(T)$ is finite and $T(c) \leqslant \rho(T) + c$, then $B(-c, -\rho(T))$ is optimal*

**Corollary**

*The tropical linear regression problem is polynomial-time equivalent to mean payoff games.*

Indeed, the above shows "regression reduces to MPG", opposite reduction derived from a result of Grigoriev and Podolski (tropical polyhedra are shadows of linear prevarieties).




$$V = \begin{pmatrix} -3 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & -1 \\ 0 & -3 & 0 & 0 & -1 & 1 & 1 & -1 & 0 \\ -1 & -1 & -4 & -2 & -1 & -1 & -2 & 0 & 0 \end{pmatrix}$$

# How we solved tropical linear regression and SVM

Solve $T(v) = \rho(T) + v$ by projective Krasnoselkii-Mann value iteration algorithm:
Fix $\beta \in (0, 1)$. Start with $v^0 = (0, \cdots, 0)^\top$, and for $k = 0, 1, \cdots, N$:

$$\tilde{v}^{k+1} = T(v^k) - (\max_{i\in[n]} T(v^k)_i), \tag{2}$$
$$v^{k+1} = (1 - \beta)v^k + \beta\tilde{v}^{k+1}. \tag{3}$$

Special case of Krasnoselskii-Mann iteration for nonexpansive mappings in Banach spaces.

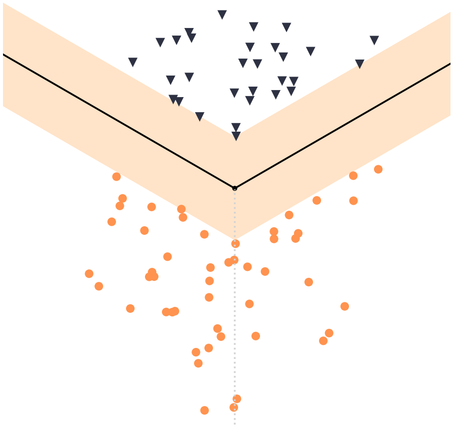
Converges if and only if there is a finite solution $v$ of $T(v) = \rho(T) + v$ (always true if the input points in the regression problem have finite entries).

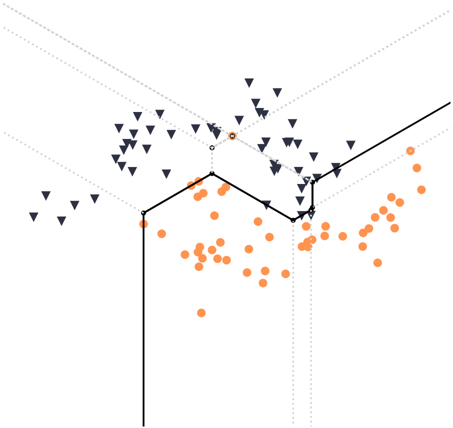
Error bound $d_H(v^{k+1}, v^k) = O(1/\sqrt{k})$ (follows from Baillon-Bruck), much faster in practice.

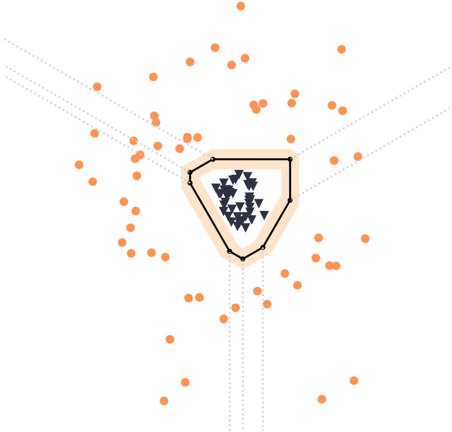
# Part VI.
# Tropical SVM

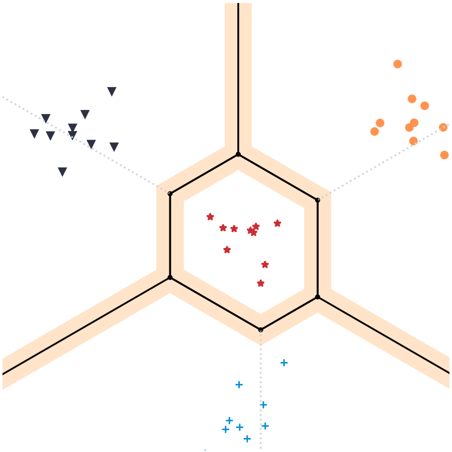
Tropical SVM : separate $n$ data sets by a tropical hyperplane; introduced by Gärtner and Jaggi (2006). Exptime algorithm. Variant considered by Yoshida et al., motivated by phylogenetic analysis.

This talk: recent work with Allamigeon, Boité, Molfessis: separating tropically data sets reduces to mean-payoff games (solvable in a highly scalable way).









multiclass separation

# Binary tropical hard-margin classifiers

We seek to separate two tropical convex hulls of points $\mathcal{V}^+ = \mathrm{Col}(V^+)$ and $\mathcal{V}^- = \mathrm{Col}(V^-)$. We choose two diagonal-free Shapley operators $T^+$ and $T^-$ such that $\mathcal{S}(T^\pm) := \{x \mid T^\pm(x) \geqslant x\} = \mathcal{V}^\pm$.
We can take

$$T_V(x) := \left[P_V^{\mathrm{DF}}(x)\right]_i \coloneqq \max_{1 \le j \le p} \left\{ V_{ij} + \min_{k \neq i}(-V_{kj} + x_k) \right\}.$$

We define:

$$T = \min(T^+, T^-),$$

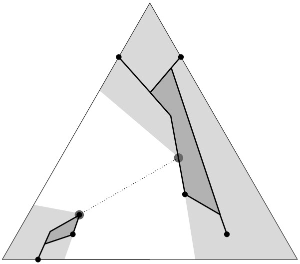
**Theorem (Allamigeon, Boité, SG, Molfessis 2024)**

*The best margin of separation of the sets $\mathcal{V}^\pm$ by a tropical hyperplane coincides with $-\rho(T)$, the opposite of the value of the mean payoff game with operator $T$. Any eigenvector of $T$ gives an apex of a separating tropical hyperplane.*

Builds on tropical analogue of Von-Neumann cyclic projections, SG, Sergeev, Fund. i priklad. mat. 07

If $\mathcal{V}_1 \cap \cdots \cap \mathcal{V}_k = \{\text{"0"}\}$, we can find half-spaces $\mathcal{H}_i$ such that $\mathcal{H}_i \supset \mathcal{V}_i$ and $\mathcal{H}_1 \cap \cdots \cap \mathcal{H}_k = \{\text{"0"}\}$. The apices of these half-spaces are obtained from an eigenvector $u$ of the cylic projector

$$\text{"}P_{\mathcal{V}_1} \cdots P_{\mathcal{V}_k}(u) = \lambda u\text{"}$$



# Separation by a tropical hypersurface

Given a collection of monomials $A \subset \mathbb{Z}^n$, find a tropical hypersurface

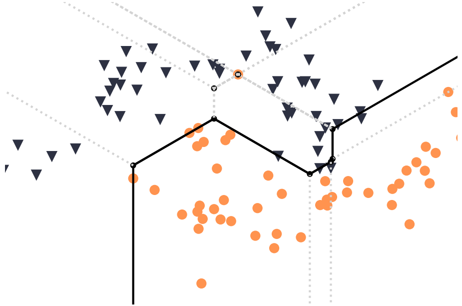
$$H^{\text{trop}}(p) = \{X \in \mathbb{R}^n \mid \max_{\alpha}(p_\alpha + \langle \alpha, X \rangle) \text{ attained twice}\} \ .$$

which separates two data clouds.
Problem introduced by Charisopoulos and P. Maragos at ISMM 2017.

We apply the above tropical SVM algorithm to the Veronese embedding of the data sets

$$\text{ver}_{\mathcal{A}}(x) := (\langle x, \alpha \rangle)_{\alpha \in \mathcal{A}} \in \mathbb{R}_{\max}^{\mathcal{A}},$$

This provides a constant factor approximation of the hard-margin of SVM for tropical hypersurfaces (when $\mathcal{A} \subset \mathbb{N}^n$, the constant is the degree $\max_{\alpha \in A} \|\alpha\|_1$). Conditioning is much better than in the clasical case



# Concluding remarks

- Equivalence between tropical polyhedra and mean-payoff games
- Solves: regression for tropical hyperplanes
- Piecewise-linear separation (tropical SVM with hard margin)
- soft margin results still to be fully explored
- regression for tropical linear spaces of higher rank is unsolved
- tropical principal component analysis is exptime to solve optimally; still very useful in applications (curse of dimensionality free methods - McEneaney-, phylogenetic analysis- Yoshida et al.-)
- advertisement. Current work with Yannis Vlassopoulos, tropical approach of LLM via directed metric spaces and tropical (alcoved) polyhedra, see arXiv:2405.12264.

Thank you !

M. Akian, S. Gaubert, and A. Guterman. Tropical polyhedra are equivalent to mean payoff games. *International Journal of Algebra and Computation*, 22(1):125001 (43 pages), 2012. doi: 10.1142/S0218196711006674.

X. Allamigeon, P. Benchimol, S. Gaubert, and M. Joswig. Tropicalizing the simplex algorithm. *SIAM J. Disc. Math.*, 29(2): 751–795, 2015a. doi: 10.1137/130936464.

X. Allamigeon, S. Gaubert, and M. Skomra. Solving generic nonarchimedean semidefinite programs using stochastic game algorithms. *Journal of Symbolic Computation*, 85:25–54, 2018. doi: 10.1016/j.jsc.2017.07.002.

Xavier Allamigeon, Pascal Benchimol, and Stéphane Gaubert. The tropical shadow-vertex algorithm solves mean payoff games in polynomial time on average. In *Proceedings of the 41st International Colloquium on Automata, Languages, and Programming (ICALP)*, number 8572 in Lecture Notes in Computer Science, pages 89–100. Springer, 2014.

Xavier Allamigeon, Pascal Benchimol, Stéphane Gaubert, and Michael Joswig. Combinatorial simplex algorithms can solve mean payoff games. *SIAM J. Opt.*, 24(4):2096–2117, 2015b. doi: 10.1137/140953800.

Xavier Allamigeon, Stéphane Gaubert, and Mateusz Skomra. The tropical analogue of the Helton–Nie conjecture is true. *Journal of Symbolic Computation*, 91:129 – 148, 2019. ISSN 0747-7171. doi: https://doi.org/10.1016/j.jsc.2018.06.017. URL http://www.sciencedirect.com/science/article/pii/S0747717118300828. MEGA 2017, Effective Methods in Algebraic Geometry, Nice (France), June 12-16, 2017.

Xavier Allamigeon, Stéphane Gaubert, and Mateusz Skomra. Tropical spectrahedra. *Discrete Comput. Geom.*, 63:507–548, 2020. doi: 10.1007/s00454-020-00176-1.

Xavier Allamigeon, Pascal Benchimol, Stéphane Gaubert, and Michael Joswig. What tropical geometry tells us about the complexity of linear programming. *SIAM Review*, 63(1):123–164, 2021. doi: 10.1137/20M1380211.

Xavier Allamigeon, Stéphane Gaubert, and Nicolas Vandame. No self-concordant barrier interior point method is strongly polynomial. Proceedings of STOC 2022: 54th Annual ACM Symposium on Theory of Computing, arXiv:2201.02186, 2022.

Jérôme Bolte, Stéphane Gaubert, and Guillaume Vigeral. Definable zero-sum stochastic games. *Mathematics of Operations Research*, 40(1):171–191, 2014. doi: 10.1287/moor.2014.0666.

Cristian S. Calude, Sanjay Jain, Bakhadyr Khoussainov, Wei Li, and Frank Stephan. Deciding parity games in quasipolynomial time. In *Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing*, STOC 2017, page 252–263, New York, NY, USA, 2017. Association for Computing Machinery. ISBN 9781450345286. doi: 10.1145/3055399.3055409. URL https://doi.org/10.1145/3055399.3055409.

G. Cohen, S. Gaubert, and J.-P. Quadrat. Duality and separation theorems in idempotent semimodules. *Linear Algebra and Appl.*, 379:395–422, 2004. URL http://arxiv.org/abs/math.FA/0212294.

J.A. De Loera, B. Sturmfels, and C. Vinzant. The central curve in linear programming. *Foundations of Computational Mathematics*, 12(4):509–540, 2012.

M. Develin and B. Sturmfels. Tropical convexity. *Doc. Math.*, 9:1–27, 2004. ISSN 1431-0635.

M. Develin, F. Santos, and B. Sturmfels. On the rank of a tropical matrix. In *Combinatorial and computational geometry*, volume 52 of *Math. Sci. Res. Inst. Publ.*, pages 213–242. Cambridge Univ. Press, Cambridge, 2005.

A. Ehrenfeucht and J. Mycielski. Positional strategies for mean payoff games. *International Journal of Game Theory*, 8(2): 109–113, 1979.

M. Forsberg, M. Passare, and A. Tsikh. Laurent determinants and arrangements of hyperplane amoebas. *Adv. Math.*, 151(1): 45–70, 2000. ISSN 0001-8708.

Bernd Gärtner and Martin Jaggi. Tropical support vector machines. Technical report, ACS-TR-362502-01.

S. Gaubert and R. Katz. Minimal half-spaces and external representation of tropical polyhedra. *Journal of Algebraic Combinatorics*, 33(3):325–348, 2011. doi: 10.1007/s10801-010-0246-4, arXiv:arXiv:0908.1586,.

S. Gaubert and G. Vigeral. A maximin characterization of the escape rate of nonexpansive mappings in metrically convex spaces. *Math. Proc. of Cambridge Phil. Soc.*, 152:341–363, 2012. doi: 10.1017/S0305004111000673.

I. Gelfand, M. Kapranov, and A. Zelevinsky. *Discriminants, resultants, and multidimensional determinants*. Birkhäuser, 1994.

I. Itenberg, G. Mikhalkin, and E. Shustin. *Tropical algebraic geometry*. Oberwolfach seminars. Birkhäuser, 2007.

M. Joswig. Tropical halfspaces. In *Combinatorial and computational geometry*, volume 52 of *Math. Sci. Res. Inst. Publ.*, pages 409–431. Cambridge Univ. Press, Cambridge, 2005. Also eprint arXiv:math.CO/0312068.

B. Lemmens and R. Nussbaum. *Nonlinear Perron-Frobenius Theory*. Cambridge university Press, 2012.

Bas Lemmens, Brian Lins, Roger Nussbaum, and Marten Wortel. Denjoy-wolff theorems for hilbert's and thompson's metric spaces. *Journal d'Analyse Mathématique*, 134(2):671–718, February 2018. ISSN 1565-8538. doi: 10.1007/s11854-018-0022-2. URL `http://dx.doi.org/10.1007/s11854-018-0022-2`.

G. Mikhalkin. Amoebas of algebraic varieties and tropical geometry. In *Different faces of geometry*, volume 3 of *Int. Math. Ser. (N. Y.)*, pages 257–300. Kluwer/Plenum, New York, 2004. URL `http://arxiv.org/abs/math.AG/0403015`.

G. Mikhalkin. Enumerative tropical algebraic geometry in $\mathbb{R}^2$. *J. Amer. Math. Soc.*, 18(2):313–377, 2005. ISSN 0894-0347.

S. Mizuno, M. J. Todd, and Y. Ye. On adaptive-step primal-dual interior-point algorithms for linear programming. *Mathematics of Operations Research*, 18(4):964–981, 1993.

U. Zwick and M. Paterson. The complexity of mean payoff games on graphs. *Theoret. Comput. Sci.*, 158(1-2):343–359, 1996. ISSN 0304-3975.