(Infinite-Dimensional) Gauge Symmetries in Al Algorithms

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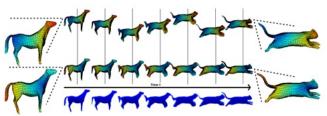


Riemannian submersion Riemannian immersion Gauge Invariant metrics Examples



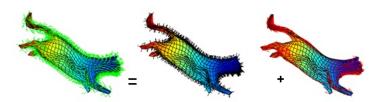
[1] A.B. Tumpach, *Some aspects of infinite-dimensional Geometry: Theory and Applications*, 212 pages, Habilitation Thesis, Lille University, 9.12.2022. http://tel.archives-ouvertes.fr/tel-00012012

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- natural objects on a finite-dimensional manifold are elements of an infinite-dimensional space (vector fields, Riemannian metrics, mesures...)

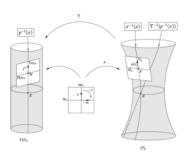


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- natural objects on a finite-dimensional manifold are elements of an infinite-dimensional space (vector fields, Riemannian metrics, mesures...)
- existence of geodesics on a finite-dimensional manifold is an infinite-dimensional phenomenon
 - initial value problem or shooting: geodesic is a solution of a Cauchy problem, i.e. a fixed point of a contraction in an appropriate infinite-dimensional space of curves
 - 2 boundary value problem: geodesic is a curve minimising an energy functional on a infinite-dimensional space of curves

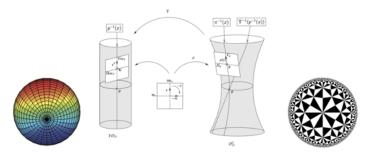


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 - 2 boundary value problem: geodesic is a curve minimising an energy functional on a infinite-dimensional space of curves
- Each time one wants to vary the geometry of a finite-dimensional manifold, one ends up with a infinite-dimensional manifold (of Riemannian metric, of connexions, of symplectic forms....)

 Some geometric structures of finite-dimensional manifolds are constructed using infinite-dimensional quotients example: hyperkähler structures of complex coadjoint orbits of complex Lie groups are constructed using quotient of the space of solutions of Nahm's equations by the action of the gauge group

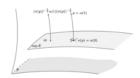


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[2] A. B. Tumpach, *Hyperkähler structures and infinite-dimensional Grassmannians*, Journal of Functional Analysis, https://doi.org/10.1016/j.jfa.2006.05.019

- time-dependant feature maps lead to infinite-dimensional feature spaces
- the correlations between time-dependant feature maps lead to infinite-dimensional covariance operators that one can study using the HS affine invariant metric



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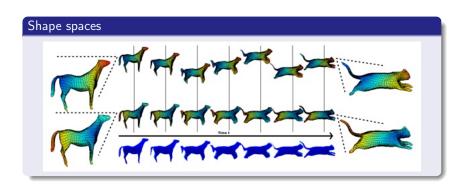
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PLAN

- Quotient spaces
- Riemannian structures on Quotient spaces: 3 Methods
 - Riemannian Submersions
 - Riemannian Immersions
 - Gauge Invariant Metrics
- Examples

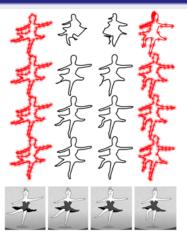
Group G		one preferred element in	
	$\operatorname{der} G$	the orbit	ferred element
\mathbb{R}^2 acting by translations		centroid at origin	curve starting at origin
SO(2) acting by rotations		axes of approximating ellipse aligned	tangent vector at starting point horizontal
R ⁺ acting by scaling		length = 1	enclosed area =1

Group G	Some elements of one orbit under the group G	a preferred element in the orbit	another choice of preferred element in the orbit
R ⁺ acting by scaling		length = 1	enclosed area =1
Diff ⁺ ([0, 1]) acting by reparameterization	lor lore lore	arc-length parameterization	curvature proportional parameterization



Pre-shape space $\mathscr{F} := \{f \text{ embedding } : \mathbb{S}^2 \to \mathbb{R}^3\} \subset \mathscr{C}^{\infty}(\mathbb{S}^2, \mathbb{R}^3)$ **Shape space** $\mathscr{S} := 2$ -dimensional submanifolds of \mathbb{R}^3

Shape spaces are non-linear manifolds



Pre-shape space $\mathscr{F}:=\{f \text{ immersion }: \mathbb{S}^1 \to \mathbb{R}^2\} \subset \mathscr{C}^\infty(\mathbb{S}^1,\mathbb{R}^2)$ Shape space $\mathscr{S}:=1$ -dimensional immersed submanifolds of \mathbb{R}^2 (I) Quotient Riemannian metric. The first way to endow the quotient space $S := \mathcal{F}/G$ with a Riemannian metric is through the quotient Riemannian metric. We recall the following classical Theorem of Riemannian geometry [5][11].

Theorem 1 (Riemannian submersion Theorem). Let \mathcal{F} be a manifold endowed with a Riemannian metric $g_{\mathcal{F}_f}$ and G a Lie group acting on \mathcal{F} in such a way that \mathcal{F}/G is a smooth manifold. Suppose $g_{\mathcal{F}}$ is G-invariant and $T_F\mathcal{F}$ splits into the direct sum of the tangent space to the fiber and its orthogonal complement, i.e.,

$$g_{\mathcal{F}}(X, Y) = g_{\mathcal{F}}(g \cdot X, g \cdot Y), \forall X, Y \in T\mathcal{F}, \forall g \in G,$$

 $T_F \mathcal{F} = \text{Ker}(dp)_F \oplus \text{Ker}(dp)_F^{\perp}, \forall F \in \mathcal{F},$

$$(5)$$

then there exists a unique Riemannian metric $g_{1,S}$ on the quotient space $S = \mathcal{F}/G$ such that the canonical projection $p : \mathcal{F} \to S$ is a Riemannian submersion, i.e. such that $dp : \operatorname{Ker}(dp)^{\perp} \to TS$ is an isometry.

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(II) Riemannian metric induced on a smooth section. Now suppose that we have chosen a preferred smooth section $s: \mathcal{S} \to \mathcal{F}$ of the fiber bundle $p: \mathcal{F} \to \mathcal{S} = \mathcal{F}/G$, for instance the space of arc-length parameterized curves in the case where G is the the group of orientation-preserving reparameterizations, or the space of centered curves when G is the group of translations. The smoothness assumption means that the range of s is a smooth manifold of \mathcal{F} , like the space of arc-length parameterized curves in the space of parameterized curves. We will denote it by $\mathcal{A} := s(\mathcal{S})$. By construction, there is a isomorphism between \mathcal{S} and \mathcal{A} which one can use to endow the quotient space \mathcal{S} with the induced Riemannian structure on \mathcal{A} by \mathcal{F} .

Theorem 2 (Riemannian immersion Theorem). Given a smooth section $s: S \to \mathcal{F}$, there exists a unique Riemannian metric g_A on A:=s(S) such that the inclusion $\iota: A \hookrightarrow \mathcal{F}$ is an isometry. Using the isomorphism $s: S \to A$, there exists a unique Riemannian metric $g_{2,S}$ on S such that $s: S \to \mathcal{F}$ is an isometry.

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(III) Gauge invariant metric. Here we suppose that we have a vector bundle Nor over \mathcal{F} which is a G-invariant subbundle of $T\mathcal{F}$ transverse to the vertical bundle Ver := $\operatorname{Ker}(dp)$. Using any G-invariant metric $g_{\mathcal{F}}$ on \mathcal{F} , one can define a G-invariant metric g_{GI} on \mathcal{F} that is degenerate along the fiber of the projection $p: \mathcal{F} \to \mathcal{S}$. We will explain the meaning of "gauge invariance" later.

Theorem 3. Let $g_{\mathcal{F}}$ be a G-invariant metric on \mathcal{F} and $\operatorname{Nor} \subset T\mathcal{F}$ be a G-invariant subbundle of $T\mathcal{F}$ such that

$$T_F \mathcal{F} = \text{Ker}(dp)_F \oplus \text{Nor}_F, \forall F \in \mathcal{F}.$$
 (6)

There exists a unique metric g_{GI} on $T\mathcal{F}$ which coincides with $g_{\mathcal{F}}$ on Nor and is degenerate exactly along the vertical fibers of $p: \mathcal{F} \to \mathcal{S}$. It induces a Riemannian metric $q_{3,\mathcal{S}}$ on shape space \mathcal{S} such that $dp: \text{Nor} \to T\mathcal{S}$ is an isometry.

$$\mathcal{C}_1(I) = \{ \gamma \in \mathcal{C}(I) : \int_0^1 |\gamma'(s)| ds = 1 \}.$$

$$\mathscr{A}_1(I) = \{ \gamma \in \mathcal{C}(I) : |\gamma'(s)| = 1, \ \forall s \in I \} \subset \mathcal{C}_1(I).$$

Theorem (A.B.T, S.Preston)

Given a curve $\gamma \in C_1(I)$, let $p(\gamma) \in \mathscr{A}_1(I)$ denote its arc-length-reparameterization, so that $p(\gamma) = \gamma \circ \psi$ where

$$\psi'(s) = \frac{1}{|\gamma'(\psi(s))|}, \qquad \psi(0) = 0. \tag{1}$$

Then p is a smooth retraction of $C_1(I)$ onto $\mathcal{A}_1(I)$.

Theorem (A.B.T, S.Preston)

 $\mathscr{A}_1([0,1])$ is diffeomorphic to the quotient Fréchet manifold $\mathcal{C}_1([0,1])/\mathscr{G}([0,1])$.

Riemannian submersion

We will consider the 2-parameter family of elastic metrics on $C_1(I)$ introduced by Mio et al. :

$$G^{a,b}(w,w) = \int_0^1 \left(a \left(D_s w \cdot \mathbf{v} \right)^2 + b \left(D_s w, \mathbf{n} \right)^2 \right) |\gamma'(t)| dt, \qquad (2)$$

where a and b are positive constants, γ is any parameterized curve in $\mathcal{C}_1(I)$, w is any element of the tangent space $T_\gamma \mathcal{C}_1(I)$, with $D_s w = \frac{w'}{|\gamma'|}$ denoting the arc-length derivative of w, $v = \gamma'/|\gamma'|$ and $n = v^\perp$.

Since the reparameterization group preserves the elastic metric $G^{a,b}$, it defines a quotient elastic metric on the quotient space $\mathcal{C}_1([0,1])/\mathscr{G}([0,1])$, which we will denote by $\overline{G}^{a,b}$.

$$\overline{G}^{a,b}([w],[w]) = \inf_{u \in T_{a}, \emptyset} G^{a,b}(w+u, w+u)$$

Since $\mathscr{A}_1([0,1])$ is diffeomorphic to the quotient Fréchet manifold $\mathcal{C}_1([0,1])/\mathscr{G}([0,1])$, we can pull-back the quotient elastic metric $\overline{G}^{a,b}$ to the space of arc-length parameterized curves $\mathscr{A}_1([0,1])$ and define

$$\widetilde{G}^{a,b}(w,w) = G^{a,b}([w],[w]) = \inf_{u \in T_{\gamma} \mathscr{O}} G^{a,b}(w+u,w+u)$$

where w is tangent to $\mathscr{A}_1([0,1])$.

If $T_{\gamma}\mathcal{C}_1([0,1])$ decomposes as $T_{\gamma}\mathcal{C}_1([0,1]) = T_{\gamma}\mathscr{O} \oplus \mathit{Hor}_{\gamma}$, this minimum is achieved by the unique vector $P_h(w) \in [w]$ belonging to the horizontal space Hor_{γ} at γ . In this case:

$$\widetilde{G}^{a,b}(w,w) = G^{a,b}(P_h(w), P_h(w)), \tag{3}$$

where $P_h(w) \in T_\gamma C_1([0,1])$ is the projection of w onto the horizontal space.

Theorem (A.B.T, S.Preston)

Let w be a tangent vector to the manifold $\mathscr{A}_1([0,1])$ at γ and write $w'=\Phi$ n, where Φ is a real function in $\mathscr{C}^\infty([0,1],\mathbb{R})$. Then the projection $P_h(w)$ of $w\in T_\gamma\mathscr{A}_1([0,1])$ onto the horizontal space Hor_γ reads $P_h(w)=w-m\nu$ where $m\in\mathscr{C}^\infty([0,1],\mathbb{R})$ is the unique solution of

$$-\frac{a}{b}m'' + \kappa^2 m = \kappa \Phi, \qquad m(0) = 0, \quad m(1) = 0$$
 (4)

where κ is the curvature function of γ .



Figure: Toy example: initial path joining a circle to the same circle via an ellipse. The 5 first shapes at the left correspond to the path at time t=0, t=0.25, t=0.5, t=0.75 and t=1. The right picture shows the entire path, with color varying from red (t=0) to blue (t=0.5) to red again (t=1).

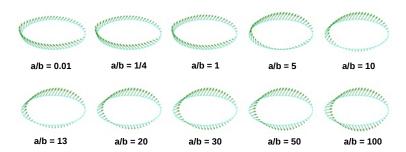


Figure: Gradient of the energy functional at the middle of the path depicted in Fig. 1 for b = 1 and different values of the parameter a/b.

Curvature-length Parameterization

Arc-length parameterization is used to compute the curvature function. One can integrate the absolute value of the curvature along the curve and renormalized to have a total integral equal to 1. The resulting function is used to define the curvature-length parameterization of the curve and resample the curve accordingly.

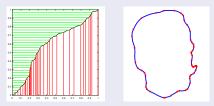


Figure: Integral of the (renormalized) absolute value of the curvature (left), and corresponding resampling of Elie's Cartan head (right).

From Curvature-length Parameterization to Arc-length Parameterization

The draw-back of curvature-length parameterization is that it does not put points at all on flat pieces of the curve. In order to fix this, instead of integrating the curvature, one can integrate λ + curvature, where λ is a parameter.

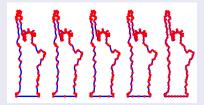
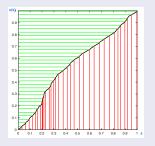


Figure:

Resampling of the statue of Liberty proportional to the intergral of λ + curvature, for (from left to right) $\lambda = 0$; $\lambda = 0.3$; $\lambda = 1$; $\lambda = 2$; $\lambda = 100$.

Curvarc-length Parameterization

The Curvarc-length parameterization is defined by $u(s) = \frac{\int_0^s (L+|\kappa(s)|)ds}{\int_0^1 (L+|\kappa(s)|)ds}$ where L is the length of the curve, and κ the curvature function.



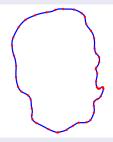


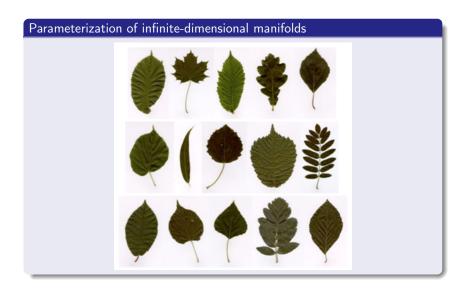
Figure: Integral of the (renormalized) curvarc length (left), and corresponding resampling of Elie's Cartan head (right).

WHAT TO REMEMBER IN ORDER TO DEFINE A MANIFOLD OF CANONICALLY PARAMETERIZED CURVES?

- ✓ the only invariant of plane curves in arc-length parameterization is the curvature
- ✓ Any strictly increasing function applied to the absolute value of the curvature defines a canonical parameterization of plane curves
- ✓ finding the right parameterization adapted to a given application = finding the right strictly increasing function (physical considerations have their space here)

EXERCICE

- Prove that the set of smooth curves parameterized proportionnally to the curvature is a smooth submanifold of the manifold of parameterized curves
- Compute the tangent space
- Compute the projection of a arbitrary parameterized curve to a curve parameterized proportionnal to curvature-length
- Compute the induced elastic metric



Parameterization of infinite-dimensional manifolds

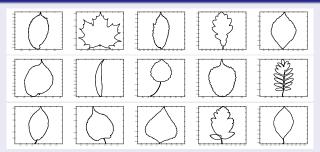


Figure: Examples of extracted boundary curves for Swedish Leaf Dataset [1]. An element of each class is represented.

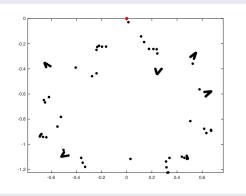


Figure: One leaf of Acer parameterized by the curvature

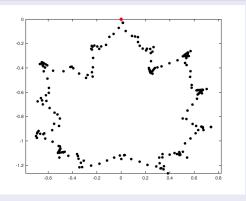


Figure: One leaf of Acer parameterized by the curvature and on top of it the same leaf parameterized by arc-length

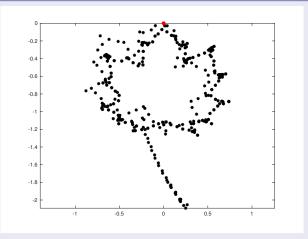


Figure: 2 leaves of Acer parameterized by arc-length

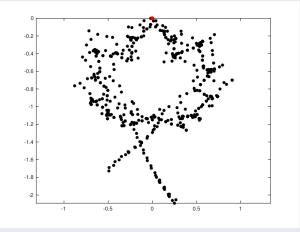


Figure: 3 leaves of Acer parameterized by arc-length

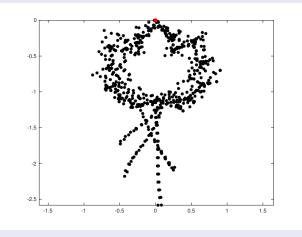


Figure: 5 leaves of Acer parameterized by arc-length

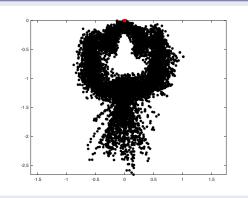


Figure: 75 leaves of Acer parameterized by arc-length

What is the best parameterization of a leave?

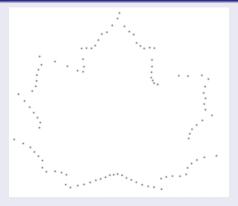
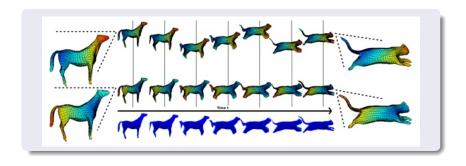


Figure: One candidate: Clock parameterization



Pre-shape space $\mathscr{F} := \{f \text{ embedding } : \mathbb{S}^2 \to \mathbb{R}^3\} \subset \mathscr{C}^{\infty}(\mathbb{S}^2, \mathbb{R}^3)$ **Shape space** $\mathscr{S} := 2$ -dimensional submanifolds of \mathbb{R}^3

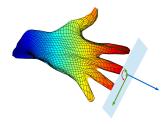
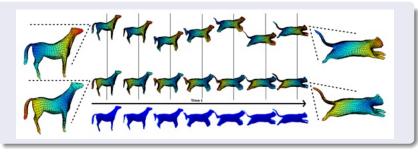


Figure: Scalar product on the tangent plan to the tip of the middle finger of a hand.

Genus-0 surfaces of \mathbb{R}^3 are *Riemann surfaces*. Since they are compact and simply connected, the Uniformization Theorem says that they are conformally equivalent to the unit sphere. This means that, given a spherical surface, there exists a homeomorphism, called the *uniformization map*, which preserves the angles and transforms the unit sphere into the surface. The uniformization maps are parameterized by $PSL(2,\mathbb{C})$.



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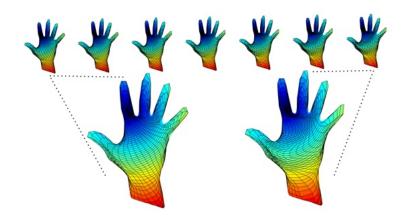


Figure: A path of zero energy (modulo computer accuracy)

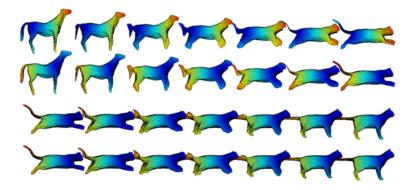


Figure: Pairs of paths projecting to the same path in Shape space, but with different parametrizations. The energies of these paths, as computed by our program, are respectively (from the upper row to the lower row): $E_{\Delta} = 225.3565$, $E_{\Delta} = 225.3216$, $E_{\Delta} = 180.8444$, $E_{\Delta} = 176.8673$.

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